Forum Geometricorum
Volume 7 (2007) 67–72.

FORUM GEOM
ISSN 1534-1178

Ceva Collineations

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Abstract. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are lines. There exists a unique point U such that if $X \in \mathcal{L}_1$, then $X^{-1} \copyright U \in \mathcal{L}_2$, where X^{-1} denotes the isogonal conjugate of X and $X^{-1} \copyright U$ is the X^{-1}-Ceva conjugate of U. The mapping $X \mapsto X^{-1} \copyright U$ is the U-Ceva collineation. It maps every line onto a line and in particular maps $\mathcal{L}_1$ onto $\mathcal{L}_2$. Examples are given involving the line at infinity, the Euler line, and the Brocard axis. Collineations map cubics to cubics, and images of selected cubics under certain U-Ceva collineations are briefly considered.

1. Introduction

One of the great geometry books of the twentieth century states [1, p.221] that "Möbius's invention of homogeneous coordinates was one of the most far-reaching ideas in the history of mathematics". In triangle geometry, two systems of homogeneous coordinates are in common use: barycentric and trilinear. Trilinears are especially useful when the angle bisectors of a reference triangle ABC play a central role, as in this note.

Suppose that $X = x : y : z$ is a point. If at most one of x, y, z is 0, then the point

$$X^{-1} = yz : zx : xy$$

is the isogonal conjugate of X, and if none of x, y, z is 0, we can write

$$X^{-1} = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

A traditional construction for X^{-1} depends on interior angle bisectors: reflect line AX in the A-bisector, BX in the B-bisector, CX in the C-bisector; then the reflected lines concur in X^{-1}.

The triangle $A_X B_X C_X$ with vertices

$$A_X = AX \cap BC, \quad B_X = BX \cap CA, \quad C_X = CX \cap AB$$

is the *cevian triangle of* X, and

$$A_X = 0 : y : z, \quad B_X = x : 0 : z, \quad C_X = x : y : 0.$$

If $U = u : v : w$ is a point, then the triangle $A^U B^U C^U$ with vertices

$$A^U = -u : v : w, \quad B^U = u : -v : w, \quad C^U = u : v : -w$$

Publication Date: February 26, 2007. Communicating Editor: Paul Yiu.

is the *anticevian triangle* of U. The lines $A_X A^U$, $B_X B^U$, $C_X C^U$ concur in the point

$$u(-uyz+vzx+wxy) : \ v(uyz-vzx+wxy) : \ w(uyz+vzx-wyz),$$

called the X-*Ceva conjugate* of U and denoted by $X©U$ (see [2, p. 57]). It is easy to verify algebraically that $X©(X©U)=U$ and that if $P=p:q:r$ is a point, then the equation $P=X©U$ is equivalent to

$$\begin{aligned} X =&(ru+pw)(pv+qu):(pv+qu)(qw+rv):(qw+rv)(ru+pw) \\ =&\text{cevapoint}(P,U). \end{aligned} \tag{1}$$

A construction of cevapoint (P,U) is given in the Glossary of [3].

One more preliminary will be needed. A *circumconic* is a conic that passes through the vertices, A,B,C. Every point $P=p:q:r$, where $pqr\neq 0$, has its own circumconic, given by the equation $p\beta\gamma+q\gamma\alpha+r\alpha\beta=0$; indeed, this curve is, loosely speaking, the isogonal conjugate of the line $p\alpha+q\beta+r\gamma=0$, and the curve is an ellipse, parabola, or hyperbola according as the line meets the circumcircle in 0, 1, or 2 points. The circumcircle is the circumconic having equation $a\beta\gamma+b\gamma\alpha+c\alpha\beta=0$.

2. The Mapping $X\mapsto X^{-1}©U$

In this section, we present first a lemma: that for given circumconic $\mathcal{P}$ and line $\mathcal{L}$, there is a point U such that the mapping $X\mapsto X©U$ takes each point X on $\mathcal{P}$ to a point on $\mathcal{L}$. The lemma easily implies the main theorem of the paper: that the mapping $X\mapsto X^{-1}©U$ takes each point of a certain line to $\mathcal{L}$.

Lemma 1. *Suppose $L=l:m:n$ and $P=p:q:r$ are points. Let $\mathcal{P}$ denote the circumconic $p\beta\gamma+q\gamma\alpha+r\alpha\beta=0$ and $\mathcal{L}$ the line $l\alpha+m\beta+n\gamma=0$. There exists a unique point U such that if $X\in\mathcal{P}$, then $X©U\in\mathcal{L}$. In fact,*

$$U=L^{-1}©P=p(-lp+mq+nr):q(lp-mq+nr):r(lp+mq-nr).$$

Proof. We wish to solve the containment $X©U\in\mathcal{L}$ for U, given that $X\in\mathcal{P}$. That is, we seek $u:v:w$ such that

$$u(-uyz+vzx+wxy)l+v(uyz-vzx+wxy)m+w(uyz+vzx-wxy)n=0, \tag{2}$$

given that $X=x:y:z$ is a point satisfying

$$pyz+qzx+qxy=0. \tag{3}$$

Equation (2) is equivalent to

$$u(-ul+vm+wn)yz+v(ul-vm+wn)zx+w(ul+vm-wn)xy=0, \tag{4}$$

so that, treating $x:y:z$ as a variable point, equations (3) and (4) represent the same circumconic. Consequently,

$$u(-lu+mv+nw)qr=v(lu-mv+nw)rp=w(lu+mv-nw)pq.$$

In order to solve for $u:v:w$, we assume, as a first of two cases, that p and q are not both 0. Then the equation

$$u(-lu+mv+nw)qr = v(lu-mv+nw)rp$$

gives

$$w = \frac{(mv-lu)\,(pv+qu)}{n(pv-qu)}. \tag{5}$$

Substituting for w in

$$u(-lu+mv+nw)qr - w(lu+mv-nw)pq = 0$$

gives

$$\frac{\left(mpqv-lpqu+nprv-nqru-lp^2v+mq^2u\right)(mv-lu)\,uv}{2nr(pv-qu)^2} = 0,$$

so that

$$u = \frac{(mq-lp+nr)\,pv}{q(lp-mq+nr)}. \tag{6}$$

Consequently, for given v, we have

$$u:v:w = \frac{(mq-lp+nr)\,pv}{q(lp-mq+nr)} : v : \frac{(mv-lu)\,(pv+qu)}{n(pv-qu)}.$$

Substituting for u from (6), canceling v, and simplifying lead to

$$u:v:w = p\,(-lp+mq+nr) : q\,(lp-mq+nr) : r\,(lp+mq-nr)\,,$$

so that $U = L^{-1}©P$.

If, as the second case, we have $p=q=0$, then $r \neq 0$ because $p:q:r$ is assumed to be a point. In this case, one can start with

$$u(-lu+mv+nw)qr = w(lu+mv-nw)pq$$

and solve for v (instead of w as in (5)) and continue as above to obtain $U = L^{-1}©P$.

The method of proof shows that the point U is unique. □

Theorem 2. *Suppose $\mathcal{L}_1$ is the line $l_1\alpha+m_1\beta+n_1\gamma=0$ and $\mathcal{L}_2$ is the line $l_2\alpha+m_2\beta+n_2\gamma=0$. There exists a unique point U such that if $X\in\mathcal{L}_1$, then $X^{-1}©U\in\mathcal{L}_2$.*

Proof. The hypothesis that $X\in\mathcal{L}_1$ is equivalent to $X^{-1}\in\mathcal{P}$, the circumconic having equation $l_1\beta\gamma+m_1\gamma\alpha+n_1\alpha\beta=0$. Therefore, the lemma applies to the circumconic $\mathcal{P}$ and the line $\mathcal{L}_2$. □

We write the mapping $X\mapsto X^{-1}©U$ as $\mathcal{C}_U(X)=X^{-1}©U$ and call $\mathcal{C}_U$ the U-Ceva collineation. That $\mathcal{C}_U$ is indeed a collineation follows as in [4] from the linearity of x, y, z in the trilinears

$$\mathcal{C}_U(X) = u(-ux+vy+wz) :\ v(ux-vy+wz) :\ w(ux+vy-wz).$$

This collineation is determined by its action on the four points A, B, C, U^{-1}, with respective images A^U, B^U, C^U, U.

Regarding the surjectivity, or onto-ness, of $\mathcal{C}^U$, suppose F is a point on $\mathcal{L}_2$; then the equation X^{-1}ⓒ$U = F$ has as solution

$$X = \text{cevapoint}(F, U))^{-1}.$$

3. Corollaries

Lemma 1 tells how to find U for given $\mathcal{L}$ and $\mathcal{P}$. Here, we tell how to find $\mathcal{L}$ from given $\mathcal{P}$ and U and how to find $\mathcal{P}$ from given U and $\mathcal{L}$.

Corollary 3. *Given a circumconic $\mathcal{P}$ and a point U, there exists a line $\mathcal{L}$ such that if $X \in \mathcal{P}$, then Xⓒ$U \in \mathcal{L}$.*

Proof. Assuming there is such a $\mathcal{L}$, we have the point $U = L^{-1}$ⓒP as Theorem 2, so that $L^{-1} = \text{cevapoint}(U, P)$, and

$$L = (\text{cevapoint}(U, P))^{-1},$$

so that $\mathcal{L}$ is the line $(wq + vr)\alpha + (ur + wp)\beta + (vp + uq)\gamma = 0$. It is easy to check that if $X \in \mathcal{P}$, then Xⓒ$U \in \mathcal{L}$. □

Corollary 4. *Given a line $\mathcal{L}$ and a point U, there exists a circumconic $\mathcal{P}$ such that if $X \in \mathcal{P}$, then Xⓒ$U \in \mathcal{L}$.*

Proof. Assuming there is such a $\mathcal{L}$, we have the point $U = L^{-1}$ⓒP, and $P = L^{-1}$ⓒU, so that $\mathcal{P}$ is the circumconic

$$u(-ul + vm + wn)\beta\gamma + v(ul - vm + wn)\gamma\alpha + w(ul + vm - wn)\alpha\beta = 0.$$

It is easy to check that if $X \in \mathcal{P}$, then Xⓒ$U \in \mathcal{L}$. □

4. Examples

4.1. Let $L = P = 1 : 1 : 1$, so that $\mathcal{L}_1 = \mathcal{L}_2$ is the line $\alpha + \beta + \gamma = 1$. We find $U = 1 : 1 : 1$, so that

$$\mathcal{C}_U(X) = -x + y + z : \ x - y + z : \ x + y - z.$$

It is easy to check that $\mathcal{C}_U(X) = X$ for every X on the line $\alpha + \beta + \gamma = 1$, such as X_{44} and X_{513}. On the line X_1X_2 we have

$$\mathcal{C}_U(X) = X \text{ for } X \in \{X_1, X_{899}\},$$

so that $\mathcal{C}_U$ maps X_1X_2 onto itself; e.g., $\mathcal{C}_U(X_2) = X_{43}$, and $\mathcal{C}_U(X_{1201}) = X_8$, and $\mathcal{C}_U(X_8) = X_{972}$. On X_1X_6 we have fixed points X_1 and X_{44}, so that $\mathcal{C}_U$ maps the line X_1X_{44} to itself. Abbreviating $\mathcal{C}_U(X_i) = X_j$ as $X_i \mapsto X_j$, we have, among points on X_1X_{44},

$$X_{1100} \mapsto X_{37} \mapsto X_6 \mapsto X_9 \mapsto X_{1743}.$$

The Euler line, X_2X_3, is a link in a chain as indicated by

$$\cdots \mapsto X_{42}X_{65} \mapsto X_2X_3 \mapsto X_{43}X_{46} \mapsto \cdots$$

4.2. Let $L = L_1 = X_6 = a : b : c$, so that $\mathcal{L}_1$ is the line at infinity and $\mathcal{P}$ is the circumcircle. Let $\mathcal{L}_2$ be the Euler line, given by taking L_2 in the statement of the theorem to be

$$X_{647} = a(b^2-c^2)(b^2+c^2-a^2) : b(c^2-a^2)(c^2+a^2-b^2) : c(a^2-b^2)(a^2+b^2-c^2).$$

The Ceva collineation $\mathcal{C}_U$ that maps $\mathcal{L}_1$ onto $\mathcal{L}_2$ is given by

$$\begin{aligned} U = X_{523} &= a(b^2-c^2) : b(c^2-a^2) : c(a^2-b^2) \\ &= \sin(B-C) : \sin(C-A) : \sin(A-B), \end{aligned}$$

and we find

$$\begin{array}{lll} X_{512} \mapsto X_2, & X_{520} \mapsto X_4, & X_{523} \mapsto X_5, \\ X_{526} \mapsto X_{30}, & X_{2574} \mapsto X_{1312}, & X_{2575} \mapsto X_{1313}. \end{array}$$

The penultimate of these, namely $X_{2574} \mapsto X_{1312}$, is of particular interest, as $X_{2574} = X_{1113}^{-1}$, where X_{1113} is a point of intersection of the Euler line and the circumcircle and X_{1312} is a point of intersection of the Euler line and the nine-point circle; and similarly for $X_{2575} \mapsto X_{1313}$. The mapping $\mathcal{C}_U$ carries the Brocard axis, X_3X_6 onto the line $X_{115}X_{125}$, where X_{115} and X_{125} are the centers of the Kiepert and Jerabek hyperbolas, respectively.

4.3. Let $L_1 = X_{523}$, so that $\mathcal{L}_1$ is the Brocard axis, X_3X_6, and let $\mathcal{L}_2$ be the Euler line, X_2X_3. Then $U = X_6 = a : b : c$. The mapping of $\mathcal{L}_1$ to $\mathcal{L}_2$ is a link in a chain:

$$\cdots \mapsto X_2X_{39} \mapsto X_2X_6 \mapsto X_3X_6 \mapsto X_2X_3 \mapsto X_6X_{25} \mapsto X_3X_{66} \mapsto \cdots$$

4.4. Here, we reverse the roles played by the Brocard axis and Euler line in Example 3: let $\mathcal{L}_1$ be the Euler line and $\mathcal{L}_2$ be the Brocard axis. Then $U = X_{184} = a^2 \cos A : b^2 \cos B : c^2 \cos C$. A few images of the X_{184}-Ceva collineation are given here:

$$\begin{array}{lll} X_2 \mapsto X_{32}, & X_3 \mapsto X_{571}, & X_4 \mapsto X_{577}, \\ X_5 \mapsto X_6, & X_{30} \mapsto X_{50}, & X_{427} \mapsto X_3. \end{array}$$

4.5. Let $\mathcal{L}_1 = \mathcal{L}_2 =$ Brocard axis. Here,

$$U = X_5 = \cos(B-C) : \cos(C-A) : \cos(A-B),$$

the center of the nine-point circle, and

$$X_{389} \mapsto X_3 \mapsto X_{52} \quad \text{and} \quad X_{570} \mapsto X_6 \mapsto X_{216}.$$

4.6. Let $\mathcal{L}_1 = \mathcal{L}_2 =$ the line at infinity, $X_{30}X_{511}$. Here,

$$U = X_3 = \cos A : \cos B : \cos C,$$

the circumcenter. Among line-to-line images under X_3-collineation are these:

$$\begin{array}{l} X_4X_{51} \mapsto \text{Euler line} \mapsto X_3X_{49}, \\ X_6X_{64} \mapsto X_4X_6 \mapsto \text{Brocard axis} \mapsto X_6X_{155}. \end{array}$$

5. Cubics

Collineations map cubics to cubics (e.g. [4, p. 23]). In particular, a U-Ceva collineation maps a cubic Λ that passes through the vertices A, B, C to a cubic $\mathcal{C}_U(\Lambda)$ that passes through the vertices A^U, B^U, C^U of the anticevian triangle of U.

5.1. Let $U = X_1$, as in §4.1, and let Λ be the Thompson cubic, $Z(X_2, X_1)$, with equation

$$bc\alpha(\beta^2 - \gamma^2) + ca\beta(\gamma^2 - \alpha^2) + ab\gamma(\alpha^2 - \beta^2) = 0.$$

Then $\mathcal{C}_U(\Lambda)$ circumscribes the excentral triangle, and for selected X_i on Λ, the image $\mathcal{C}_U(X_i)$ is as shown here:

X_i	1	2	3	4	6	9	57	223
$\mathcal{C}_U(X_i)$	1	43	46	1745	9	1743	165	1750

5.2. Let $U = X_1$, and let Λ be the cubic $Z(X_1, X_{75})$, with equation

$$\alpha(c^2\beta^2 - b^2\gamma^2) + \beta(a^2\gamma^2 - c^2\alpha^2) + \gamma(b^2\alpha^2 - a^2\beta^2) = 0.$$

For selected X_i on Λ, the image $\mathcal{C}_U(X_i)$ is as shown here:

X_i	1	6	19	31	48	55	56	204	221
$\mathcal{C}_U(X_i)$	1	9	610	63	19	57	40	2184	84

5.3. Let $U = X_6$, as in §4.3, and let Λ be the Thompson cubic. Then $\mathcal{C}_U(\Lambda)$ circumscribes the tangential triangle, and for selected X_i on Λ, the image $\mathcal{C}_U(X_i)$ is as shown here:

X_i	1	2	3	4	6	9	57	223	282	1073	1249
$\mathcal{C}_U(X_i)$	55	6	25	154	3	56	198	1436	1035	1033	64

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