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Orthocycles, Bicentrics, and Orthodiagonals

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Abstract. We study configurations involving a circle (orthocycle) intimately related to a cyclic quadrilateral. As an illustration of the usefulness of this circle we explore its connexions with bicentric (bicentrics) and orthodiagonal quadrilaterals (orthodiagonals) reviewing the more or less known facts and revealing some other properties of these classes of quadrilaterals.

1. Introduction

Consider a generic convex cyclic quadrilateral $q = ABCD$ inscribed in the circle $k(K, r)$ and having finite intersection points F, G of opposite sides. Line $e = FG$ is the polar of the intersection point E of the diagonals AC, BD. The circle c with diameter FG is orthogonal to k. Also, the midpoints X, Y of the diagonals and the center H of c are collinear. We call c the **orthocycle** of the cyclic quadrilateral q. Consider also the circle f with diameter EK. This is the

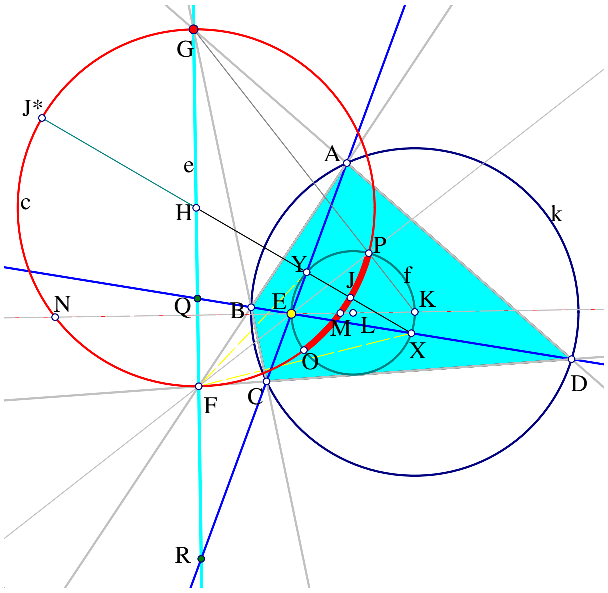

Figure 1. The orthocycle c of the cyclic quadrilateral $ABCD$

locus of the midpoints of chords of k passing through E. It is also the inverse of e with respect to k and is orthogonal to c. Thus, c belongs to the circle-bundle $\mathscr{C}'$, which is orthogonal to the bundle $\mathscr{C}(k, f)$ generated by k and f. The bundle $\mathscr{C}$ is of non intersecting type with limit points M, N, symmetric with respect to

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e, and $\mathscr{C}'$ is a bundle of intersecting type, all of whose members pass through M and N. If we fix the data (k, E, c), then all cyclic quadrilaterals q having these as *circumcircle, diagonals-intersection-point, orthocycle* respectively form a one-parameter family. A member q of this family is uniquely determined by a point J on the circular arc (OMP) of the orthocycle c. Thus the set of all q inscribed in the circle k and having diagonals through E is parameterized through pairs (c, J), c (the orthocycle) being a circle of bundle $\mathscr{C}'$ and J a point on the corresponding arc (OMP) intercepted on the orthocycle by f. In the following sections we consider these facts more closely and investigate (i) the bicentrics inscribed in k, and (ii) a certain 1-1 correspondence of cyclics to orthodiagonals in which the orthocycle plays an essential role.

Regarding the proofs of the statements made, everything (is or) follows immediately from standard, well known material. In fact, the statement on the polar relies on its usual construction from two intersecting chords ([3, p.103]). The statement on the collinearity follows from Newton's theorem on a complete quadrilateral ([3, p.62]). From the harmonic ratios appearing in complete quadrilaterals follows also that the intersection points Q, R of the diagonals with line e divide F, G harmonically. Consequently the circle with diameter QR is also orthogonal to c ([2, §1237]). The orthogonality of c, k follows from the fact that PF is the polar of G, which implies that P, G are inverse with respect to k. Besides, by measuring angles at P, circles f, c are shown to be orthogonal. The statement on the parametrization is analyzed in the following section.

The orthocycle gives a means to establish unity in apparently unrelated properties. For example the well known formula

$$\frac{1}{r^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}$$

is proved to be, essentially, a case of *Stewart's formula* (see next paragraph).

Furthermore, the orthogonality of c to f can be used to characterize the cyclics. To formulate the characterization we consider more general the *orthocycle* of a generic convex quadrilateral to be the circle on the diameter defined by the two intersection points of its pairs of opposite sides.

Proposition 1. *The quadrilateral $q = ABCD$ is cyclic if and only if its orthocycle c is orthogonal to the circle f passing through the midpoints X, Y of its diagonals and their intersection point E.*

Proof. If q is cyclic, then we have already seen that its orthocycle c belongs to the bundle $\mathscr{C}'$ which is orthogonal to the one generated by its circumcircle k and the circle f passing through the diagonal midpoints and their intersection point.

Conversely, if the orthocycle c and f intersect orthogonally, then Y and X are inverse with respect to c. Since the same is true with the intersection points R, Q of the diagonals of q with line FG (see Figure 2), there is a circle a passing through the four points X, Y, R and Q. Then we have

$$|ER| \cdot |EX| = |EY| \cdot |EQ|. \tag{1}$$

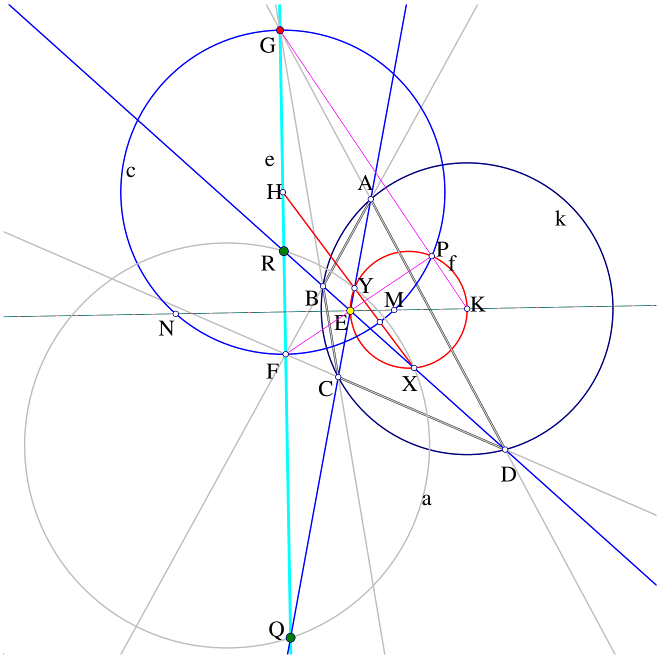

Figure 2. Cyclic characterization

But from the general properties of the complete quadrilaterals we have also that $(Q, E, C, A) = -1$ is a harmonic division, hence

$$|EC| \cdot |EA| = |EQ| \cdot |EY|. \tag{2}$$

Analogously, $(R, E, B, D) = -1$ implies

$$|EB| \cdot |ED| = |ER| \cdot |EX|. \tag{3}$$

Relations (1) to (3) imply that $|EB| \cdot |ED| = |EC| \cdot |EA|$, proving the proposition. □

For a classical treatment of the properties discussed below see Chapter 10 of Paul Yiu's Geometry Notes [5]. Zaslavsky (see [6], [1]) uses the term *orthodiagonal* for a map between quadrangles and gives characterizations of cyclics in another context than the one discussed below.

2. Bicentrics

Denote by (k, E, c) the family of quadrilaterals characterized by these elements (*circumcircle, diagonal-intersection-point, orthocycle*) correspondingly. Referring to Figure 1 we have the following properties ([2, §§674, 675, 1276]).

Proposition 2. (1) *There is a 1-1 correspondance between the members of the family* (k, E, c) *and the points* J *of the open arc* (OMP) *of circle* c*.*

(2) *Let* X, Y *be the intersection points of* f *with line* HJ*.* X, Y *are the midpoints of the diagonals of* q *and are inverse with respect to* c*.*

(3) *Then FJ bisects angles AFD and XFY. Analogously, GJ bisects angles BGD and XGY.*

Proof. In fact, from the Introduction, it is plain that each member q of the family (k, E, c) defines a J as required. Conversely, a point J on arc (OMP) of c defines two intersection points X, Y of HJ with f, which are inverse with respect to c, since f and c are orthogonal. The chords EX and EY define the cyclic $q = ABCD$, having these as diagonals and X, Y as the midpoints of these diagonals. Consider the orthocycle c' of this q. By the analysis made in the Introduction, c' belongs to the bundle $\mathscr{C}'$ and is also orthogonal to the circle with diameter QR. Thus c' is uniquely defined by the chords XE, YE and must coincide with c. This proves (1).

(2) is already discussed in the Introduction.

(3) follows from the orthogonality of circles k, c. In fact, this implies that J, J^* divide X, Y harmonically. Then (FJ^*, FJ, FY, FX) is a harmonic bundle of lines and FJ^*, FJ are orthogonal. Hence, they bisect $\angle XFY$. They also bisect $\angle BFC$. This follows immediately from the similarity of triangles AFC and DFB. Analogous is the situation with the angles at G. □

Referring to figure 1, denote by $q(c)$ the particular quadrilateral of the family (k, E, c), constructed with the recipe of the previous proposition, for $J \equiv M$. The following two lemmas imply that $q(c)$ is bicentric.

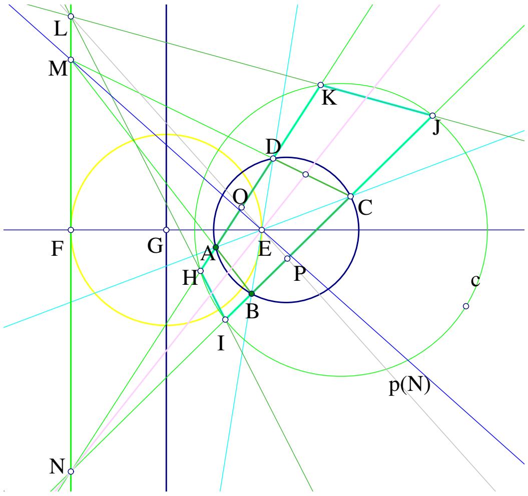

Figure 3. Bundle quadrilaterals

Lemma 3. *Consider a circle bundle of non intersecting type and two chords of a member circle passing through the limit point E of the bundle* (see Figure 3). *The chords define a quadrilateral $q = ABCD$ having these as diagonals. Extend two opposite sides AD, BC until they intersect a second circle member c of the bundle. The intersection points form a quadrilateral $r = HIJK$. Then the intersection* *point L of the sides HI, JK lies on the polar MN of E with respect to a circle of the bundle (all circles c of the bundle have the same polar with respect to E).*

Indeed, N, M can be taken as the intersection points of opposite sides of q. Then N is on the polar of E, hence the polar $p(N)$ of N contains E. Consider the intersection points O, P of this polar with sides HK, IJ respectively. Then,
(a) these sides intersect at a point L lying on $p(N)$,
(b) L is also on line MN.

(a) follows from the standard theorem on cyclic quadrilaterals.

(b) follows from the fact that the quadruple of lines (NL, NH, NE, NI) at N is harmonic. But (NM, NH, NE, NI) is also harmonic, hence L is contained in line MN.

Lemma 4. *q(c) is bicentric.*

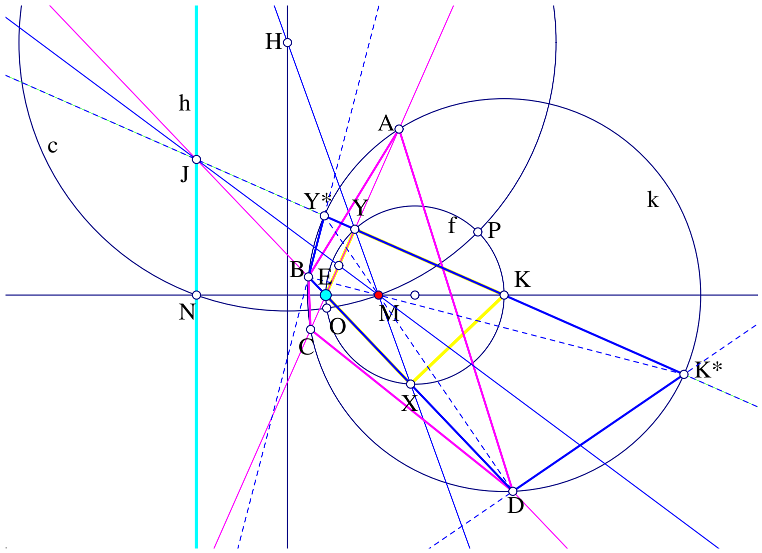

Figure 4. Bisectors of $q(c)$

Indeed, by Proposition 2 the bisectors of angles $\angle AGB, \angle BFC$ will intersect at M. It suffices to show that the bisectors of two opposite angles of $q(c)$ intersect also at M. Let us show that the bisector of angle $\angle ABC$ passes through M (Figure 4). We start with the quadrilateral $q_1 = EXKY$. Its diagonals intersect at M. According to the previous lemma the extensions of its sides will define a quadrilateral $q_2 = BDK^*Y^*$ inscribed in k and having its opposite sides intersecting on line h the common polar of E with respect to every member circle of the bundle I. This implies that the diagonals of q_2 intersect at the pole E of h. But BK^* joins B to the middle K^* of the arc (CK^*B), hence is the bisector of angle $\angle ABC$ and passes through M.

Proposition 5. (1) *There is a unique member $q = q(c) = ABCD$, of the family (k, E, c) which is bicentric. The corresponding J is the limit point M of bundle I contained in the circle k.*

(2) *There is a unique member $o = o(c) = A^*B^*C^*D^*$, of the family (k, E, c) which is orthodiagonal. The corresponding HJ passes through the center L of the circle f.*

(3) *For every bicentric the incenter M is on the line joining the intersection point of the diagonals with the circumcenter.*

(4) *For every bicentric the incenter M is on the line joining the midpoints of the diagonals.*

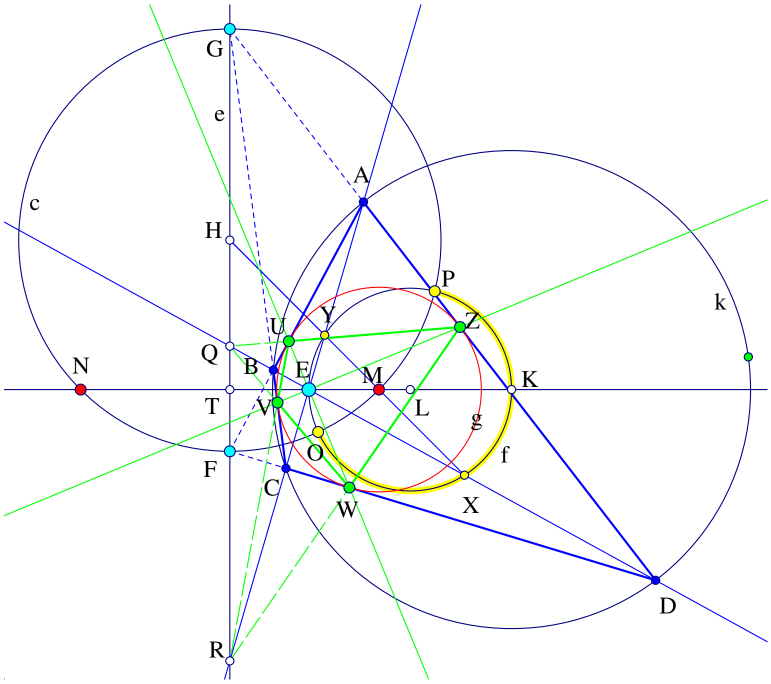

Figure 5. The bicentric member in (k, E, c)

Proof. In fact, by the previous lemmas we know that $q(c)$ is bicentric. To prove the uniqueness we assume that $q = ABCD$ is bicircular and consider the incircle g and the tangential quadrilateral $q' = UVWZ$. From Brianchon's theorem the diagonals of q' intersect also at E. Thus, the poles of the diagonals UW, VZ being correspondingly F, G, line e will be also polar of E with respect to g. In particular the center of g will be on line MN and the pairs of opposite sides of the tangential q' will intersect on e at points Q, R say. The diagonals of q pass through Q, R respectively. In fact, D being the pole of line WZ and B the pole of UV, BD is the polar of R with respect to g. By the standard construction of the polar it follows that Q is on BD. Analogously R is on AC. The center of g will be the intersection M' of the bisectors of angles BGA and BFC. By measuring the angles at M' we find easily that the bisectors form there a right angle. Thus, M' will be on the orthocycle c, hence, being also on line MN, it will coincide with M. In that case line HYX passes through M. This follows from proposition 1 which identifies the bisector of angle YFX with FM. This proves (1).

To prove (2) consider the quadrangle $s = EXKY$. If the diagonals intersect orthogonally then s is a rectangle. Consequently XY is a diameter of f and passes through L. The converse is also valid. If XY passes through L then s is a rectangle and o is orthodiagonal.

The other statements are immediate consequences. Notice that property 2 holds more generally for every circumscriptible quadrilateral ([2, §1614]). □

Proposition 6. *Consider all tripples (k, E, c) with fixed k, E and c running through the members of the circle bundle $\mathscr{C}'$. Denote by $q(c)$ the bicentric member of the corresponding family (k, E, c) and by $q' = UVWZ$ the tangential quadrangle of $q(c)$* (see Figure 5). *The following statements are consequences of the previous considerations:*

(1) *All tangential quadrilaterals $q' = UVWZ$ are orthodiagonal, the diagonals being each time parallel to the bisectors of angles $\angle BGA, \angle BFC$.*

(2) *The pairs of opposite sides or q' intersect at the points Q, R, which are the intersection points of the diagonals of $q(c)$ with e.*

(3) *The orthocycle c' of the tangential q' is the circle on the diameter QR and intersects the incircle g of $q(c)$ orthogonally. The radius r_g of the incircle satisfies $r_g^2 = |ME||MT|$.*

(4) *The bicentrics $\{q(c) : c \in \mathscr{C}'\}$ are precisely the inscribed in circle k and having their diagonals pass through E. They, all, have the same incircle g, depending only on k and E.*

(5) *The radii r_g of the inscribed circle g, r of circumscribed k, and the distance $d = |MK|$ of their centers satisfy the relation* $\dfrac{1}{r_g^2} = \dfrac{1}{(r+d)^2} + \dfrac{1}{(r-d)^2}$.

Proof. (1) follows from the fact that UV is orthogonal to the bisector FM of angle BFC. Analogously VZ is orthogonal to GM and FM, GM are orthogonal ([2, §674]).

(2) follows from the standard construction of the polar of E with respect to g. Thus e is also the polar of E with respect to the incircle g ([2, §1274]).

(3) follows also from (2) and the definition of the orthocycle. The relation for r_g is a consequence of the orthogonality of circles c', g.

(4) is a consequence of (3) and (5) is proved below by specializing to a particular bicentric $q(c)$ which is simultaneously orthodiagonal ([5, p.159]). Since the radius and the center of the incircle g is the same for all $q(c)$ this is legitimate. □

Proposition 7. (1) *For fixed (k, E), the set of all bicentrics $\{q(c) : c \in \mathscr{C}'\}$ contains exactly one member which is simultaneously bicentric and orthodiagonal. It corresponds to the minimum circle of bundle $\mathscr{C}'$, is kite-shaped and symmetric with respect to MN* (see Figure 6).

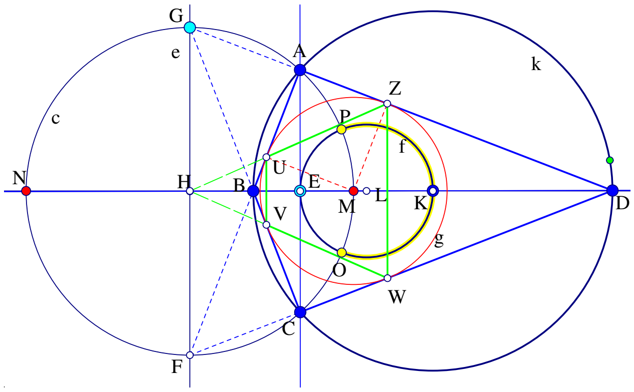

Figure 6. The orthodiagonal bicentric

(2) *All the bicentric orthodiagonals are constructed by reflecting an arbitrary right-angled triangle ABD on its hypotenuse* (see Figure 7). *The center of the incircle coincides with the trace E of the bisector with the hypotenuse, the length of this bisector w satisying $\frac{2}{w^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}$. Here R is the circumradius of ABD and $d = |EK|$ is the distance of the bisector's trace from the middle of BD.*

(3) *There is a particular bicentric orthodiagonal constructed directly from a regular octagon, with inradius $r = \frac{R}{\sqrt{2+\sqrt{2}}}$ and sides equal to w and $w + 2r$ respectively* (see Figure 8).

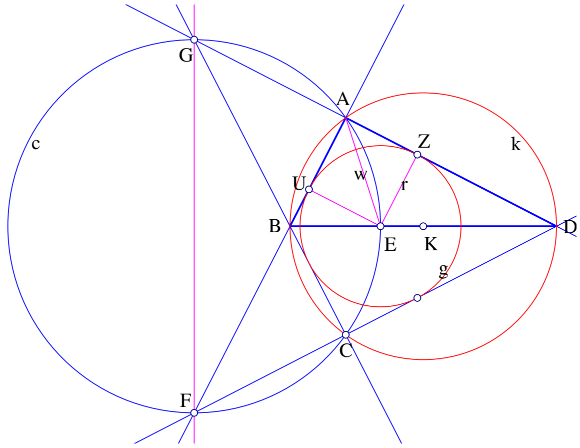

Figure 7. The general orthodiagonal bicentric

Proof. In fact, by applying the previous results to the tangential quadrilateral q' of $q(c)$, we know that the orthocycle of q' is orthogonal to the fixed incircle g and passes through two fixed points, depending only on (k, E) (the limit points of the corresponding circle bundle $\mathscr{C}'$ for the pair (g, E)). If the diagonals EQ and ER become orthogonal then E must be on the orthocycle of q' and this is possible only in the limiting position in which it coincides with line MN. Then the orthocycle of $q(c)$ has MN as diameter and this implies (1).

(2) follows immediately from (1). The formula is an application of Stewart's general formula (see [5, p.14]) on this particular configuration plus a simple calculation. The formula implies trivially the formula of the previous proposition, since all the bicirculars characterized by the fixed pair (k, E) have the same incircle g and from the square $EZAU$ (Figure 6) we have $w^2 = 2r^2$.

(3) is obvious and underlines the existence of a particular nice kite. □

Notice the necessary inequality between the distance $d = MK$ and the distance $d_1 = |EK|$ of circumcenter from the intersection point of diagonals: $2|MK| > |EK|$, holding for every bicentric ([6, p.44]) and being a consequence of a general propety of circle bundles of non intersecting type.

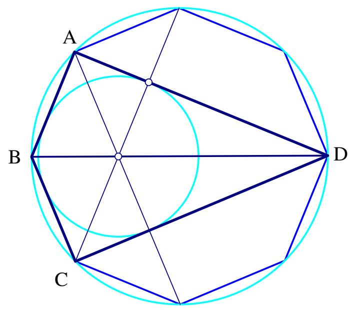

Figure 8. A distinguished kite

3. Circumscribed Quadrilateral

The following proposition give some well known properties of quadrilaterals circumscribed on circles by adding the ingredient of the orthocycle. For convenience we review here these properties and specialize in a subsequent proposition to the case of a bicentric circumscribed.

Proposition 8. *Consider the tangential quadrilateral $q' = QRST$ circumscribed on the circumcircle k of the cyclic quadrilateral $q = ABCD$* (Figure 9). *The following facts are true:*

(1) *The diagonals of q' and q intersect at the same point E.*

(2) *The pairs of opposite sides of q' and pairs of opposite sides of q intersect on the same line e, which is the polar of E with respect to the circumcircle k of q.*

(3) *The diameter UV of the orthocycle of q' is divided harmonically by the diameter FG of the orthocycle of q.*

(4) *The orthocycle of q' is orthogonal to the orthocycle of q.*

(5) *The diagonals of q' (respectively q) pass through the intersection points of opposite sides of q (respectively q').*

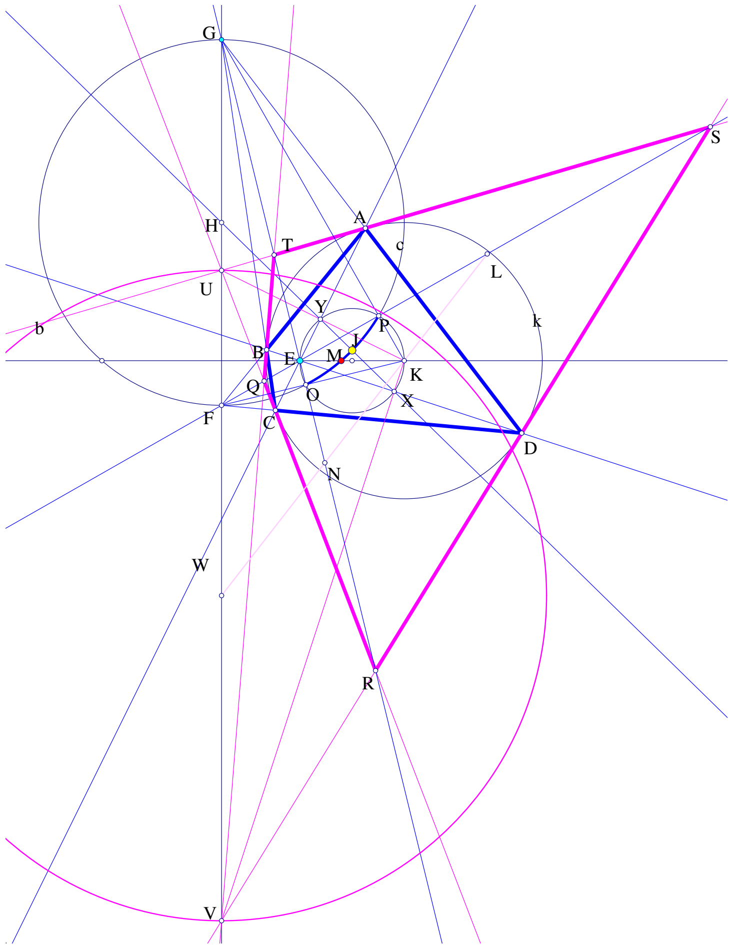

Figure 9. Circumscribed on cyclic

Proof. (1) is a consequence of Brianchon's theorem (see a simpler proof in [5, p.157]). Identify the polar of E with the diameter $e = FG$ of the orthocycle of q. The polar of F is PG and the polar of G is OF. T is the pole of AB which contains F. Hence the polar of F will pass through T, analogously it will pass through R. This proves (2) (see [2, §1275]) and (5).

To show (3) it suffices to see that lines (KP, KY, KO, KX) form a harmonic bundle. But the cross ratio of these four lines through K is independent of the location of K on the circle with diameter EK. Hence is the same with the cross ratio of lines (EP, EY, EO, EX) which is -1 by the general properties of complete quadrilaterals.

(4) is a consequence of (3). Note that the line LN joining the midpoints of the diagonals of q' passes through the center W of the orthocycle and the center of k ([2, §1614]). □

Proposition 9. *For each quadrangle of the family $q \in (k, E, c)$ construct the tangential quadrangle $q' = QRST$ of $q = ABCD$ (Figure 10). The following facts are true.*

(1) *There is exactly one $q_0 \in (k, E, c)$ whose corresponding tangential q' is cyclic. The corresponding line of diagonal midpoints of q' passes through the center K of circle k.*

(2) *The line of diagonal midpoints of q_0 is orthogonal to the corresponding line of diagonal midpoints of q'.*

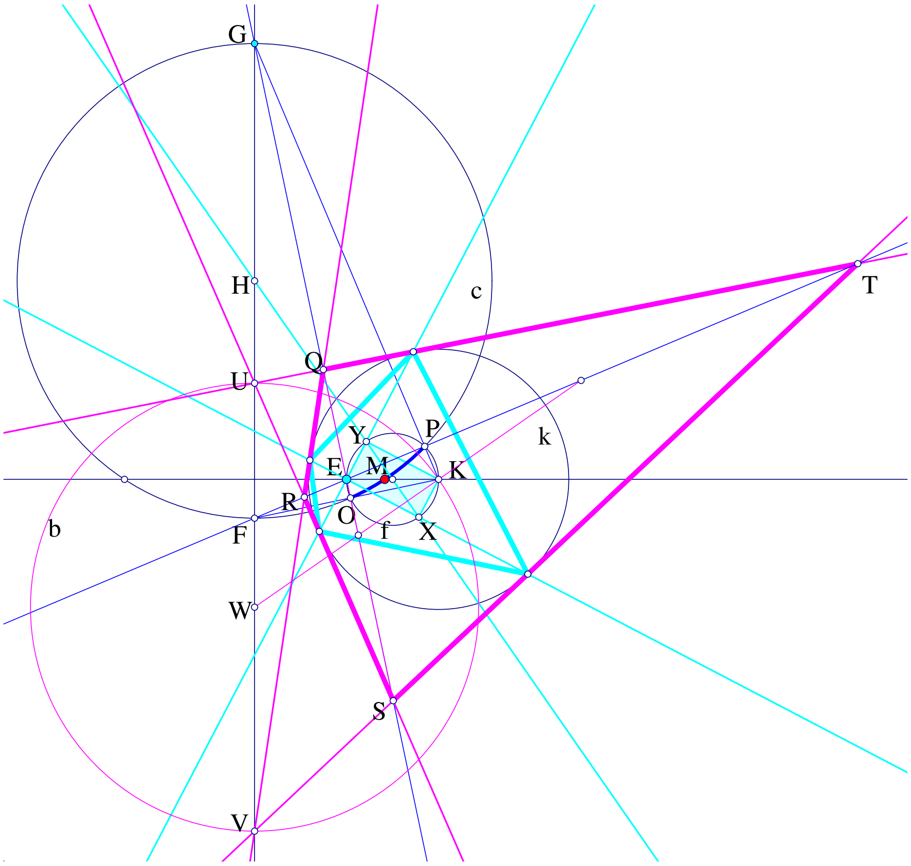

Figure 10. Bicentric circumscribed

Proof. q' being cyclic and circumscriptible it is bicentric. Hence the lines joining opposite contact points must be orthogonal and the orthocycle of q' passes through K. This follows from Proposition 2. Thus $p = EXKY$ is a rectangle, XY being a diameter of the circle f. Inversely, by Proposition 3, if q' is bicentric, then p is a rectangle and K is the limit point of the corresponding bundle I, and K is the center of the incircle. For the other statement notice that circle c being orthogonal simultaneously to circle k and b has its center on the radical axis of b and k. In the particular case of bicentric q', the angles WKK, XYK and EKY are equal and this implies that WL is then orthogonal to HXY which becomes the radical axis of k and b.

Note that the diagonals of all q' are the same and identical with the lines EF, EG which remain fixed for all members q of the family (k, E, c). Also combining this proposition and Proposition 3 we have (see [5, p.162]) that q' is cyclic, if and only if q is orthodiagonal. □

4. Orthodiagonals

The first part of the following proposition constructs an orthodiagonal from a cyclic. This is the inverse procedure of the well known one, which produces a cyclic by projecting the diagonals intersection point of an orthodiagonal to its sides ([4, vol. II p. 358], [6], [1]).

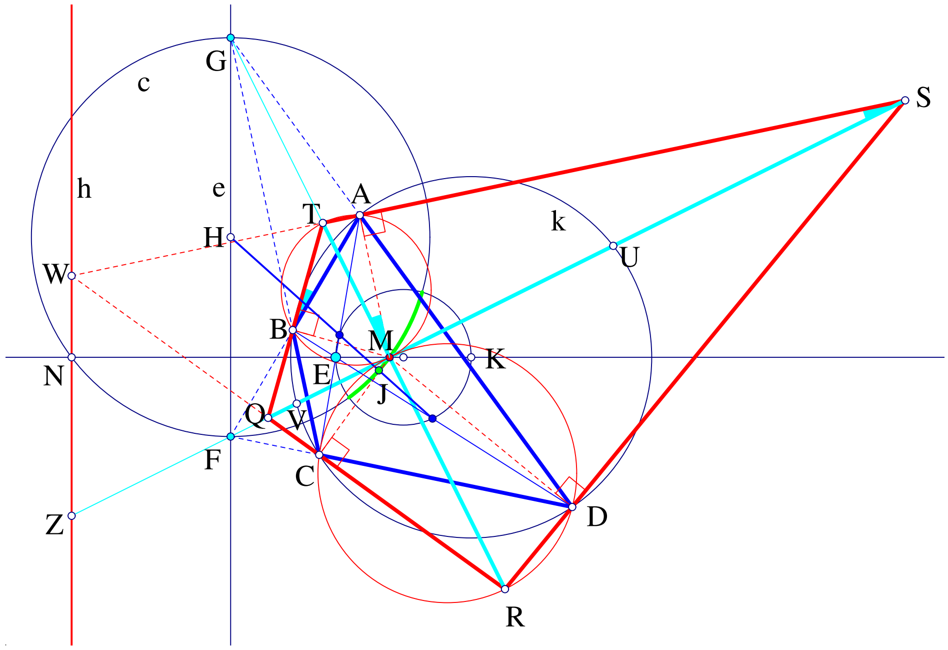

Figure 11. Orthodiagonal from cyclic

Proposition 10. (1) *For each cyclic quadrilateral* $q = ABCD$ *of the family* (k, E, c) *there is an orthodiagonal* $p = QRST$ *whose diagonals coincide with the sides of the right angled triangle* $t = FGM$*, defined by the limit point* M *of bundle* $\mathscr{C}$ *and the intersection points* F, G *of the pairs of opposite sides of* q*. The vertices of* q *are the projections of the intersection point* M *of the diagonals of* p *on its sides.*

(2) *The pairs of opposite sides of* p *intersect at points* W, W^* *on line* h *which is the common polar of all circles of bundle* I *with respect to its limit point* M.

(3) *The orthodiagonal p is cyclic if and only if the corresponding q is bicentric,* i.e., *point J is identical with M.*

(4) *The circumcircle of the orthodiagonal and cyclic p belongs to bundle I.*

Proof. Consider the lines orthogonal to MA, MB, MC, MD at the vertices of q (Figure 11). They build a quadrilateral. To show the statement on the diagonals consider the two resulting cyclic quadrilaterals $q_1 = MATB$ and $q_2 = MCRD$. Point F lies on the radical axis of their circumcircles since lines FBA, FCD are chords through F of circle k. Besides, for the same reason $|FB| \cdot |FA| = |FV| \cdot |FU| = |CF| \cdot |FD| = |FM|^2$. The last because circles c, k are orthogonal and M is the limit point of bundle I. From $|FB| \cdot |FA| = |FM|^2$ follows that line FM is tangent to the circumcircle of q_1. Analogously it is tangent to q_2 at M. Thus points G, T, M, R are collinear. Analogously points F, Q, M, S are collinear. This proves (1).

For (2), note that quadrangle $ABQS$ is cyclic, since $\angle TBA = \angle TMA = \angle MSA$. Thus $|FM|^2 = |FQ| \cdot |FS|$ and this implies that points M, Z divide harmonically Q, S, Z being the intersection point of h with the diagonal QS. Analogously the intersection point Z^* of h with the diagonal TR and M will divide T, R harmonically. Thus, by the characteristic property of the diagonals of a complete quadrilateral ZZ^* will be identical with line WW^*.

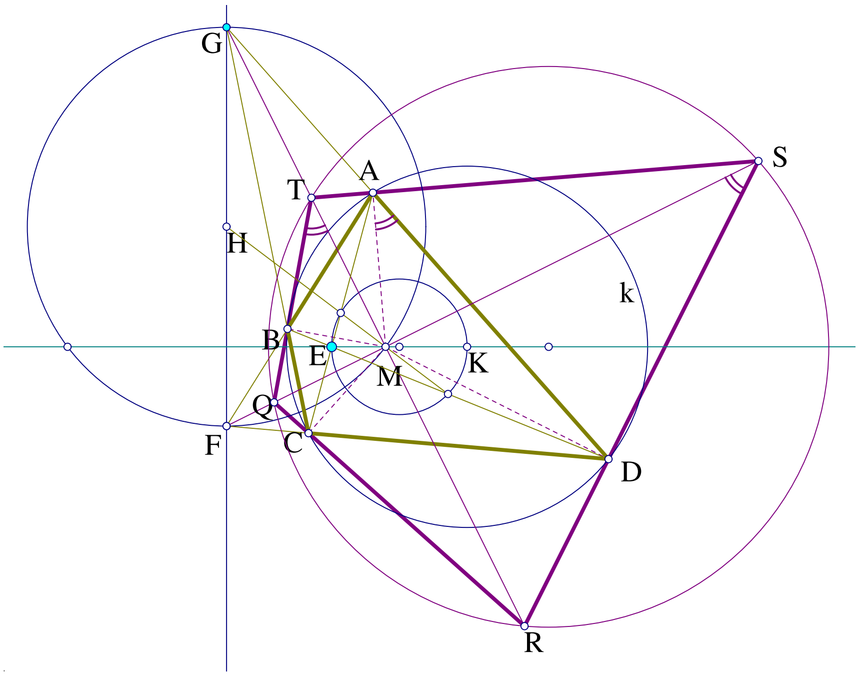

Figure 12. Orthodiagonal and cyclic

For (3), note that p is cyclic if and only if angles $\angle QTR = \angle QSR$ (Figure 12). By the definition of p this is equivalent to $\angle BAM = \angle MAD$, *i.e.*, AM being the bisector of angle A of q. Analogously MB, MC, MD must be bisectors of the corresponding angles of q.

For (4), note that the circumcenter of p must be on the line EM. This follows from the discussion in the first paragraph and the second statement. Indeed, the circumcenter must be on the line which is orthogonal from M to the diameter of the orthocycle of p. Besides the circle with center F and radius FM is a circle of bundle $\mathscr{C}'$ and, according to the proof of first statement, is orthogonal to this circumcircle. Thus the circumcircle of p, being orthogonal to two circles of bundle $\mathscr{C}'$, belongs to bundle $\mathscr{C}$. □

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