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Bicevian Tucker Circles

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Abstract. We prove that there are exactly ten bicevian Tucker circles and show several curves containing the Tucker bicevian perspectors.

1. Bicevian Tucker circles

The literature is abundant concerning Tucker circles. We simply recall that a Tucker circle is centered at T on the Brocard axis OK and meets the sidelines of ABC at six points A_b, A_c, B_c, B_a, C_a, C_b such that
(i) the lines X_yY_x are parallel to the sidelines of ABC,
(ii) the lines Y_xZ_x are antiparallel to the sidelines of ABC, *i.e.*, parallel to the sidelines of the orthic triangle $H_aH_bH_c$.

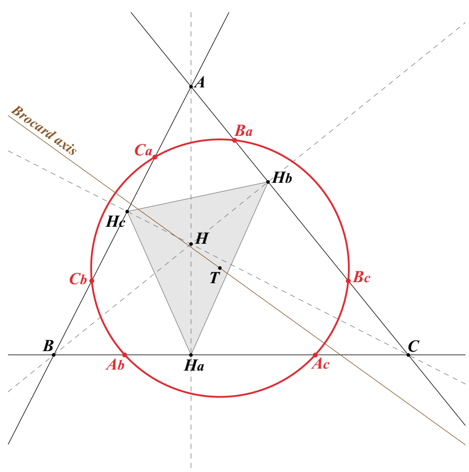

Figure 1. A Tucker circle

If T is defined by $\overrightarrow{OT} = t \cdot \overrightarrow{OK}$, we have

$$B_aC_a = C_bA_b = A_cB_c = \frac{2abc}{a^2+b^2+c^2}\,|t| = R|t|\tan\omega,$$

and the radius of the Tucker circle is

$$R_T = R\sqrt{(1-t)^2 + t^2\,\tan^2\omega}$$

where R is the circumradius and ω is the Brocard angle. See Figure 1.

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One obvious and trivial example consists of the circumcircle of ABC which we do not consider in the sequel. From now on, we assume that the six points are not all the vertices of ABC.

In this paper we characterize the *bicevian Tucker circles*, namely those for which a *Tucker triangle* formed by three of the six points (one on each sideline) is perspective to ABC. It is known that if a Tucker triangle is perspective to ABC, its companion triangle formed by the remaining three points is also perspective to ABC. The two perspectors are then said to be cyclocevian conjugates.

There are basically two kinds of Tucker triangles:
(i) those having one sideline parallel to a sideline of ABC: there are three pairs of such triangles e.g. $A_bB_cC_b$ and its companion $A_cB_aC_a$,
(ii) those not having one sideline parallel to a sideline of ABC: there is only one such pair namely $A_bB_cC_a$ and its companion $A_cB_aC_b$. These are the proper Tucker triangles of the literature.

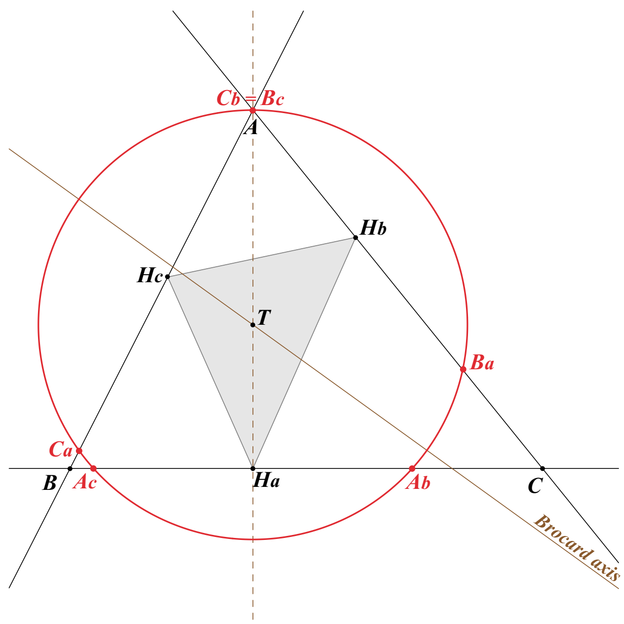

Figure 2. A Tucker circle through a vertex of ABC

In the former case, there are six bicevian Tucker circles which are obtained when T is the intersection of the Brocard axis with an altitude of ABC (which gives a Tucker circle passing through one vertex of ABC, see Figure 2) or with a perpendicular bisector of the medial triangle (which gives a Tucker circle passing through two midpoints of ABC, see Figure 3).

The latter case is more interesting but more difficult. Let us consider the Tucker triangle $A_bB_cC_a$ and denote by X_a the intersection of the lines BB_c and CC_a; define X_b and X_c similarly. Thus, ABC and $A_bB_cC_a$ are perspective (at X) if and only if the three lines AA_b, BB_c and CC_a are concurrent or equivalently the three

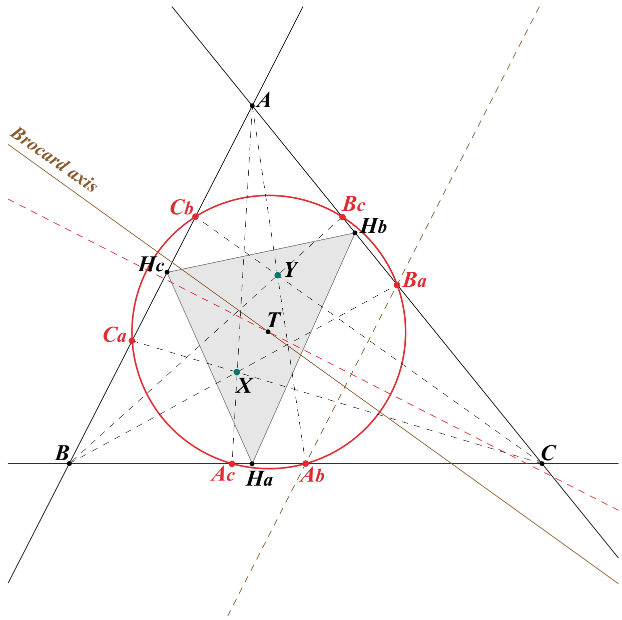

Figure 3. A Tucker circle through two midpoints of ABC

points X_a, X_b and X_c coincide. Consequently, the triangles ABC and $A_cB_aC_b$ are also perspective at Y, the cyclocevian conjugate of X.

Lemma 1. *When T traverses the Brocard axis, the locus of X_a is a conic γ_a.*

Proof. This can be obtained through easy calculation. Here is a synthetic proof. Consider the projections π_1 from the line AC onto the line BC in the direction of H_aH_b, and π_2 from the line BC onto the line AB in the direction of AC. Clearly, $\pi_2(\pi_1(B_c)) = \pi_2(A_c) = C_a$. Hence, the tranformation which associates the line BB_c to the line CC_a is a homography between the pencils of lines passing through B and C. It follows from the theorem of Chasles-Steiner that their intersection X_a must lie on a conic. □

This conic γ_a is easy to draw since it contains B, C, the anticomplement G_a of A, the intersection of the median AG and the symmedian CK and since the tangent at C is the line CA. Hence the perspector X we are seeking must lie on the three conics γ_a, γ_b, γ_c and Y must lie on three other similar conics γ'_a, γ'_b, γ'_c. See Figure 4.

Lemma 2. *γ_a, γ_b and γ_c have three common points X_i, $i = 1, 2, 3$, and one of them is always real.*

Proof. Indeed, γ_b and γ_c for example meet at A and three other points, one of them being necessarily real. On the other hand, it is clear that any point X lying on two conics must lie on the third one. □

This yields the following

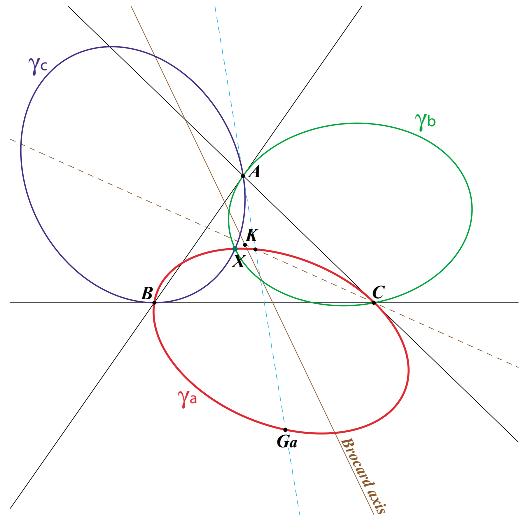

Figure 4. γ_a, γ_b and γ_c with only one real common point X

Theorem 3. *There are three* (*proper*) *bicevian Tucker circles and one of them is always real.*

2. Bicevian Tucker perspectors

The points X_i are not ruler and compass constructible since we need intersect two conics having only one known common point. For each X_i there is a corresponding Y_i which is its cyclocevian conjugate and the Tucker circle passes through the vertices of the cevian triangles of these two points. We call these six points X_i, Y_i the *Tucker bicevian perspectors*.

When X_i is known, it is easy to find the corresponding center T_i of the Tucker circle on the line OK : the perpendicular at T_i to the line H_bH_c meets AK at a point and the parallel through this point to H_bH_c meets the lines AB, AC at two points on the required circle. See Figure 5 where only one X is real and Figure 6 where all three points X_i are real.

We recall that the bicevian conic $\mathcal{C}(P, Q)$ is the conic passing through the vertices of the cevian triangles of P and Q. See [3] for general bicevian conics and their properties.

Theorem 4. *The three lines $\mathcal{L}_i$ passing through X_i, Y_i are parallel and perpendicular to the Brocard axis OK.*

Proof. We know (see [3]) that, for any bicevian conic $\mathcal{C}(P, Q)$, there is an inscribed conic bitangent to $\mathcal{C}(P, Q)$ at two points lying on the line PQ. On the other hand, any Tucker circle is bitangent to the Brocard ellipse and the line through the contacts is perpendicular to the Brocard axis. Hence, any bicevian Tucker circle must be tangent to the Brocard ellipse at two points lying on the line X_iY_i and this completes the proof. □

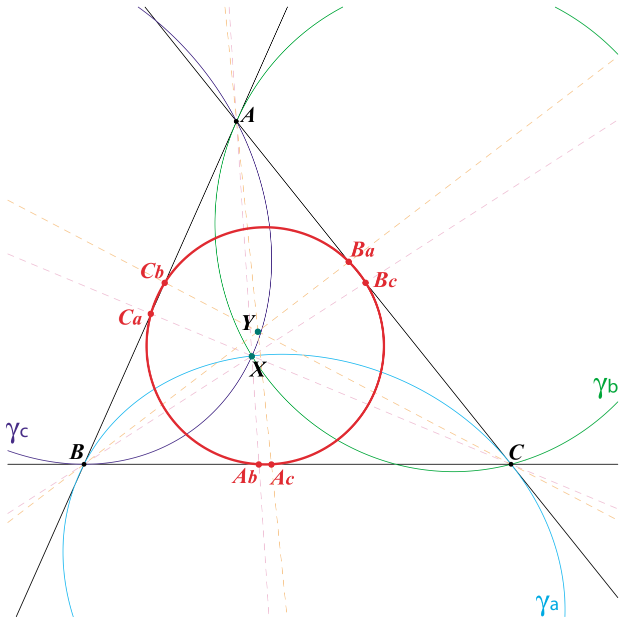

Figure 5. One real bicevian Tucker circle

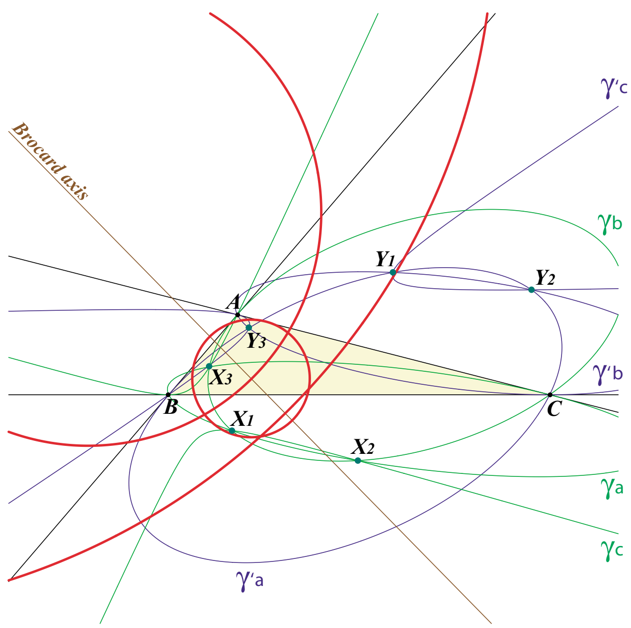

Figure 6. Three real bicevian Tucker circles

Corollary 5. *The two triangles $X_1X_2X_3$ and $Y_1Y_2Y_3$ are perspective at X_{512} and the axis of perspective is the line GK.*

Conversely, any bicevian conic $\mathcal{C}(P,Q)$ bitangent to the Brocard ellipse must verify $Q = K/P$. Such conic has its center on the Brocard line if and only if P lies either
(i) on $\mathrm{p}\mathcal{K}(X_{3051}, K)$ in which case the conic has always its center at the Brocard midpoint X_{39}, but the Tucker circle with center X_{39} is not a bicevian conic, or
(ii) on $\mathrm{p}\mathcal{K}(X_{669}, K)$ = K367 in [4].

This gives the following

Theorem 6. *The six Tucker bicevian perspectors X_i, Y_i lie on $\mathrm{p}\mathcal{K}(X_{669}, X_6)$, the pivotal cubic with pivot the Lemoine point K which is invariant in the isoconjugation swapping K and the infinite point X_{512} of the Lemoine axis.*

See Figure 7. We give another proof and more details on this cubic below.

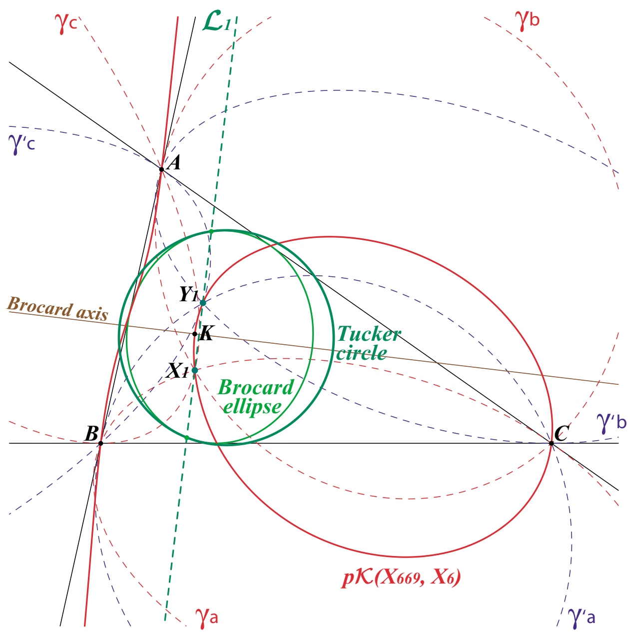

Figure 7. Bicevian Tucker circle and Brocard ellipse

3. Nets of conics associated with the Tucker bicevian perspectors

We now consider curves passing through the six Tucker bicevian perspectors X_i, Y_i. Recall that two of these points are always real and that all six points are two by two cyclocevian conjugates on three lines $\mathcal{L}_i$ perpendicular to the Brocard axis. We already know two nets of conics containing these points:
(i) the net $\mathcal{N}$ generated by γ_a, γ_b, γ_c which contain the points X_i, $i = 1, 2, 3$;
(ii) the net $\mathcal{N}'$ generated by γ'_a, γ'_b, γ'_c which contain the points Y_i, $i = 1, 2, 3$.

The equations of the conics are

$$\gamma_a : \quad a^2 y(x+z) - b^2 x(x+y) = 0,$$

$$\gamma'_a: \quad a^2 z(x+y) - c^2 x(x+z) = 0;$$

the other equations are obtained through cyclic permutations.

Thus, for any point $P = u : v : w$ in the plane, a conic in $\mathcal{N}$ is

$$\mathcal{N}(P) = u\,\gamma_a + v\,\gamma_b + w\,\gamma_c;$$

similarly for $\mathcal{N}'(P)$. Clearly, $\mathcal{N}(A) = \gamma_a$, etc.

Proposition 7. *Each net of conics ($\mathcal{N}$ and $\mathcal{N}'$) contains one and only one circle. These circles Γ and Γ' contain X_{110}, the focus of the Kiepert parabola.*

These circles are

$$\Gamma: \quad \sum_{\text{cyclic}} b^2c^2(b^2-c^2)(a^2-b^2)x^2 + a^2(b^2-c^2)(c^4+a^2b^2-2a^2c^2)yz = 0$$

and

$$\Gamma': \quad \sum_{\text{cyclic}} b^2c^2(b^2-c^2)(c^2-a^2)x^2 - a^2(b^2-c^2)(b^4+a^2c^2-2a^2b^2)yz = 0.$$

In fact, $\Gamma = \mathcal{N}(P')$ and $\Gamma' = \mathcal{N}'(P'')$ where

$$P' = \frac{c^2}{c^2-a^2} : \frac{a^2}{a^2-b^2} : \frac{b^2}{b^2-c^2},$$
$$P'' = \frac{b^2}{b^2-a^2} : \frac{c^2}{c^2-b^2} : \frac{a^2}{a^2-c^2}.$$

These points lie on the trilinear polar of X_{523}, the line through the centers of the Kiepert and Jerabek hyperbolas and on the circum-conic with perspector X_{76}, which is the isotomic transform of the Lemoine axis. See Figure 8.

Proposition 8. *Each net of conics contains a pencil of rectangular hyperbolas. Each pencil contains one rectangular hyperbola passing through X_{110}.*

Note that these two rectangular hyperbolas have the same asymptotic directions which are those of the rectangular circum-hyperbola passing through X_{110}. See Figure 9.

4. Cubics associated with the Tucker bicevian perspectors

When P has coordinates which are linear in x, y, z, the curves $\mathcal{N}(P)$ and $\mathcal{N}'(P)$ are in general cubics but $\mathcal{N}(z : x : y)$ and $\mathcal{N}'(y : z : x)$ are degenerate. In other words, for any point $x : y : z$ of the plane, we (loosely) may write

$$z\,\gamma_a + x\,\gamma_b + y\,\gamma_c = 0$$

and

$$y\,\gamma'_a + z\,\gamma'_b + x\,\gamma'_c = 0.$$

We obtain two circum-cubics $\mathcal{K}(P)$ and $\mathcal{K}'(P)$ when P takes the form

$$P = q\,z - r\,y : r\,x - p\,z : p\,y - q\,x$$

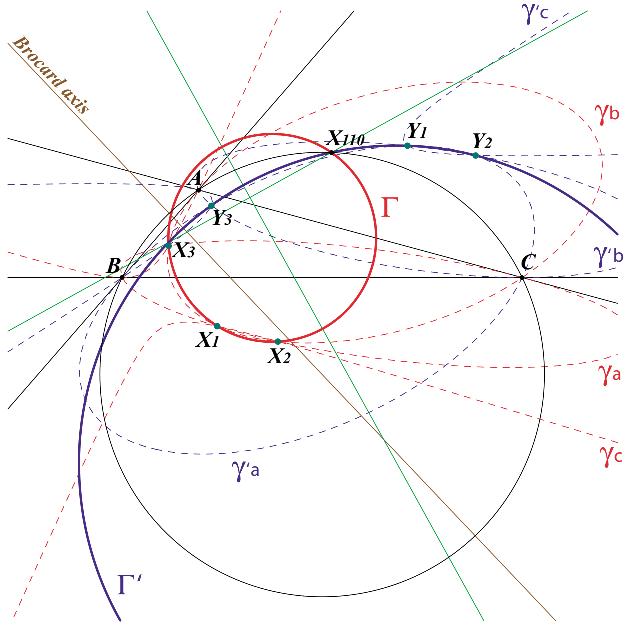

Figure 8. Circles through the Tucker bicevian perspectors

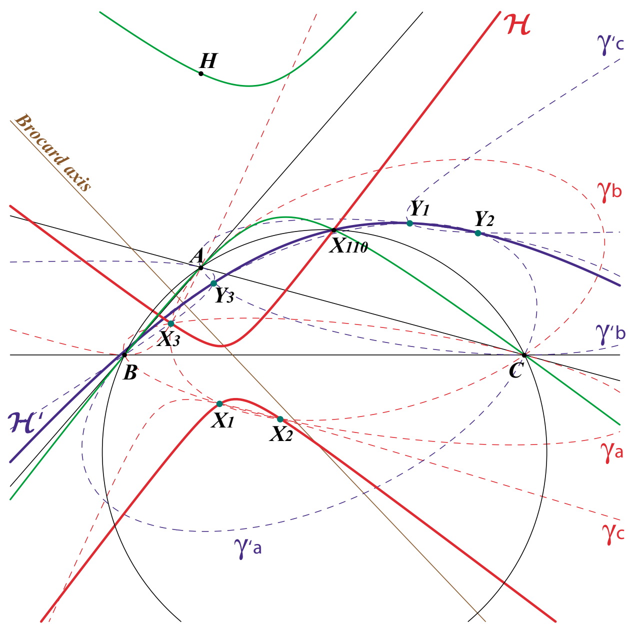

Figure 9. Rectangular hyperbolas through the Tucker bicevian perspectors

associated to the cevian lines of the point $Q = p : q : r$ and both cubics contain Q. Obviously, $\mathcal{K}(P)$ contains the points X_i and $\mathcal{K}'(P)$ contains the points Y_i.

For example, with $Q = G$, we obtain the two cubics $\mathcal{K}(G)$ and $\mathcal{K}'(G)$ passing through G and the vertices of the antimedial triangle $G_aG_bG_c$. See Figure 10.

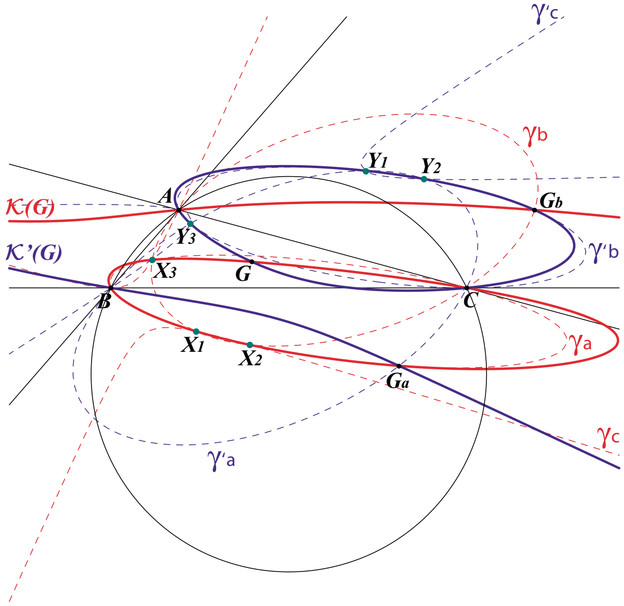

Figure 10. The two cubics $\mathcal{K}(G)$ and $\mathcal{K}'(G)$

These two cubics $\mathcal{K}(P)$ and $\mathcal{K}'(P)$ are isotomic pivotal cubics with pivots the bicentric companions (see [5, p.47] and [2]) of X_{523} respectively

$$X'_{523} = a^2 - b^2 : b^2 - c^2 : c^2 - a^2$$

and

$$X''_{523} = c^2 - a^2 : a^2 - b^2 : b^2 - c^2$$

both on the line at infinity. The two other points at infinity of the cubics are those of the Steiner ellipse.

4.1. *An alternative proof of Theorem 6.* We already know (Theorem 6) that the six Tucker bicevian perspectors X_i, Y_i lie on the cubic $\mathrm{p}\mathcal{K}(X_{669}, X_6)$. Here is an alternative proof. See Figure 11.

Proof. Let U, V, W be the traces of the perpendicular at G to the Brocard axis. We denote by Γ_a the decomposed cubic which is the union of the line AU and the conic γ_a. Γ_a contains the vertices of ABC and the points X_i. Γ_b and Γ_c are defined similarly and contain the same points.

The cubic $c^2\,\Gamma_a + a^2\,\Gamma_b + b^2\,\Gamma_c$ is another cubic through the same points since it belongs to the net of cubics. It is easy to verify that this latter cubic is $\mathrm{p}\mathcal{K}(X_{669}, X_6)$.

Now, if Γ'_a, Γ'_b, Γ'_c are defined likewise, the cubic $b^2\,\Gamma'_a + c^2\,\Gamma'_b + a^2\,\Gamma'_c$ is $\mathrm{p}\mathcal{K}(X_{669}, X_6)$ again and this shows that the six Tucker bicevian perspectors lie on the curve. □

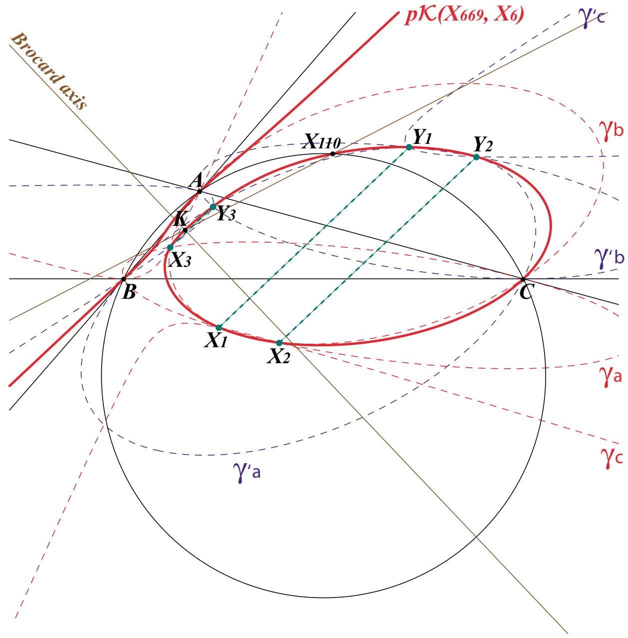

Figure 11. $\text{p}\mathcal{K}(X_{669}, X_6)$ and the three lines $\mathcal{L}_i$

4.2. *More on the cubic* $\text{p}\mathcal{K}(X_{669}, X_6)$. The cubic $\text{p}\mathcal{K}(X_{669}, X_6)$ also contains K, X_{110}, X_{512}, X_{3124} and meets the sidelines of ABC at the feet of the symmedians. Note that the pole X_{669} is the barycentric product of K and X_{512}, the isopivot or secondary pivot (see [1], §1.4). This shows that, for any point M on the cubic, the point K/M (cevian quotient or Ceva conjugate) lies on the cubic and the line $M\ K/M$ contains X_{512} i.e. is perpendicular to the Brocard axis.

We can apply to the Tucker bicevian perspectors the usual group law on the cubic. For any two points P, Q on $\text{p}\mathcal{K}(X_{669}, X_6)$ we define $P \oplus Q$ as the third intersection of the line through K and the third point on the line PQ.

For a permuation $i,\ j,\ k$ of $1,\ 2,\ 3$, we have

$$X_i \oplus X_j = Y_k, \qquad Y_i \oplus Y_j = X_k.$$

Furthermore, $X_i \oplus Y_i = K$. These properties are obvious since the pivot of the cubic is K and the secondary pivot is X_{512}.

The third point of $\text{p}\mathcal{K}(X_{669}, X_6)$ on the line KX_{110} is $X_{3124} = a^2(b^2-c^2)^2\ :\ b^2(c^2-a^2)^2\ :\ c^2(a^2-b^2)^2$, the cevian quotient of K and X_{512} and the third point on the line $X_{110}X_{512}$ is the cevian quotient of K and X_{110}.

The infinite points of $\text{p}\mathcal{K}(X_{669}, X_6)$ are X_{512} and two imaginary points, those of the bicevian ellipse $\mathcal{C}(G, K)$ or, equivalently, those of the circum-ellipse $\mathcal{C}(X_{39})$ with perspector X_{39} and center X_{141}.

The real asymptote is perpendicular to the Brocard axis and meets the curve at $X = K/X_{512}$, the third point on the line KX_{110} seen above. X also lies on the Brocard ellipse, on $\mathcal{C}(G, K)$. See Figure 12.

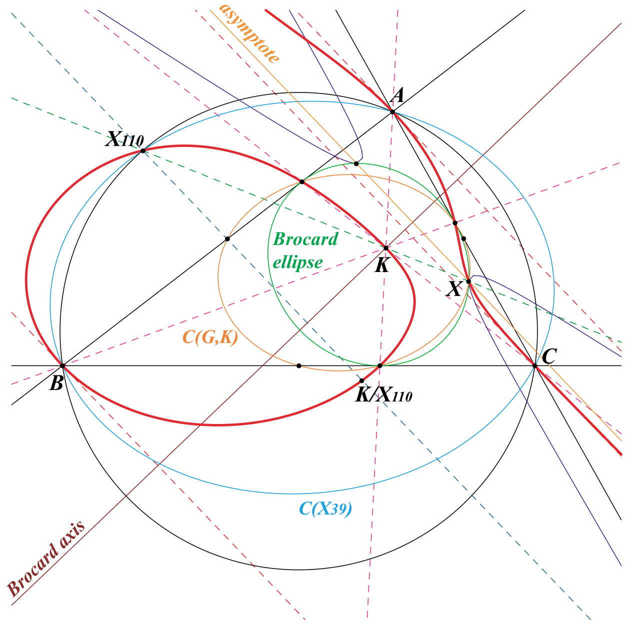

Figure 12. $K367 = \mathrm{p}\mathcal{K}(X_{669}, X_6)$

$\mathrm{p}\mathcal{K}(X_{669}, X_6)$ is the isogonal transform of $\mathrm{p}\mathcal{K}(X_{99}, X_{99})$, a member of the class CL007 in [4]. These are the $\mathrm{p}\mathcal{K}(W, W)$ cubics or parallel tripolars cubics.

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