Construction of Triangle from a Vertex and the Feet of Two Angle Bisectors

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Abstract. We give two simple constructions of a triangle given one vertex and the feet of two angle bisectors.

1. Construction from \((A, T_a, T_b)\)

We present two simple solutions of the following construction problem (number 58) in the list compiled by W. Wernick [2]: Given three noncollinear points \(A, T_a\) and \(T_b\), to construct a triangle \(ABC\) with \(T_a, T_b\) on \(BC, CA\) respectively such that \(AT_a\) and \(BT_b\) are bisectors of the triangle. L. E. Meyers [1] has indicated the constructibility of such a triangle. Let \(\ell\) be the half line \(AT_b\). Both solutions we present here make use of the reflection \(\ell'\) of \(\ell\) in \(AT_a\). The vertex \(B\) necessarily lies on \(\ell'\). In what follows \(P'(Q)\) denotes the circle, center \(P\), passing through the point \(Q\).

Construction 1. Let \(Z\) be the pedal of \(T_b\) on \(\ell'\). Construct two circles, one \(T_b(Z)\), and the other with \(T_aT_b\) as diameter. Let \(X\) be an intersection, if any, of these two circles. If the line \(XT_a\) intersects the half lines \(\ell'\) at \(B\) and \(\ell\) at \(C\), then \(ABC\) is a desired triangle.

Construction 2. Let $P$ be an intersection, if any, of the circle $T_b(T_a)$ with the half line $\ell'$. Construct the perpendicular bisector of $PT_a$. If this intersects $\ell'$ at a point $B$, and if the half line $BT_a$ intersects $\ell$ at $C$, then $ABC$ is a desired triangle.

We study the number of solutions for various relative positions of $A$, $T_a$ and $T_b$. Set up a polar coordinate system with $A$ at the pole and $T_b$ at $(1, 0)$. Suppose $T_a$ has polar coordinates $(\rho, \theta)$ for $\rho > 0$ and $0 < \theta < \frac{\pi}{2}$. The half line $\ell'$ has polar angle $2\theta$. The circle $T_b(T_a)$ intersects $\ell'$ if the equation

$$\sigma^2 - 2\sigma \cos 2\theta = \rho^2 - 2\rho \cos \theta$$

has a positive root $\sigma$. This is the case when

(i) $\rho > 2\cos \theta$, or
(ii) $\rho \leq 2\cos \theta$, $\cos 2\theta > 0$ and $4\cos^2 2\theta + 4\rho(\rho - 2\cos \theta) \geq 0$. Equivalently, $\rho_+ \leq \rho \leq 2\cos \theta$, where

$$\rho_\pm = \cos \theta \pm \sqrt{\sin \theta \sin 3\theta}$$

are the roots of the equation $\rho^2 - 2\rho \cos \theta + \cos^2 2\theta = 0$ for $0 < \theta < \frac{\pi}{3}$.

Now, the perpendicular bisector of $PT_a$ intersects the line $\ell'$ at the point $B$ with polar coordinates $(\beta, 2\theta)$, where

$$\beta = \frac{\rho \cos \theta - \sigma \cos 2\theta}{\rho \cos \theta - \sigma}.$$ 

The requirement $\beta > 0$ is equivalent to $\sigma < \rho \cos \theta$. From (1), this is equivalent to $\rho < 4\cos \theta$.

For $0 < \theta < \frac{\pi}{3}$, let $P_\pm$ be the points with polar coordinates $(\rho_\pm, \theta)$. These points bound a closed curve $\mathcal{C}$ as shown in Figure 3. If $T_a$ lies inside the curve $\mathcal{C}$,
then the circle $T_b(T_a)$ does not intersect the half line $\ell'$. We summarize the results with reference to Figure 3.

The construction problem of $ABC$ from $(A, T_a, T_b)$ has

(1) a unique solution if $T_a$ lies in the region between the two semicircles $\rho = 2 \cos \theta$ and $\rho = 4 \cos \theta$,

(2) two solutions if $T_a$ lies between the semicircle $\rho = 2 \cos \theta$ and the curve $\mathcal{C}$ for $\theta < \frac{\pi}{4}$.

2. Construction from $(A, T_b, T_c)$

The construction of triangle $ABC$ from $(A, T_a, T_b)$ is Problem 60 in Wernick’s list [2]. Wernick has indicated constructibility. We present two simple solutions.

Construction 3. Given $A$, $T_b$, $T_c$, construct the circles with centers $T_b$ and $T_c$, tangent to $AT_c$ and $AT_b$ respectively. The common tangent of these circles that lies opposite to $A$ with respect to the line $T_bT_c$ is the line $BC$ of the required triangle $ABC$. The construction of the vertices $B, C$ is obvious.
Construction 4. Given $A, T_b, T_c$, construct

(i) the circle through the three points
(ii) the bisector of angle $T_bAT_c$ to intersect the circle at $M$,
(iii) the reflection $M'$ of $M$ in the line $T_bT_c$,
(iv) the circle $M'(T_b)$ to intersect the bisector at $I$ (so that $A$ and $I$ are on opposite sides of $T_bT_c$),
(v) the half line $T_bI$ to intersect the half line $AT_c$ at $B$,
(vi) the half line $T_cI$ to intersect the half line $AT_b$ at $C$.

$ABC$ is the required triangle with incenter $I$.

References


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