Some Powerian Pairs in the Arbelos

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Abstract. Frank Power has presented two pairs of Archimedean circles in the arbelos. In each case the two Archimedean circles are tangent to each other and tangent to a given circle. We give some more of these Powerian pairs.

1. Introduction

We consider an arbelos with greater semicircle \((O)\) of radius \(r\) and smaller semicircles \((O_1)\) and \((O_2)\) of radii \(r_1\) and \(r_2\) respectively. The semicircles \((O_1)\) and \((O)\) meet in \(A\), \((O_2)\) and \((O)\) in \(B\), \((O_1)\) and \((O_2)\) in \(C\) and the line through \(C\) perpendicular to \(AB\) meets \((O)\) in \(D\). Beginning with Leon Bankoff [1], a number of interesting circles congruent to the Archimedean twin circles has been found associated with the arbelos. These have radii \(\frac{r_1}{r}\). See [2]. Frank Power [5] has presented two pairs of Archimedean circles in the Arbelos with a definition unlike the other known ones given for instance in [2, 3, 4].

![Figure 1](image)

**Proposition 1** (Power [5]). Let \(M_1\) and \(M_2\) be the 'highest' points of \((O_1)\) and \((O_2)\) respectively. Then the pairs of congruent circles tangent to \((O)\) and tangent to each other at \(M_1\) and \(M_2\) respectively, are pairs of Archimedean circles.

To pairs of Archimedean circles tangent to a given circle and to each other at a given point we will give the name Powerian pairs.

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1The pair of Archimedean circles \((A_5a)\) and \((A_5b)\), with numbering as in [4], qualifies for what we will later in the paper refer to as Powerian pair, as they are tangent to each other at \(C\) and to the circular hull of Archimedes’ twin circles. This however is not how they were originally defined.*
2. Three double Powerian pairs

2.1. Let $M$ be the midpoint of $CD$. Consider the endpoints $U_1$ and $U_2$ of the diameter of $(CD)$ perpendicular to $OM$.

![Figure 2](image2.png)

Note that $OC^2 = (r_1 - r_2)^2$ and as $CD = 2\sqrt{r_1 r_2}$ that $OD^2 = r_1^2 - r_1 r_2 + r_2^2$ and $OU_1^2 = r_1^2 + r_2^2$.

Now consider the pairs of congruent circles tangent to each other at $U_1$ and $U_2$ and tangent to $(O)$. The radii $\rho$ of these circles satisfy

$$(r_1 + r_2 - \rho)^2 = OU_1^2 + \rho^2$$

from which we see that $\rho = \frac{r_1 r_2}{r}$. This pair is thus Powerian. By symmetry the other pair is Powerian as well.

2.2. Let $T_1$ and $T_2$ be the points of tangency of the common tangent of $(O_1)$ and $(O_2)$ not through $C$. Now consider the midpoint $O'$ of $O_1 O_2$, also the center of the semicircle $(O_1 O_2)$, which is tangent to segment $T_1 T_2$ at its midpoint.

![Figure 3](image3.png)

As $T_1 T_2 = 2\sqrt{r_1 r_2}$ we see that $O'T_1^2 = \left(\frac{r_1 + r_2}{2}\right)^2 + r_1 r_2$. Now consider the pairs of congruent circles tangent to each other at $T_1$ and tangent to $(O_1 O_2)$. The
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Radii $\rho$ of these circles satisfy

$$\left(\frac{r_1 + r_2}{2}\right)^2 + \rho^2 - \rho^2 = O'T_1^2$$

from which we see that $\rho = \frac{r_1 r_2}{r}$ and this pair is Powerian. By symmetry the pair of congruent circles tangent to each other at $T_2$ and to $(O_1 O_2)$ is Powerian.

Remark: These pairs are also tangent to the circle with center $O'$ through the point where the Schoch line meets $(O)$.

2.3. Note that $AD = 2\sqrt{r_1}$, hence

$$AT_1 = \frac{r_1}{r} \cdot AD = \frac{2r_1 \sqrt{r_1}}{\sqrt{r}}.$$

Now consider the pair of congruent circles tangent to each other at $T_1$ and to the circle with center $A$ through $C$. The radii of these circles satisfy

$$AT_1^2 + \rho^2 = (2r_1 - \rho)^2$$

from which we see that $\rho = \frac{r_1 r_2}{r}$ and this pair is Powerian. In the same way the pair of congruent circles tangent to each other at $T_2$ and to the circle with center $B$ through $C$ is Powerian.

References


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