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Some Powerian Pairs in the Arbelos

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Abstract. Frank Power has presented two pairs of Archimedean circles in the arbelos. In each case the two Archimedean circles are tangent to each other and tangent to a given circle. We give some more of these Powerian pairs.

1. Introduction

We consider an arbelos with greater semicircle (O) of radius r and smaller semicircle (O_1) and (O_2) of radii r_1 and r_2 respectively. The semicircles (O_1) and (O) meet in A, (O_2) and (O) in B, (O_1) and (O_2) in C and the line through C perpendicular to AB meets (O) in D. Beginning with Leon Bankoff [1], a number of interesting circles congruent to the Archimedean twin circles has been found associated with the arbelos. These have radii $\frac{r_1 r_2}{r}$. See [2]. Frank Power [5] has presented two pairs of Archimedean circles in the Arbelos with a definition unlike the other known ones given for instance in [2, 3, 4]. [1]

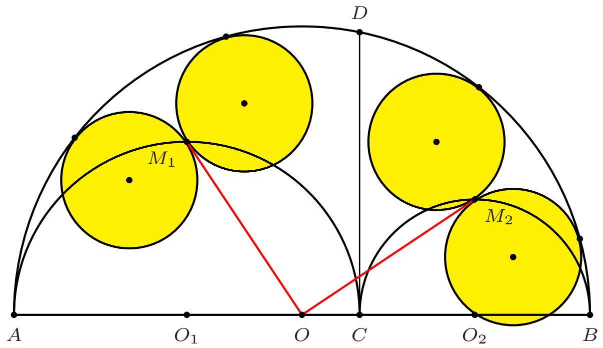

Figure 1

Proposition 1 (Power [5])**.** *Let M_1 and M_2 be the 'highest' points of (O_1) and (O_2) respectively. Then the pairs of congruent circles tangent to (O) and tangent to each other at M_1 and M_2 respectively, are pairs of Archimedean circles.*

To pairs of Archimedean circles tangent to a given circle and to each other at a given point we will give the name *Powerian pairs*.

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[1] The pair of Archimedean circles (A_{5a}) and (A_{5b}), with numbering as in [4], qualifies for what we will later in the paper refer to as *Powerian pair*, as they are tangent to each other at C and to the circular hull of Archimedes' twin circles. This however is not how they were originally defined.

2. Three double Powerian pairs

2.1. Let M be the midpoint of CD. Consider the endpoints U_1 and U_2 of the diameter of (CD) perpendicular to OM.

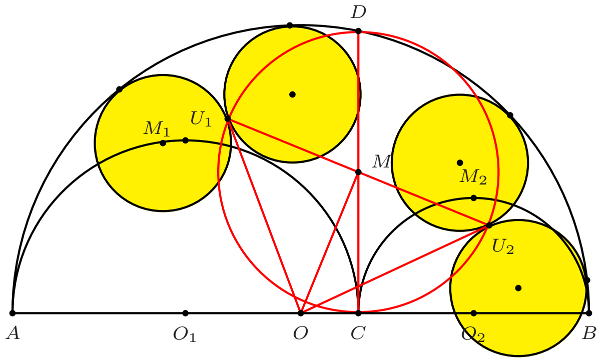

Figure 2

Note that $OC^2 = (r_1 - r_2)^2$ and as $CD = 2\sqrt{r_1 r_2}$ that $OD^2 = r_1^2 - r_1 r_2 + r_2^2$ and $OU_1^2 = r_1^2 + r_2^2$.

Now consider the pairs of congruent circles tangent to each other at U_1 and U_2 and tangent to (O). The radii ρ of these circles satisfy

$$(r_1 + r_2 - \rho)^2 = OU_1^2 + \rho^2$$

from which we see that $\rho = \frac{r_1 r_2}{r}$. This pair is thus Powerian. By symmetry the other pair is Powerian as well.

2.2. Let T_1 and T_2 be the points of tangency of the common tangent of (O_1) and (O_2) not through C. Now consider the midpoint O' of O_1O_2, also the center of the semicircle (O_1O_2), which is tangent to segment T_1T_2 at its midpoint.

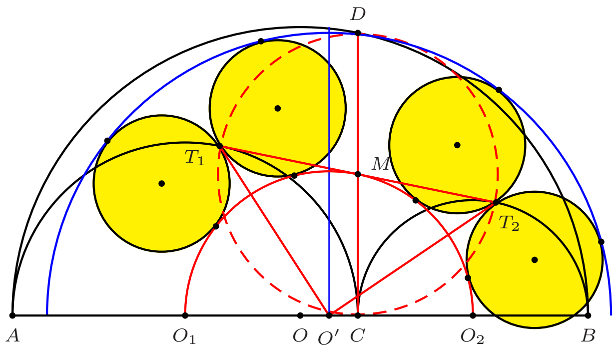

Figure 3

As $T_1T_2 = 2\sqrt{r_1 r_2}$ we see that $O'T_1^2 = \left(\frac{r_1+r_2}{2}\right)^2 + r_1 r_2$. Now consider the pairs of congruent circles tangent to each other at T_1 and tangent to (O_1O_2). The radii ρ of these circles satisfy

$$\left(\frac{r_1+r_2}{2}\right)+\rho)^2-\rho^2=O'T_1^2$$

from which we see that $\rho=\frac{r_1r_2}{r}$ and this pair is Powerian. By symmetry the pair of congruent circles tangent to each other at T_2 and to (O_1O_2) is Powerian.

Remark: These pairs are also tangent to the circle with center O' through the point where the Schoch line meets (O).

2.3. Note that $AD=2\sqrt{rr_1}$, hence

$$AT_1=\frac{r_1}{r}AD=\frac{2r_1\sqrt{r_1}}{\sqrt{r}}.$$

Now consider the pair of congruent circles tangent to each other at T_1 and to the circle with center A through C. The radii of these circles satisfy

$$AT_1^2+\rho^2=(2r_1-\rho)^2$$

from which we see that $\rho=\frac{r_1r_2}{r}$ and this pair is Powerian. In the same way the pair of congruent circles tangent to each other at T_2 and to the circle with center B through C is Powerian.

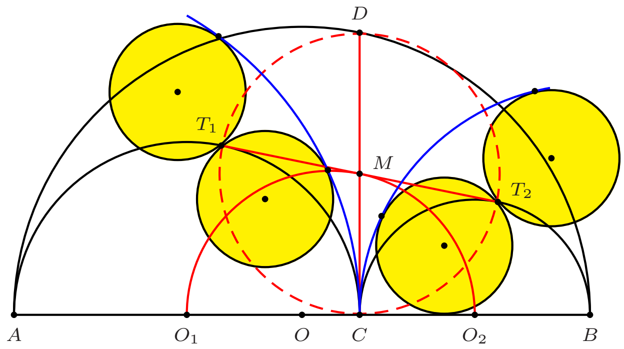

Figure 4

References

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