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The Arbelos and Nine-Point Circles

Quang Tuan Bui

Abstract. We construct some new Archimedean circles in an arbelos in connection with the nine-point circles of some appropriate triangles. We also construct two new pairs of Archimedes circles analogous to those of Frank Power, and one pair of Archimedean circles related to the tangents of the arbelos.

1. Introduction

We consider an arbelos consisting of three semicircles (O_1), (O_2), (O), with points of tangency A, B, P. Denote by r_1, r_2 the radii of (O_1), (O_2) respectively. Archimedes has shown that the two circles, each tangent to (O), the common tangent PQ of (O_1), (O_2), and one of (O_1), (O_2), have congruent radius $r = \frac{r_1 r_2}{r_1+r_2}$. See [1, 2]. Let C be a point on the half line PQ such that $PC = h$. We consider the nine-point circle (N) of triangle ABC. This clearly passes through O, the midpoint of AB, and P, the altitude foot of C on AB. Let AC intersect (O_1) again at A', and BC intersect (O_2) again at B'. Let O_e and H be the circumcenter and orthocenter of triangle ABC. Note that C and H are on opposite sides of the semicircular arc (O), and the triangles ABC and ABH have the same nine-point circle. We shall therefore assume C beyond the point Q on the half line PQ. See Figure 1. In this paper the labeling of knowing Archimedean circles follows [2].

2. Archimedean circles with centers on the nine-point circle

Let the perpendicular bisector of AB cut (N) at O and M_e, and the altitude CP cut (N) at P and M_h. See Figure 1.

2.1. It is easy to show that $POM_\text{e}M_\text{h}$ is a rectangle so M_e is the reflection of P in N. Because O_e is also the reflection of H in N, $HPO_\text{e}M_\text{e}$ is a parallelogram, and we have

$$O_\text{e}M_\text{e} = PH. \tag{1}$$

Furthermore, from the similarity of triangles HPB and APC, we have $\frac{PH}{PB} = \frac{PA}{PC}$. Hence,

$$PH = \frac{4r_1 r_2}{h}. \tag{2}$$

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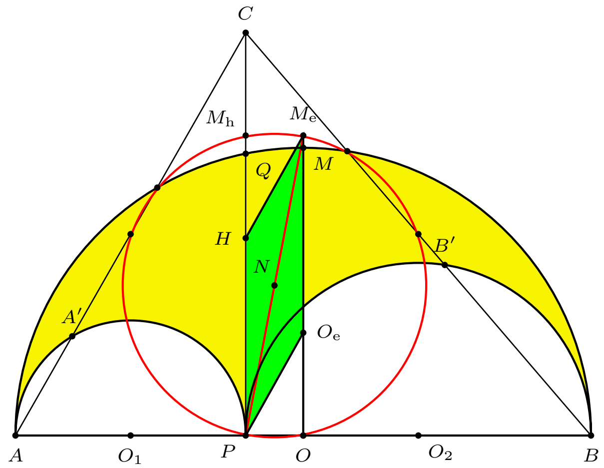

Figure 1.

2.2. Since C is beyond Q on the half line PQ, the intersection F of $A'O_1$ and $B'O_2$ is a point F below the arbelos. Denote by (I) the incircle of triangle FO_1O_2. See Figure 2. The line IO_1 bisects both angles O_2O_1F and $A'O_1A$. Because $O_1A' = O_1A$, IO_1 is perpendicular to AC, and therefore is parallel to BH. Similarly, IO_2 is parallel to AH. From these, two triangles AHB and O_2IO_1 are homothetic with ratio $\frac{AB}{O_2O_1} = 2$. It is easy to show that O is the touch point of (I) with AB and that the inradius is

$$IO = \frac{1}{2} \cdot PH. \tag{3}$$

In fact, if F' is the reflection of F in the midpoint of O_1O_2 then O_1FO_2F' is a parallelogram and the circle (PH) (with PH as diameter) is the incircle of $F'O_1O_2$. It is the reflection of (I) in midpoint of O_1O_2.

2.3. Now we apply these results to the arbelos. From (2), $\frac{1}{2} \cdot PH = \frac{2r_1r_2}{h} =$ Archimedean radius $\frac{r_1r_2}{r_1+r_2}$ if and only if

$$CP = h = 2(r_1 + r_2) = AB.$$

In this case, point C and the orthocenter H of ABC are easy constructed and the circle with diameter PH is the Bankoff triplet circle (W_3). From this we can also construct also the incircle of the arbelos. In this case $F' =$ incenter of the arbelos. From (3) we can show that when $CP = AB$, the incircle of FO_1O_2 is also Archimedean. See Figure 3.

Let M be the intersection of OO_e and the semicircle (O), *i.e.*, the highest point of (O). When $CP = h = 2(r_1 + r_2) = AB$,

$$OO_e = M_hH = M_hC = \frac{h - PH}{2} = (r_1 + r_2) - \frac{r_1r_2}{r_1 + r_2}.$$

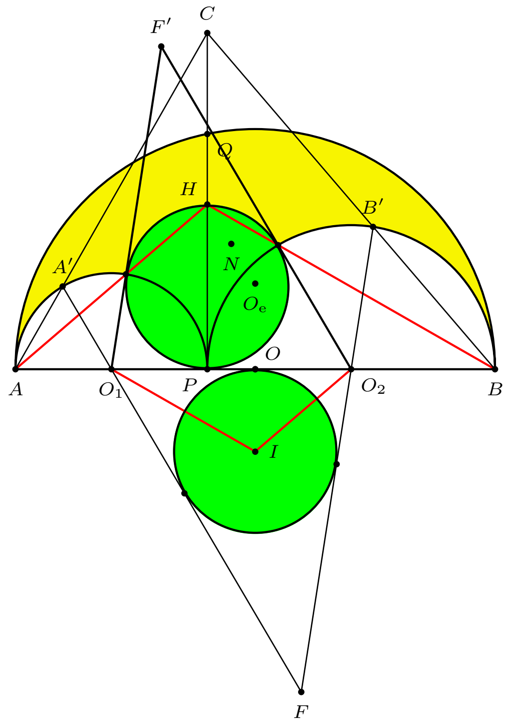

Figure 2.

Therefore,

$$O_eM = (r_1 + r_2) - \left(r_1 + r_2 - \frac{r_1 r_2}{r_1 + r_2}\right) = \frac{r_1 r_2}{r_1 + r_2}.$$

From (1), $O_eM_e = PH = \frac{2r_1r_2}{r_1+r_2}$. This means that M is the midpoint of O_eM_e, or the two circles centered at O_e and M_e and touching (O) at M are also Archimedean circles. See Figure 3.

We summarize the results as follows.

Proposition 1. *In the arbelos (O_1), (O_2), (O), if C is any point on the half line PQ beyond Q and H is orthocenter of ABC, then the circle (PH) is Archimedean if and only if $CP = AB = 2(r_1 + r_2)$. In this case, we have the following results.*
(1). *The orthocenter H of ABC is the intersection point of Bankoff triplet circle (W_3) with PQ (other than P).*
(2). *The incircle of triangle FO_1O_2 is an Archimedean circle touching AB at O; it is reflection of (W_3) in the midpoint of O_1O_2.*
(3). *The circle centered at circumcenter O_e of ABC and touching (O) at its highest point M is an Archimedean circle. This circle is (W_{20}).*
(4). *The circle centered on nine point circle of ABC and touching (O) at M is an Archimedean circle; it is the reflection of (W_{20}) in M.*
(5). *The reflection F' of F in midpoint of O_1O_2 is the incenter of the arbelos.*

Remarks. (a) The Archimedean circles in (2) and (4) above are new.

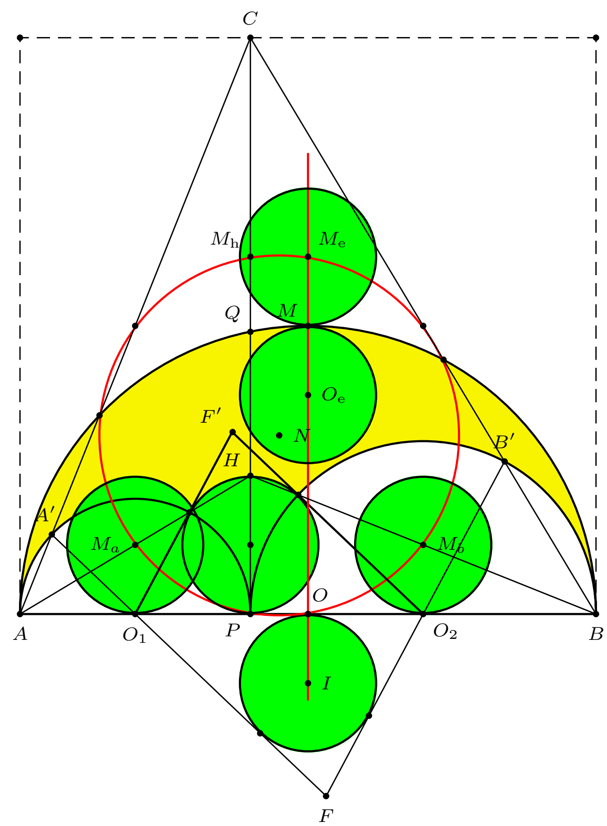

Figure 3.

(b) There are two more obvious Archimedean circles with centers on the nine-point circle. These are (M_a) and (M_b), where M_a and M_b are the midpoints of AH and BH respectively. See Figure 3.

(c) The midpoints M_a, M_b of HA, HB are on nine point circle of ABC and are two vertices of Eulerian triangle of ABC. Two circles centered at M_a, M_b and touch AB at O_1, O_2 respectively are congruent with (W_3) so they are also Archimedean circles (see [2]).

3. Two new pairs of Archimedean circles

If T is a point such that $OT^2 = r_1^2 + r_2^2$, then there is a pair of Archimedean circles mutually tangent at T, and each tangent internally to (O). Frank Power [5]. constructed two such pairs with $T = M_1$, M_2, the highest points of (O_1) and (O_2) respectively. Allowing tangency with other circles, Floor van Lamoen [4] called such a pair Powerian. We construct two new Powerian pairs.

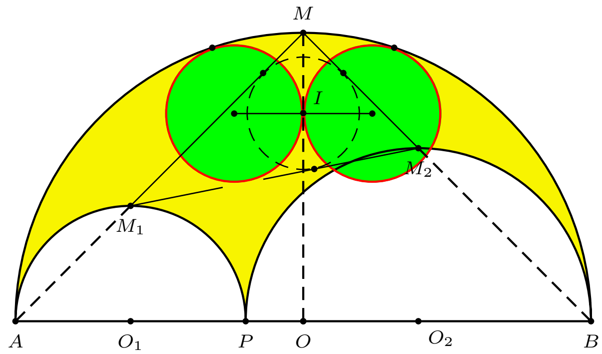

Figure 4.

3.1. The triangle MM_1M_2 has $MM_1 = \sqrt{2} \cdot r_2$, $MM_2 = \sqrt{2} \cdot r_1$, and a right angle at M. Its incenter is the point I on OM such that

$$MI = \sqrt{2} \cdot \frac{1}{2}(MM_1 + MM_2 - M_1M_2) = (r_1 + r_2) - \sqrt{r_1^2 + r_2^2}.$$

Therefore, $OI^2 = r_1^2 + r_2^2$, and we have a Powerian pair. See Figure 4.

3.2. Consider also the semicircles (T_1) and (T_2) with diameters AO_2 and BO_1. The intersection J of (T_1) and (T_2) satisfies

$$OJ^2 = OP^2 + PJ^2 = (r_1 - r_2)^2 + 2r_1r_2 = r_1^2 + r_2^2.$$

Therefore, we have another Powerian pair. See Figure 5.

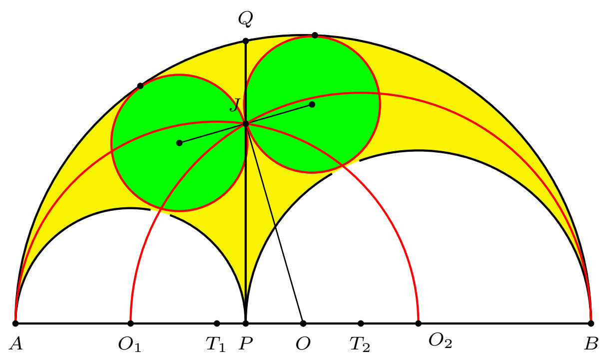

Figure 5.

4. Two Archimedean circles related to the tangents of the arbelos

We give two more Archimedean circles related to the tangents of the arbelos.

Let $\mathcal{L}$ be the tangent of (O) at Q, and Q_1, Q_2 the orthogonal projections of O_1, O_2 on $\mathcal{L}$. The lines O_1Q_1 and O_2Q_2 intersect the semicircles (O_1) and (O_2) at R_1 and R_2 respectively. Note that R_1R_2 is a common tangent of the semicircles (O_1) and (O_2). The circles (N_1), (N_2) with diameters Q_1R_1 and Q_2R_2 are Archimedean. Indeed, if (W_6) and (W_7) are the two Archimedean circles through

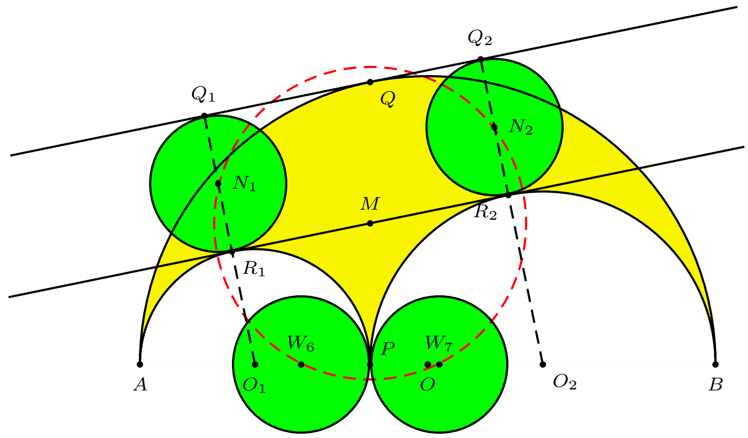

Figure 6.

P with centers on AB (see [2]), then N_1, N_2, W_6, W_7 lie on the same circle with center the midpoint M of PQ. See Figure 6. We leave the details to the reader.

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Quang Tuan Bui: 45B, 296/86 by-street, Minh Khai Street, Hanoi, Vietnam

E-mail address: bqtuan1962@yahoo.com