Forum Geometricorum
Volume 7 (2007) 121–123.

FORUM GEOM
ISSN 1534-1178

Characterizations of an Infinite Set of Archimedean Circles

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Abstract. For an arbelos with the two inner circles touching at a point O, we give necessary and sufficient conditions that a circle passing through O is Archimedean.

Consider an arbelos with two inner circles α and β with radii a and b respectively touching externally at a point O. A circle of radius $r_A = ab/(a+b)$ is called Archimedean. In [3], we have constructed three infinite sets of Archimedean circles. One of these consists of circles passing through the point O. In this note we give some characterizations of Archimedean circles passing through O. We set up a rectangular coordinate system with origin O and the positive x-axis along a diameter OA of α (see Figure 1).

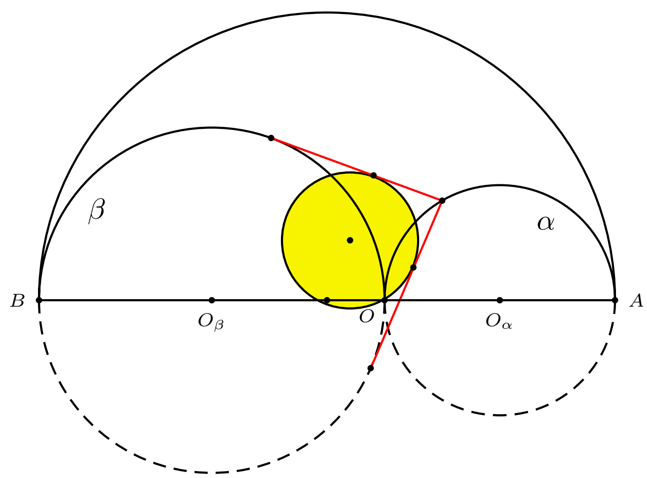

Figure 1

Theorem 1. *A circle through O (not tangent internally to β) is Archimedean if and only if its external common tangents with β intersect at a point on α.*

Proof. Consider a circle δ with radius $r \neq b$ and center $(r\cos\theta, r\sin\theta)$ for some real number θ with $\cos\theta \neq -1$. The intersection of the common external tangents

Publication Date: July 2, 2007. Communicating Editor: Paul Yiu.

of β and δ is the external center of similitude of the two circles, which divides the segment joining their centers externally in the ratio $b : r$. This is the point

$$\left(\frac{br(1+\cos\theta)}{b-r}, \frac{br\sin\theta}{b-r}\right). \tag{1}$$

The theorem follows from

$$\left(\frac{br(1+\cos\theta)}{b-r} - a\right)^2 + \left(\frac{br\sin\theta}{b-r}\right)^2 - a^2 = \frac{2br(a+b)(1+\cos\theta)}{(b-r)^2}(r - r_A).$$

□

Let O_α and O_β be the centers of the circles α and β respectively.

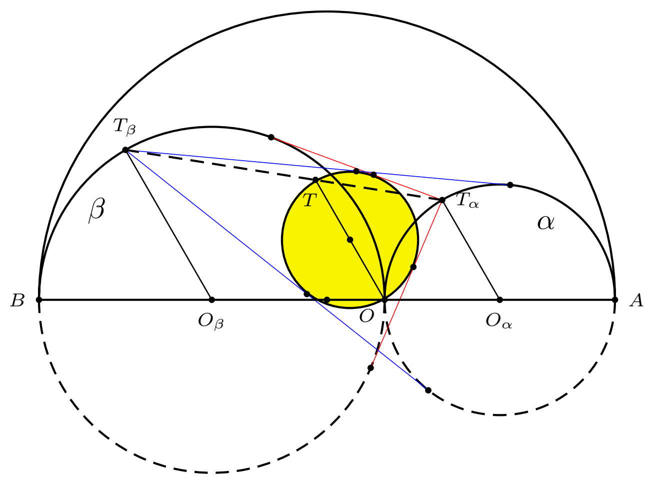

Figure 2

Corollary 2. *Let δ be an Archimedean circle with a diameter OT, and T_α the intersection of the external common tangents of the circles δ and β; similarly define T_β.*
(i) *The vectors $\overrightarrow{OT}$ and $\overrightarrow{O_\alpha T_\alpha}$ are parallel with the same direction.*
(ii) *The point T divides the segment $T_\alpha T_\beta$ internally in the ratio $a : b$.*

Proof. We describe the center of δ by $(r_A\cos\theta, r_A\sin\theta)$ for some real number θ (see Figure 2). Then the point T_α is described by

$$\left(\frac{br_A(1+\cos\theta)}{b-r_A}, \frac{br_A\sin\theta}{b-r_A}\right) = (a(1+\cos\theta), a\sin\theta)$$

by (1). This implies $\overrightarrow{O_\alpha T_\alpha} = a(\cos\theta, \sin\theta)$. (ii) is obtained directly, since T_β is expressed by $(b(-1+\cos\theta), b\sin\theta)$. □

In Theorem 1, we exclude the Archimedean circle which touches β internally at the point O. But this corollary holds even if the circle δ touches β internally. If δ is the Bankoff circle touching the line OA at the origin O [1], then T_α is the highest point on α. If δ is the Archimedean circle touching β externally at the point O, then T_α obviously coincides with the point A. This fact is referred in [2] using the circle labeled W_6. Another notable Archimedean circle passing through O is that having center on the Schoch line $x = \frac{b-a}{b+a} r_A$, which is labeled as U_0 in [2]. We have showed that the intersection of the external common tangents of β and this circle is the intersection of the line $x = 2r_A$ and the circle α [3].

By the uniqueness of the figure, we get the following characterizations of the Archimedean circles passing through the point O.

Corollary 3. *Let δ be a circle with a diameter OT, and let T_α and T_β be points on α and β respectively such that $\overrightarrow{O_\alpha T_\alpha}$ and $\overrightarrow{O_\beta T_\beta}$ are parallel to $\overrightarrow{OT}$ with the same direction.* (i) *The circle δ is Archimedean if and only if the points T divides the line segment $T_\alpha T_\beta$ internally in the ratio $a : b$.* (ii) *If the center of δ does not lie on the line OA, then δ is Archimedean if and only if the three points T_α, T_β and T are collinear.*

The statement (i) in this corollary also holds when δ touches β internally.

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