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Remarks on Woo's Archimedean Circles

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Abstract. The property of Woo's Archimedean circles does not hold only for Archimedean circles but circles with any radii. The exceptional case of this has a close connection to Archimedean circles.

1. Introduction

Let A and B be points with coordinates $(2a, 0)$ and $(-2b, 0)$ on the x-axis with the origin O and positive real numbers a and b. Let α, β and γ be semicircles forming an arbelos with diameters OA, OB and AB respectively. We follow the notations in [4]. For a real number n, let $\alpha(n)$ and $\beta(n)$ be the semicircles in the upper half-plane with centers $(n, 0)$ and $(-n, 0)$ respectively and passing through the origin O. A circle with radius $r = \frac{ab}{a+b}$ is called an Archimedean circle. Thomas Schoch has found that the circle touching the circles $\alpha(2a)$ and $\beta(2b)$ externally and γ internally is Archimedean [2] (see Figure 1). Peter Woo called the Schoch line the one passing through the center of this circle and perpendicular to the x-axis, and found that the circle U_n touching the circles $\alpha(na)$ and $\beta(nb)$ externally with center on the Schoch line is Archimedean for a nonnegative real number n. In this note we consider the property of Woo's Archimedean circles in a general way.

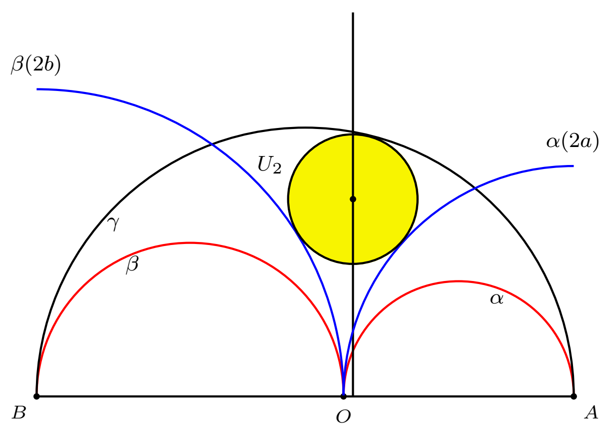

Figure 1.

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2. A generalization of Woo's Archimedean circles

We show that the property of Woo's Archimedean circles does not only hold for Archimedean circles. Indeed circles with any radii can be obtained in a similar way. We say that a circle touches $\alpha(na)$ *appropriately* if they touch externally (respectively internally) for a positive (respectively negative) number n. If one of the two circles is a point circle and lies on the other, we also say that the circle touches $\alpha(na)$ appropriately. The same notion of appropriate tangency applies to $\beta(nb)$.

Theorem 1. *Let s and t be nonzero real numbers such that $tb \pm sa \neq 0$. If there is a circle of radius ρ touching the circles $\alpha(nsa)$ and $\beta(ntb)$ appropriately for a real number n, then its center lies on the line*

$$x = \frac{tb - sa}{tb + sa}\rho. \tag{1}$$

Proof. Consider the center (x, y) of the circle with radius ρ touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately. The distance between (x, y) and the centers of $\alpha(nsa)$ and $\beta(ntb)$ are $|\rho + nsa|$ and $|\rho + ntb|$ respectively. Therefore by the Pythagorean theorem,

$$y^2 = (\rho + nsa)^2 - (x - nsa)^2 = (\rho + ntb)^2 - (x + ntb)^2.$$

Solving the equations, we get (1) above. □

For a real number k different from 0 and $\pm\rho$, we can choose the real numbers s and t so that (1) expresses the line $x = k$. Let us assume $st > 0$. Then the circles $\alpha(nsa)$ and $\beta(ntb)$ lie on opposite sides of the y-axis. If $sz > 0$ and $tz > 0$, there is always a circle of radius ρ touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately. If $ns < 0$ and $nt < 0$, such a circle exists when $-2n(sa + tb) \leq 2\rho$. Hence in the case $st > 0$, the tangent circle exists if $n(sa + tb) + \rho \geq 0$. Now let us assume $st < 0$. Then circles $\alpha(nsa)$ and $\beta(ntb)$ lie on the same side of the y-axis. The circle of radius ρ touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately exists if $-2n(sa + tb) \geq 2\rho$. Hence in the case $st < 0$, the tangent circle exists if $n(sa+tb)+\rho \leq 0$. In any case the center of the circle with radius ρ touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately is

$$\left(\frac{tb - sa}{tb + sa}\rho,\ \pm\frac{2\sqrt{nabst((sa + tb) + \rho)\rho}}{|sa + tb|}\right).$$

Therefore, for every point P not on the lines $x = 0, \pm\rho$, we can choose real numbers s, t and n so that the circle, center P, radius ρ, is touching $\alpha(nsa)$ and $\beta(tzb)$ appropriately.

The Schoch line is the line $x = \frac{b-a}{b+a}r$ (see [4]). Therefore Woo's Archimedean circles and the Schoch line are obtained when $s = t$ and $\rho = r$ in Theorem 1. If $st > 0$, then $-1 < \frac{tb-sa}{tb+sa} < 1$. Hence the line (1) lies in the region $-\rho < x < \rho$ in this case.

The external center of similitude of β and a circle with radius ρ and center on the line (1) lies on the line

$$x = \frac{2tb^2\rho}{(b-\rho)(sa+tb)}$$

by similarity. In particular, the external centers of similitude of Woo's Archimedean circles and β lie on the line $x = 2r$. See [4].

3. Circles with centers on the y-axis

We have excluded the cases $tb \pm sa \neq 0$ in Theorem 1. The case $tb + sa = 0$ is indeed trivial since the circles $\alpha(nsa)$ and $\beta(ntb)$ coincide. By Theorem 1, for $k \neq 0$, the circle touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately and with center on the line $x = k$ has radius $\frac{tb+sa}{tb-sa}k$. On the other hand, if $tb = sa$, the circles $\alpha(nsa)$ and $\beta(ntb)$ are congruent and lie on opposite sides of the y-axis, and the line (1) coincides with the y-axis. Therefore the radii of circles touching the two circles appropriately and having the center on this line cannot be determined uniquely.

We show that this exceptional case $(tb = sa)$ has a close connection with Archimedean circles. Since $\alpha(nsa)$ and $\beta(ntb)$ are congruent, we now define $\alpha[n] = \alpha(n(a+b))$ and $\beta[n] = \beta(n(a+b))$. The circles $\alpha[n]$ and $\beta[n]$ are congruent, and their radii are n times of the radius of γ. For two circles of radii ρ_1, ρ_2 and with distance d between their centers, consider their inclination [3] given by

$$\frac{\rho_1^2 + \rho_2^2 - d^2}{2\rho_1\rho_2}.$$

This is the cosine of the angle between the circles if they intersect, and is 0, $+1$, -1 according as they are orthogonal or tangent internally or externally.

Theorem 2. *If a circle $\mathcal{C}$ of radius ρ touches $\alpha[n]$ and $\beta[n]$ appropriately for a real number n, then the inclination of $\mathcal{C}$ and γ is $\frac{2r}{\rho} - n$.*

Proof. The square of the distance between the centers of the circles $\mathcal{C}$ and γ is $(\rho + n(a+b))^2 - (n(a+b))^2 + (a-b)^2$ by the Pythagorean theorem. Therefore their inclination is

$$\frac{\rho^2 + (a+b)^2 - (\rho + n(a+b))^2 + (n(a+b))^2 - (a-b)^2}{2\rho(a+b)} = \frac{2r}{\rho} - n.$$

□

Let k be a positive real number. The radius of a circle touching $\alpha[n]$ and $\beta[n]$ appropriately is kr if and only if the inclination of the circle and γ is $\frac{2}{k} - n$ for a real number n.

Corollary 3. *A circle touching $\alpha[n]$ and $\beta[n]$ appropriately for a real number n is Archimedean if and only if the inclination of this circle and γ is $2 - n$.*

This gives an infinite set of Archimedean circles δ_n with centers on the positive y-axis. The circle δ_n exists if $n \geq \frac{-r}{2(a+b)}$, and the maximal value of the inclination of γ and δ_n is $2 + \frac{r}{2(a+b)}$. The circle δ_1 touches γ internally, δ_2 is orthogonal to

γ, and δ_3 touches γ externally by the corollary (see Figure 2). The circle δ_0 is the Bankoff circle [1], whose inclination with γ is 2.

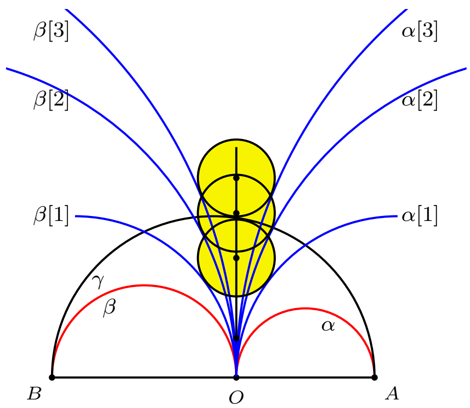

Figure 2

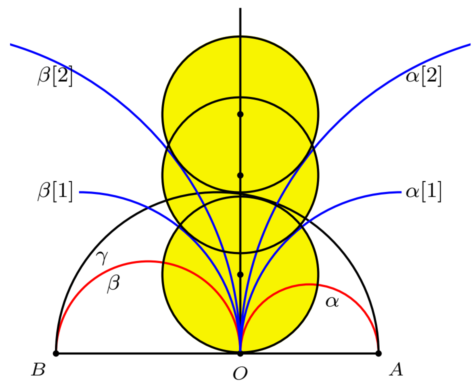

Figure 3

By the remark preceding Corollary 3 we can get circles with various radii and centers on the y-axis tangent or orthogonal to γ. Figure 3 shows some such examples. The three circles all have radii $2r$. One touches the degenerate circles $\alpha[0]$ and $\beta[0]$ (and the line AB) at O, and γ internally. A second circle touches $\alpha[1]$ and $\beta[1]$ externally and are orthogonal to γ. Finally, a third circle touches $\alpha[2]$, $\beta[2]$, and γ externally.

From [4], the center of the Woo circle U_n is the point

$$\left(\frac{b-a}{b+a}r,\ 2r\sqrt{n+\frac{r}{a+b}}\right).$$

The inclination of U_n and γ is $1+\frac{2(2-n)r}{a+b}$. This depends on the radii of α and β except the case $n=2$. In contrast to this, Corollary 3 shows that the inclination of δ_n and γ does not depend on the radii of α and β.

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