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Heronian Triangles Whose Areas Are Integer Multiples of Their Perimeters

Lubomir Markov

Abstract. We present an improved algorithm for finding all solutions to Goehl's problem $A = mP$ for triangles, *i.e.*, the problem of finding all Heronian triangles whose area (A) is an integer multiple (m) of the perimeter (P). The new algorithm does not involve elimination of extraneous rational triangles, and is a true extension of Goehl's original method.

1. Introduction and main result

In a recent paper [3], we presented a solution to the problem of finding all Heronian triangles (triangles with integer sides and area) for which the area A is a multiple m of the perimeter P, where $m \in \mathbb{N}$. The problem was introduced by Goehl [2] and is of interest because although its solution is exceedingly simple in the special case of right triangles, the general case remained unsolved for about 20 years despite considerable effort. It is also remarkable and somewhat contrary to intuition that for each m there are only finitely many triangles with the property $A = mP$; for instance, the triangles $(6, 8, 10)$, $(5, 12, 13)$, $(6, 25, 29)$, $(7, 15, 20)$ and $(9, 10, 17)$ are the only ones whose area equals their perimeter (the case $m = 1$). Reproducing Goehl's solution to the problem in the special case of right triangles is a simple matter: Suppose that a and b are the legs of a right triangle and $c = \sqrt{a^2 + b^2}$ is the hypotenuse. Setting the area equal to a multiple m of the perimeter and manipulating, one immediately obtains the identities $8m^2 = (a - 4m)(b - 4m)$ and $c = a + b - 4m$. These allow us to determine a, b and c after finding all possible factorizations of the left-hand side of the form $8m^2 = d_1 \cdot d_2$ and matching d_1 and d_2 with $(a - 4m)$ and $(b - 4m)$, respectively; restricting d_1 to those integers that do not exceed $\sqrt{8m^2} = 2\sqrt{2}m$ assures $a < b$ and avoids repetitions. We state Goehl's result in the following form:

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Theorem 1. *For a given m, the right-triangle solutions (a, b, c) to the problem $A = mP$ are determined from the relations*

$$8m^2 = (a - 4m)(b - 4m), \tag{1}$$
$$c = a + b - 4m. \tag{2}$$

Each factorization

$$8m^2 = d_1 \cdot d_2, \tag{3}$$

where

$$d_1 \leq \lfloor 2m\sqrt{2} \rfloor, \tag{4}$$

generates a solution triangle with sides given by the formulas

$$\begin{cases} a = d_1 + 4m, \\ b = d_2 + 4m, \\ c = d_1 + d_2 + 4m. \end{cases} \tag{5}$$

Our paper [3] extended Goehl's result to general triangles, but the solution involved extraneous rational triangles, which then had to be eliminated. The aim of this work is to present a radical simplification of our previous solution, which does not introduce extraneous triangles and is a direct generalization of Goehl's method. Our main goal is to prove the following theorem:

Theorem 2. *For a given m, all solutions (a, b, c) to the problem $A = mP$ are determined as follows: Find all divisors u of $2m$; for each u, find all numbers v relatively prime to u and such that $1 \leq v \leq \lfloor \sqrt{3}u \rfloor$; to each pair u and v, there correspond a factorization identity*

$$4m^2(u^2 + v^2) = \left[v\left(a - \frac{2m}{u}v\right) - 2mu\right]\left[v\left(b - \frac{2m}{u}v\right) - 2mu\right], \tag{6}$$

and a relation

$$c = a + b - \frac{4mv}{u}. \tag{7}$$

Each factorization

$$4m^2(u^2 + v^2) = \delta_1 \cdot \delta_2, \tag{8}$$

where

$$\delta_1 \leq \left\lfloor 2m\sqrt{u^2 + v^2} \right\rfloor \tag{9}$$

and only those factors δ_1, δ_2 for which $v \mid \delta_1 + 2mu$ and $v \mid \delta_2 + 2mu$ are considered, generates a solution triangle with sides given by the formulas

$$\begin{cases} a = \frac{\delta_1 + 2mu}{v} + \frac{2mv}{u}, \\ b = \frac{\delta_2 + 2mu}{v} + \frac{2mv}{u}, \\ c = \frac{\delta_1 + \delta_2 + 4mu}{v}. \end{cases} \tag{10}$$

Furthermore, for each fixed u, one concludes from the corresponding v's that
(1) *the obtuse-triangle solutions are obtained exactly when $v < u$;*
(2) *the acute-triangle solutions are obtained exactly when $u < v \leq \lfloor \sqrt{3}u \rfloor$, with*

the further restriction $\dfrac{2m}{u}(v^2-u^2) \le \delta_1 \le \left\lfloor 2m\sqrt{u^2+v^2} \right\rfloor$;
(3) *the right-triangle solutions are obtained exactly when* $u = v = 1$.

Note that Theorem 1 is a special case of Theorem 2 and that the substitution $u = v = 1$ transforms relations (6) through (10) into relations (1) through (5), respectively.

2. Summary of preliminary facts

Let A be the area and P the perimeter of a triangle with sides a, b, c, with the agreement that c shall always denote the largest side. Our problem (we call it $A = mP$ for short) is to find all Heronian triangles whose area equals an integer multiple m of the perimeter. We state all preliminaries as a sequence of lemmas whose proofs can either be easily reproduced by the reader, or can be found (except for Lemma 5) in [3].

First we note that Heron's formula

$$4A = \sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$$

and simple trigonometry easily imply the following lemma:

Lemma 3. *Assume that the triple* (a, b, c) *solves the problem* $A = mP$.
(1) $a + b - c$ *is an even integer.*
(2) $a + b - c < 4m\sqrt{3}$.
(3) *The resulting triangle is* $\begin{Bmatrix} obtuse \\ acute \\ right \end{Bmatrix}$ *if and only if* $a + b - c \begin{Bmatrix} < 4m \\ > 4m \\ = 4m \end{Bmatrix}$.

Next, we need a crucial rearrangement of Heron's formula:

Lemma 4. *The following doubly-Pythagorean form of Heron's formula holds:*

$$\left[c^2 - (a^2 + b^2)\right]^2 + (4A)^2 = (2ab)^2. \tag{11}$$

This representation allows the problem $A = mP$ to be reduced to a problem about Pythagorean triples; for our purposes, a Pythagorean triple (x, y, z) shall consist of nonnegative integers such that z (the "hypotenuse") shall always represent the largest number, whereas x and y (the "legs") need not appear in any particular order. The following parametric representation of *primitive* Pythagorean triples (*i.e.*, such that the components do not have a common factor greater than 1) is the only preliminary statement not proved in [3]; a self-contained proof can be found in [1]:

Lemma 5. *Depending on whether the first leg* x *is odd or even, every primitive Pythagorean triple* (x, y, z) *is uniquely expressed as* $(u^2 - v^2, 2uv, u^2 + v^2)$ *where* u *and* v *are relatively prime of opposite parity, or* $\left(\dfrac{u^2 - v^2}{2}, uv, \dfrac{u^2 + v^2}{2}\right)$ *where* u *and* v *are relatively prime and odd.*

A combination of Lemmas 4 and 5 easily yields

Lemma 6. *For a fixed m, solving the problem $A = mP$ is equivalent to determining all integer a, b, c that satisfy the equation*

$$\left[c^2 - (a^2 + b^2)\right]^2 + [4m(a + b + c)]^2 = (2ab)^2, \tag{12}$$

or equivalently, to solving in positive integers the following system of three equations in six unknowns:

$$\begin{cases} \pm\left[c^2 - (a^2 + b^2)\right] &= k(u^2 - v^2); \\ 4m(a + b + c) &= 2kuv; \\ 2ab &= k(u^2 + v^2). \end{cases} \tag{13}$$

It is easy to see that the first equation in (13) can be interpreted as follows.

Lemma 7. *Assume that, corresponding to certain values of u and v, there is a triple (a, b, c) which solves the problem $A = mP$. Then the triangle (a, b, c) is* $\left\{\begin{matrix} obtuse \\ acute \\ right \end{matrix}\right\}$ *if and only if* $\left\{\begin{matrix} u > v \\ u < v \\ u = v = 1 \end{matrix}\right\}$.

3. Proof of Theorem 2

Let us first investigate the case of an obtuse triangle (the case $u > v$); thus, the system (13) is $c^2 - (a^2+b^2) = k(u^2-v^2)$, $4m(a+b+c) = 2kuv$, $2ab = k(u^2 + v^2)$. For completeness, we reproduce the crucial proof of the main factorization identity from [3] (equation (17) below), which in essence solves the problem $A = mP$. Indeed, from the first and the third equations in (13) we get $(a + b)^2 - c^2 = 2kv^2$, and after factoring the left-hand side and using the second equation we get $a + b - c = \frac{4mv}{u}$. This implies that u must divide $2m$ because $a + b - c$ is even, and u, v are relatively prime. Combining the last relation with $a + b + c = \frac{kuv}{2m}$ and solving the resulting system yields

$$b + a = \frac{ku^2v + 8m^2v}{4mu}, \quad c = \frac{ku^2v - 8m^2v}{4mu}.$$

Similarly, adding the first and second equations and rearranging terms gives $(a - b)^2 = c^2 - 2ku^2$. Let us assume for a moment that $b \geq a$; then we have $b - a = \sqrt{c^2 - 2ku^2}$, and it is clear that the radicand must be a square. Put $Q = \frac{2m}{u}$ and substitute it in the expressions for c, $b + a$ and $b - a$. After simplification, one gets

$$c = \frac{kv - 2Q^2v}{2Q}, \ b + a = \frac{kv + 2Q^2v}{2Q}, \ b - a = \frac{1}{2Q}\sqrt{\left(kv - 2Q^2v\right)^2 - 32km^2}, \tag{14}$$

where the radicand must be a square. Put $\left(kv - 2Q^2v\right)^2 - 32km^2 = X^2$, and get

$$c = \frac{kv - 2Q^2v}{2Q}, \qquad b + a = \frac{kv + 2Q^2v}{2Q}, \qquad b - a = \frac{1}{2Q}X. \tag{15}$$

On the other hand, consider $\left(kv - 2Q^2v\right)^2 - 32km^2 = X^2$ as an equation in the variables X and k. Expanding the square and rearranging yields

$$k^2v^2 - 4k(v^2Q^2 + 8m^2) + 4Q^4v^2 = X^2.$$

The last equation is a Diophantine equation solvable by factoring: subtract the quantity $\left(kv - \dfrac{2(v^2Q^2 + 8m^2)}{v}\right)^2$ from both sides, simplify and rearrange terms; the result is

$$\left[2(v^2Q^2 + 8m^2)\right]^2 - \left(2v^2Q^2\right)^2 = \left(v^2k - 2v^2Q^2 - 16m^2\right)^2 - (Xv)^2. \quad (16)$$

In (16), factor both sides, substitute $Q = \dfrac{2m}{u}$ and simplify. This gives

$$\left(\frac{16m^2}{u}\right)^2(u^2+v^2) = \left[v^2(k - 2Q^2) - 16m^2 - Xv\right]\left[v^2(k - 2Q^2) - 16m^2 + Xv\right] \quad (17)$$

which is the main factorization identity mentioned above.

Now, the new idea is to eliminate k and X in (17), using (15) and the crucial fact that $a + b - c = \dfrac{4mv}{u}$. Indeed, from (15) we immediately obtain

$$X = 2Q(b - a), \quad k = \frac{2Qc + 2Q^2v}{v}, \quad (18)$$

which we substitute in (17) and simplify to get

$$16m^2(u^2 + v^2) = [v(c - b + a) - 4mu]\,[v(c + b - a) - 4mu]\,. \quad (19)$$

In the last relation, substitute $c = a + b - \dfrac{4mv}{u}$ and simplify again. The result is

$$4m^2(u^2 + v^2) = \left[v\Big(a - \frac{2m}{u}v\Big) - 2mu\right]\left[v\Big(b - \frac{2m}{u}v\Big) - 2mu\right],$$

which is exactly (6). This identity allows us to find sides a and b by directly matching factors of the left-hand side to respective quantities on the right; then c will be determined from $c = a + b - \dfrac{4mv}{u}$. Suppose $4m^2(u^2 + v^2) = \delta_1 \cdot \delta_2$. Since we want $\delta_1 = v\Big(a - \dfrac{2m}{u}v\Big) - 2mu$, it is clear that for a to be an integer, we necessarily must have $v \mid \delta_1 + 2mu$. Similarly, the requirement $v \mid \delta_2 + 2mu$ will ensure that b is an integer. Imposing these additional restrictions will produce *only* the integer solutions to the problem. Furthermore, choosing $\delta_1 \le \delta_2$ (or equivalently, $\delta_1 \le 2m\sqrt{u^2 + v^2}$) will guarantee that $a \le b$.

Next, solve

$$\delta_1 = v\left(a - \frac{2m}{u}v\right) - 2mu, \quad \delta_2 = v\left(a - \frac{2m}{u}v\right) - 2mu$$

for a and b, express c in terms of them and thus obtain formulas for the sides:

$$\begin{cases} a = \dfrac{\delta_1 + 2mu}{v} + \dfrac{2mv}{u}, \\ b = \dfrac{\delta_2 + 2mu}{v} + \dfrac{2mv}{u}, \\ c = \dfrac{\delta_1 + \delta_2 + 4mu}{v}; \end{cases}$$

these are exactly the formulas (10). To ensure $c \geq b$, we solve the inequality

$$\frac{\delta_1 + \delta_2 + 4mu}{v} \geq \frac{\delta_2 + 2mu}{v} + \frac{2mv}{u}$$

and obtain, after simplification,

$$\delta_1 \geq \frac{2m}{u}(v^2 - u^2). \tag{20}$$

The last relation will always be true if $u > v$, and thus the proof of the obtuse-case part of the theorem is concluded. Now, consider the acute case; i.e., the case $v > u$. The first equation in (13) is again $c^2 - (a^2 + b^2) = k(u^2 - v^2)$ (both sides are negative), and all the above derivations continue to hold true; it is now crucial to use the important bound $a + b - c < 4m\sqrt{3}$ which, combined with $a + b - c = \dfrac{4mv}{u}$, implies that $u < v < \sqrt{3}u$. The only difference from the obtuse case is that the bound (20) does not hold automatically; now it must be imposed to avoid repetitions and guarantee that $b \leq c$. Since the right-triangle case is obviously incorporated in the theorem, the proof is complete.

4. An example

We again examine the case $m = 2$ (cf. [3]). Let $m = 2$ in the algorithm suggested by Theorem 2; then $2m = 4$ and thus u could be 4, 2 or 1. For each u, determine the corresponding v's:

(A) $u = 4 \Rightarrow v = 1, 3;\ 5$

(B) $u = 2 \Rightarrow v = 1;\ 3$

(C) $u = 1 \Rightarrow v = 1$.

Now observe how the case $u = 4,\ v = 5$ has to be discarded since we have $4m^2(u^2 + v^2) = 656 = 2^4 \cdot 41,\ 9 \leq \delta_1 \leq 25$, the only factor in that range is 16, and it must be thrown out because $v = 5$ does not divide $\delta_1 + 2mu = 32$. The working factorizations are shown in the table below.

u	v	type of triangle	δ_1 range	$4m^2(u^2+v^2)$	$\delta_1 \cdot \delta_2$	(a,b,c)
4	1	obtuse	$\delta_1 \leq 16$	272	$1 \cdot 272$	$(18, 289, 305)$
					$2 \cdot 136$	$(19, 153, 170)$
					$4 \cdot 68$	$(21, 85, 104)$
					$8 \cdot 34$	$(25, 51, 74)$
					$16 \cdot 17$	$(33, 34, 65)$
4	3	obtuse	$\delta_1 \leq 20$	400	$2 \cdot 200$	$(9, 75, 78)$
					$5 \cdot 80$	$(10, 35, 39)$
					$8 \cdot 50$	$(11, 25, 30)$
					$20 \cdot 20$	$(15, 15, 24)$
2	1	obtuse	$\delta_1 \leq 8$	80	$1 \cdot 80$	$(11, 90, 97)$
					$2 \cdot 40$	$(12, 50, 58)$
					$4 \cdot 20$	$(14, 30, 40)$
					$5 \cdot 16$	$(15, 26, 37)$
					$8 \cdot 10$	$(18, 20, 34)$
2	3	acute	$10 \leq \delta_1 \leq 14$	208	$13 \cdot 16$	$(13, 14, 15)$
1	1	right	$\delta_1 \leq 5$	32	$1 \cdot 32$	$(9, 40, 41)$
					$2 \cdot 16$	$(10, 24, 26)$
					$4 \cdot 8$	$(12, 16, 20)$

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Lubomir Markov: Department of Mathematics and Computer Science, Barry University, 11300 NE Second Avenue, Miami Shores, FL 33161

E-mail address: `lmarkov@mail.barry.edu`