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Coincidence of Centers for Scalene Triangles

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Abstract. A *center function* is a function $\mathcal{Z}$ that assigns to every triangle T in a Euclidean plane $\mathbf{E}$ a point $\mathcal{Z}(T)$ in $\mathbf{E}$ in a manner that is symmetric and that respects isometries and dilations. A family $\mathbf{F}$ of center functions is said to be *complete* if for every scalene triangle ABC and every point P in its plane, there is $\mathcal{Z} \in \mathbf{F}$ such that $\mathcal{Z}(ABC) = P$. It is said to be *separating* if no two center functions in $\mathbf{F}$ coincide for any scalene triangle. In this note, we give simple examples of complete separating families of continuous triangle center functions. Regarding the impression that no two different center functions can coincide on a scalene triangle, we show that for every center function $\mathcal{Z}$ and every scalene triangle T, there is another center function $\mathcal{Z}'$, of a simple type, such that $\mathcal{Z}(T) = \mathcal{Z}'(T)$.

1. Introduction

Exercise 1 of [33, p. 37] states that if any two of the four classical centers coincide for a triangle, then it is equilateral. This can be seen by proving each of the 6 substatements involved, as is done for example in [26, pp. 78–79], and it also follows from more interesting considerations as described in Remark 5 below. The statement is still true if one adds the Gergonne, the Nagel, and the Fermat-Torricelli centers to the list. Here again, one proves each of the relevant 21 substatements; see [15], where variants of these 21 substatements are proved. If one wishes to extend the above statement to include the hundreds of centers catalogued in Kimberling's encyclopaedic work [25], then one must be prepared to test the tens of thousands of relevant substatements. This raises the question whether it is possible to design a definition of the term *triangle center* that encompasses the well-known centers and that allows one to prove in one stroke that no two centers coincide for a scalene triangle. We do not attempt to answer this expectedly very difficult question. Instead, we adhere to the standard definition of what a center is, and we look at maximal families of centers within which no two centers coincide for a scalene triangle.

In Section 2, we review the standard definition of triangle centers and introduce the necessary terminology pertaining to them. Sections 3 and 4 are independent.

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In Section 3, we examine the family of polynomial centers of degree 1. Noting the similarity between the line that these centers form and the Euler line, we digress to discuss issues related to these two lines. In Section 4, we exhibit maximal families of continuous, in fact polynomial, centers within which no two centers coincide for a scalene triangle. We also show that for every scalene triangle T and for every center function $\mathcal{Z}$, there is another center function of a fairly simple type that coincides with $\mathcal{Z}$ on T.

2. Terminology

By a *non-degenerate* triangle ABC, we mean an ordered triple (A, B, C) of non-collinear points in a fixed Euclidean plane $\mathbf{E}$. Non-degenerate triangles form a subset of $\mathbf{E}^3$ that we denote by $\mathbf{T}$. For a subset $\mathbf{U}$ of $\mathbf{T}$, the set of triples $(a, b, c) \in \mathbf{R}^3$ that occur as the side-lengths of a triangle in $\mathbf{U}$ is denoted by $\mathbf{U}_0$. Thus

$$\begin{aligned} \mathbf{U}_0 &= \{(a,b,c) \in \mathbf{R}^3 : a,b,c \text{ are the side-lengths of some triangle } ABC \text{ in } \mathbf{U}\}, \\ \mathbf{T}_0 &= \{(a,b,c) \in \mathbf{R}^3 : 0 < a < b+c,\ 0 < b < c+a,\ 0 < c < a+b\}. \end{aligned}$$

In the spirit of [23] – [25], a *symmetric triangle center function* (or simply, a *center function*, or a *center*) is defined as a function that assigns to every triangle in $\mathbf{T}$ (or more generally in some subset $\mathbf{U}$ of $\mathbf{T}$) a point in its plane in a manner that is symmetric and that respects isometries and dilations. Writing $\mathcal{Z}(A, B, C)$ as a barycentric combination of the position vectors A, B, and C, and letting a, b, and c denote the side-lengths of ABC in the standard order, we see that a center function $\mathcal{Z}$ on $\mathbf{U}$ is of the form

$$\mathcal{Z}(A,B,C) = f(a,b,c)A + f(c,a,b)B + f(b,c,a)C, \tag{1}$$

where f is a real-valued function on $\mathbf{U}_0$ having the following properties:

$$f(a,b,c) = f(a,c,b), \tag{2}$$

$$f(a,b,c) + f(b,c,a) + f(c,a,b) = 1, \tag{3}$$

$$f(\lambda a, \lambda b, \lambda c) = f(a,b,c)\ \forall\ \lambda > 0. \tag{4}$$

Here, we have treated the points in our plane $\mathbf{E}$ as position vectors relative to a fixed but arbitrary origin. We will refer to the center $\mathcal{Z}$ defined by (1) as *the center function defined by* f without referring explicitly to (1). The function f may be an explicit function of other elements of the triangle (such as its angles) that are themselves functions of a, b and c.

Also, we will always assume that the domain $\mathbf{U}$ of $\mathcal{Z}$ is closed under permutations, isometries and dilations, and has non-empty interior. In other words, we assume that $\mathbf{U}_0$ is closed under permutations and multiplication by a positive number, and that it has a non-empty interior.

According to this definition of a center $\mathcal{Z}$, one need only define $\mathcal{Z}$ on the similarity classes of triangles. On the other hand, the values that $\mathcal{Z}$ assigns to two triangles in different similarity classes are completely independent of each other. To reflect more faithfully our intuitive picture of centers, one must impose the condition that a center function be continuous. Thus a center function $\mathcal{Z}$ on $\mathbf{U}$ is called *continuous* if it is defined by a function f that is continuous on $\mathbf{U}_0$. If f can be chosen to be a rational function, then $\mathcal{Z}$ is called a *polynomial center function*. Since two rational functions cannot coincide on a non-empty open set, it follows that the rational function that defines a polynomial center function is unique. Also, a rational function $f(x, y, z)$ that satisfies (4) is necessarily of the form $f = g/h$, where g and h are d-forms, i.e., homogeneous polynomials of the same degree d. If $d = 1$, f is called a *projective linear function*. *Projective quadratic functions* correspond to $d = 2$, and so on. Thus a polynomial center $\mathcal{Z}$ is a center defined by a projective function.

A family $\mathbf{F}$ of center functions on $\mathbf{U}$ is said to be *separating* if no two elements in $\mathbf{F}$ coincide on any scalene triangle. It is said to be *complete* if for every scalene triangle T in $\mathbf{U}$, $\{\mathcal{Z}(T) : \mathcal{Z} \in \mathbf{F}\}$ is all of $\mathbf{E}$. The assumption that T is scalene is necessary here. In fact, if a triangle $T = ABC$ is such that $AB = AC$, then $\{\mathcal{Z}(T) : \mathcal{Z} \in \mathbf{F}\}$ will be contained in the line that bisects angle A, being a line of symmetry of ABC, and thus cannot cover $\mathbf{E}$.

3. Polynomial centers of degree 1

We start by characterizing the simplest polynomial center functions, i.e., those defined by projective linear functions. We note the similarity between the line these centers form and the Euler line and we discuss issues related to these two lines.

Theorem 1. *A projective linear function $f(x, y, z)$ satisfies (2), (3), and (4) if and only if*

$$f(x, y, z) = \frac{(1 - 2t)x + t(y + z)}{x + y + z} \tag{5}$$

for some t. If $\mathcal{S}_t$ is the center function defined by (5) (and(1)), then $\mathcal{S}_0$, $\mathcal{S}_{1/3}$, $\mathcal{S}_{1/2}$, and $\mathcal{S}_1$ are the incenter, centroid, Spieker center, and Nagel center, respectively. Also, the centers $\{\mathcal{S}_t(ABC) : t \in \mathbf{R}\}$ of a non-equilateral triangle ABC in $\mathbf{T}$ form the straight line whose trilinear equation is

$$a(b - c)\alpha + b(c - a)\beta + c(a - b)\gamma = 0.$$

Furthermore, the distance $|\mathcal{S}_t\mathcal{S}_u|$ between $\mathcal{S}_t$ and $\mathcal{S}_u$ is given by

$$|\mathcal{S}_t\mathcal{S}_u| = \frac{|t - u|\sqrt{H}}{a + b + c}, \tag{6}$$

where

$$\begin{aligned} H &= (-a + b + c)(a - b + c)(a + b - c) + (a + b)(b + c)(c + a) - 9abc \\ &= -(a^3 + b^3 + c^3) + 2(a^2 b + b^2 c + c^2 a + ab^2 + bc^2 + ca^2) - 9abc. \end{aligned} \tag{7}$$

Proof. Let $f(x, y, z) = L_0/M_0$, where L_0 and M_0 are linear forms in x, y, and z, and suppose that f satisfies (2), (3), and (4). Let σ be the cycle $(x\ y\ z)$, and let $L_i = \sigma^i(L_0)$ and $M_i = \sigma^i(M_0)$. Since f satisfies (3), it follows that $L_0M_1M_2 + L_1M_0M_2 + L_2M_0M_1 - M_0M_1M_2$ vanishes on $\mathbf{U}_0$ and hence vanishes identically. Thus M_0 divides $L_0M_1M_2$. If M_0 divides L_0, then f is a constant, and hence of the desired form, with $t = 1/3$. If M_0 divides M_1, then it follows easily that M_1 is a constant multiple of M_0 and that M_0 is a constant multiple of $x + y + z$. The same holds if M_0 divides M_2. Finally, we use (3) and (4) to see that L_0 is of the desired form.

Let $\mathcal{S}_t$ be as given. The barycentric coordinates of $\mathcal{S}_t(ABC)$ are given by

$$f(a,b,c) : f(b,c,a) : f(c,a,b)$$

and therefore the trilinear coordinates $\alpha : \beta : \gamma$ of $\mathcal{S}_t(ABC)$ are given by

$$\begin{aligned} \alpha a : \beta b : \gamma c &= (1-2t)a + t(b+c) : (1-2t)b + t(c+a) : (1-2t)c + t(a+b) \\ &= a + t(b+c-2a) : b + t(c+a-2b) : c + t(a+b-2c) \end{aligned}$$

Therefore there exists non-zero λ such that

$$\lambda\alpha a - a = t(b+c-2a),\ \lambda\beta b - b = t(c+a-2b),\ \lambda\gamma c - c = t(a+b-2c).$$

It is clear that the value $t = 0$ corresponds to the incenter. Thus we assume $t \neq 0$. Eliminating t, we obtain

$$\begin{aligned} (\lambda\alpha - 1)a(c+a-2b) &= (\lambda\beta - 1)b(b+c-2a), \\ (\lambda\beta - 1)b(a+b-2c) &= (\lambda\gamma - 1)c(c+a-2b). \end{aligned}$$

Eliminating λ and simplifying, we obtain

$$(a-2b+c)\left[a(b-c)\alpha + b(c-a)\beta + c(a-b)\gamma\right] = 0.$$

Dividing by $a - 2b + c$, we get the desired equation.

Finally, the last statement follows after routine, though tedious, calculations. We simply note that the actual trilinear coordinates of $\mathcal{S}_t$ are given by

$$\frac{2K((1-2t)a + t(b+c))}{a(a+b+c)} : \frac{2K((1-2t)b + t(c+a))}{b(a+b+c)} : \frac{2K((1-2t)c + t(a+b))}{c(a+b+c)},$$

where K is the area of the triangle, and we use the fact that the distance $|PP'|$ between the points P and P' whose actual trilinear coordinates are $\alpha : \beta : \gamma$ and $\alpha' : \beta' : \gamma'$ is given by

$$|PP'| = \frac{1}{2K}\sqrt{-abc[a(\beta-\beta')(\gamma-\gamma') + b(\gamma-\gamma')(\alpha-\alpha') + c(\alpha-\alpha')(\beta-\beta')]};$$

see [25, Theorem 1B, p. 31]. □

4. The Euler-like line $L(\mathcal{I}, \mathcal{G})$

The straight line $\{\mathcal{S}_t : t \in \mathbf{R}\}$ in Theorem 1 is the first central line in the list of [25, p. 128], where it is denoted by $L(1, 2, 8, 10)$. The notation $L(1, 2, 8, 10)$ reflects the fact that it passes through the centers catalogued in [25] as X_1, X_2, X_8, and X_{10}. These are the incenter, centroid, Nagel center, and Spieker center, and they correspond in $\{\mathcal{S}_t : t \in \mathbf{R}\}$ to the values $t = 0, 1/3, 1$, and $1/2$, respectively. We shall denote this line by $L(\mathcal{I}, \mathcal{G})$ to indicate that it is the line joining the incenter $\mathcal{I}$ and the centroid $\mathcal{G}$. Letting $\mathcal{O}$ be the circumcenter, the line $L(\mathcal{G}, \mathcal{O})$ is then nothing but the Euler line. In this section, we survey similarities between these lines. For the third line $L(\mathcal{O}, \mathcal{I})$ and a natural context in which it occurs, we refer the reader to [17].

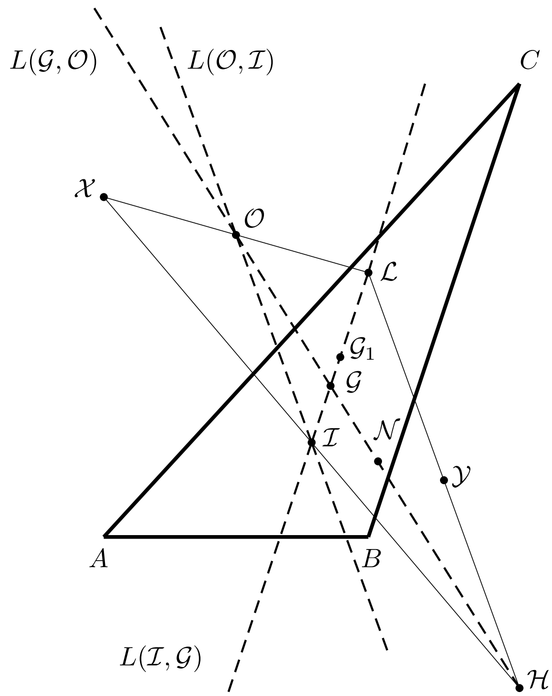

Figure 1

It follows from (6) that the Spieker center $\mathcal{G}_1 = \mathcal{S}_{1/2}$ and the centroid $\mathcal{G} = \mathcal{S}_{1/3}$ of a triangle are at distances in the ratio $3:2$ from the incenter $\mathcal{I} = \mathcal{S}_0$. This is shown in Figure 1 which is taken from [17]. The collinearity of $\mathcal{G}_1$, $\mathcal{I}$, and $\mathcal{G}$ and the ratio $3:2$ are highlighted in [6, pp. 137–138] and [27], and they also appear in [22, pp. 225–227] and the first row in [25, Table 5.5, p. 143]. In spite of this, we feel that these elegant facts and the striking similarity between the line $L(\mathcal{I},\mathcal{G})$ and the Euler line $L(\mathcal{O},\mathcal{G})$ deserve to be better known. Unaware of the aforementioned references, the authors of [3] rediscovered the collinearity of the incenter, the Spieker center, and the centroid and the ratio 3 : 2, and they proved, in Theorems 6 and 7, that the same thing holds for any polygon that admits an incircle, i.e., a circle that touches the sides of the polygon internally. Here, the centroid of a polygon is the center of mass of a lamina of uniform density that is laid on the polygon, the Spieker center is the centroid of wires of uniform density placed on the sides, and the incenter is the center of the incircle. Later, the same authors, again unaware of [8, p. 69], rediscovered (in [4]) similar properties of $L(\mathcal{I},\mathcal{G})$ in dimension 3 and made interesting generalizations to solids admitting inspheres. For a deeper explanation of the similarity between the Euler line and its rival $L(\mathcal{I},\mathcal{G})$ and for affine and other generalizations, see [29] and [28].

We should also mention that the special case of (6) pertaining to the distance between the incenter and the centroid appeared in [7]. Also, the fact that the Spieker center $\mathcal{G}_1$ is the midpoint of the segment joining the incenter $\mathcal{I}$ and the Nagel point $\mathcal{L}$ is the subject matter of [12], [30], and [31]. In each of these references, $\mathcal{L}$ (respectively, $\mathcal{G}_1$) is described as the point of intersection of the lines that bisect the perimeter and that pass through the vertices (respectively, the midpoints of the sides). It is not apparent that the authors of these references are aware that $\mathcal{L}$ and $\mathcal{G}_1$ are the Nagel and Spieker centers. For the interesting part that $\mathcal{G}_1$ is indeed the Spieker center, see [5] and [20, pp. 1–14]. One may also expect that the Euler line and the line $L(\mathcal{I},\mathcal{G})$ cannot coincide unless the triangle is isosceles. This is indeed so, as is proved in [21, Problem 4, Section 11, pp. 142–144]. It also follows from the fact that the area of the triangle $\mathcal{GOI}$ is given by the elegant formula

$$[\mathcal{GOI}] = \left|\frac{s(b-c)(c-a)(a-b)}{24K}\right|,$$

where s is the semiperimeter and K the area of ABC; see [34, Exercise 5.7].

We also note that the Euler line consists of the centers $\mathcal{T}_t$ defined by the function

$$g = \frac{(1-2t)\tan A + t(\tan B + \tan C)}{\tan A + \tan B + \tan C} \tag{8}$$

obtained from f of (5) by replacing a, b, and c by $\tan A$, $\tan B$, and $\tan C$, respectively. Then $\mathcal{T}_0$, $\mathcal{T}_{1/3}$, $\mathcal{T}_{1/2}$, and $\mathcal{T}_1$ are nothing but the circumcenter, centroid, the center of the nine-point circle, and the orthocenter, respectively. The distance $|\mathcal{T}_t\mathcal{T}_u|$ between $\mathcal{T}_t$ and $\mathcal{T}_u$ is given by

$$|\mathcal{T}_t\mathcal{T}_u| = \frac{|t-u|\sqrt{H^*}}{a+b+c},$$

where H^* is obtained from H in (7) by replacing a, b, and c with $\tan A$, $\tan B$, and $\tan C$, respectively. Letting K be the area of the triangle with side-lengths a, b, and c, and using the identity $\tan A = 4K/(b^2+c^2-a^2)$ and its iterates, H^* reduces to a rational function in a, b, and c. In view of the formula $144K^2r^2 = E$ given in [32], where

$$E = a^2b^2c^2 - (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2), \tag{9}$$

and where r is the distance between the circumcenter $\mathcal{T}_0$ and the centroid $\mathcal{T}_{1/3}$, H^* is expected to simplify into

$$H^* = \frac{(a+b+c)^2E}{16K^2},$$

where E is as given in (9), and where $16K^2$ is given by Heron's formula

$$16K^2 = 2(a^2b^2+b^2c^2+c^2a^2) - (a^4+b^4+c^4). \tag{10}$$

Referring to Figure 1, let $\mathcal{X}$ be the point where the lines $\mathcal{LO}$ and $\mathcal{HI}$ meet, and let $\mathcal{Y}$ be the midpoint of $\mathcal{HL}$. Then the Euler line and the line $L(\mathcal{I},\mathcal{G})$ are medians of both triangles $\mathcal{XHL}$ and $\mathcal{OIY}$. The points $\mathcal{X}$ and $\mathcal{Y}$ do not seem to be catalogued in [25]. Also, of the many lines that can be formed in Figure 1, the line $\mathcal{IN}$ is catalogued in [25] as the line joining $\mathcal{I}$, $\mathcal{N}$, and the Feuerbach point. As for distances between various points in Figure 1, formulas for the distances $\mathcal{IN}$, $\mathcal{IO}$, $\mathcal{IH}$, and $\mathcal{OH}$ can be found in [9, pp. 6–7]. The first two are quite well-known and they are associated with Euler, Steiner, Chapple and Feuerbach. Also, formulas for the distances $\mathcal{GI}$ and $\mathcal{GO}$ appeared in [7] and [32], as mentioned earlier. These formulas, as well as other formulas for distances between several other pairs of centers, had already been found by Euler [35, Section XIB, pp. 88–90].

5. Complete separating families of polynomial centers

In the next theorem, we exhibit a complete separating family of polynomial center functions that contains the functions used to define the line $L(\mathcal{I}, \mathcal{G})$ encountered in Theorem 1.

Theorem 2. *Let ABC be a scalene triangle and let V be any point in its plane. Then there exist unique real numbers t and v such that V is the center of ABC with respect to the center function $\mathcal{Q}_{t,v}$ defined by the projective quadratic function f given by*

$$f(x, y, z) \;=\; \frac{(1-2t)x^2 + t(y^2+z^2) + 2(1-v)yz + vx(y+z)}{(x+y+z)^2}. \qquad (11)$$

Consequently, the family $\mathbf{F} = \{\mathcal{Q}_{t,v} : t, v \in \mathbf{R}\}$ is a complete separating family. Also, $\mathbf{F}$ contains the line $L(\mathcal{I}, \mathcal{G})$ described in Theorem 1.

Proof. Clearly f satisfies the conditions (2), (3), and (4). Since V is in the plane of ABC, it follows that $V = \xi A + \eta B + \zeta C$ for some ξ, η, and ζ with $\xi + \eta + \zeta = 1$. Let a, b, and c be the side-lengths of ABC as usual. The system $f(a, b, c) = \xi$, $f(b, c, a) = \eta$, $f(c, a, b) = \zeta$ of equations is equivalent to the system

$$\begin{aligned}(b^2 + c^2 - 2a^2)t + (-2bc + ca + ab)v &= \xi(a+b+c)^2 - a^2 - 2bc,\\ (a^2 + b^2 - 2c^2)t + (-2ab + bc + ca)v &= \zeta(a+b+c)^2 - c^2 - 2ab.\end{aligned}$$

The existence of a (unique) solution (t, v) to this system now follows from the fact that its determinant $-3(a-b)(b-c)(c-a)(a+b+c)$ is not zero.

The last statement follows from the observation that if $v = 1 - t$, then the expression of $f(x, y, z)$ in (11) reduces to the projective linear function $f(x, y, z)$ given in (5). □

Remarks. (1) According to [25, p. 46], the Fermat-Torricelli point is not a polynomial center. Therefore it does not belong to the family $\mathbf{F}$ defined in Theorem 2. Also, the circumcenter, the orthocenter, and the Gergonne point do not belong to $\mathbf{F}$, although they are polynomial centers. In fact, these centers are defined by the functions f given by

$$\frac{x^2(y^2+z^2-x^2)}{16K^2},\ \frac{y^2+z^2-x^2}{x^2+y^2+z^2},\ \frac{(x-y+z)(x+y-z)}{2(xy+yz+zx)-(x^2+y^2+z^2)},$$

respectively, where K is the area of the triangle whose side-lengths are x, y, and z, and is given by Heron's formula as in (10); see [24, pp. 172–173].

(2) One may replace the denominator of f in (11) by an arbitrary symmetric quadratic form that does not vanish on any point in $\mathbf{T}_0$, and obtain a different

separating complete family of center functions. Thus if we replace f by the similar function

$$g(x,y,z) \;=\; \frac{(-1-2t)x^2+t(y^2+z^2)+2vyz+(1-v)x(y+z)}{2(xy+yz+zx)-(x^2+y^2+z^2)},$$

then we would obtain a complete separating family **G** of center functions that contains the centroid, the Gergonne center and the Mittenpunkt, but not any of the other well known traditional centers. Here the Mittenpunkt is the center defined by the function

$$g(x,y,z)=\frac{xy+xz-x^2}{2(xy+yz+zx)-(x^2+y^2+z^2)}.$$

(3) It is clear that complete families are maximal separating families. However, it is not clear whether the converse is true. It also follows from Zorn's Lemma that every separating family of center functions can be imbedded in a maximal separating family. Thus the seven centers mentioned at the beginning of this note belong to some maximal separating family of centers. The question is whether such a family can be defined in a natural way.

The next theorem shows that pairs of center functions that coincide on scalene triangles exist in abundance. However, it does not answer the question whether such a pair can be chosen from the hundreds of centers that are catalogued in [25]. In case this is not possible, the question arises whether this is due to certain intrinsic properties of the centers in [25].

Theorem 3. *Let $\mathcal{Z}$ be a center function, and let ABC be any scalene triangle in the domain of $\mathcal{Z}$. Then there exists another center function $\mathcal{Z}'$ defined by a projective function f such that $\mathcal{Z}(A,B,C)=\mathcal{Z}'(A,B,C)$.*

Moreover, if $\mathcal{Z}$ is not the centroid, then f can be chosen to be quadratic. If $\mathcal{Z}$ is the centroid, then f can be chosen to be quartic.

Proof. Let **F** and **G** be the families of centers defined in Theorem 2 and in Remark 4. Clearly, the centroid is the only center function that these two families have in common.

If $\mathcal{Z}\notin\mathbf{F}$, then we use Theorem 2 to produce the center $\mathcal{Z}'=\mathcal{Z}_{t,v}$ for which $\mathcal{Z}_{t,v}(A,B,C)=\mathcal{Z}(A,B,C)$, and we take $\mathcal{Z}'=\mathcal{Z}_{t,v}$. If $\mathcal{Z}\notin\mathbf{G}$, then we argue similarly as indicated in Remark 2 to produce the desired center function.

It remains to deal with the case when $\mathcal{Z}$ is the centroid. In this case, we let $f(x,y,z)=g(x,y,z)/h(x,y,z)$, where

$$\begin{aligned}
h(x,y,z) &= (x^4+y^4+z^4)+(x^3y+y^3z+z^3x+x^3z+y^3x+z^3y)\\
&\quad +(x^2y^2+y^2z^2+z^2x^2)\\
g(x,y,z) &= (1-2t)x^4+t(y^4+z^4)+vx^3(y+z)+wx(y^3+z^3)\\
&\quad +(1-v-w)x(y^3+z^3)+sx^2(y^2+z^2)+(1-2s)y^2z^2,
\end{aligned}$$

and we consider the equations

$$f(a,b,c) = f(b,c,a) = f(c,a,b) = \frac{1}{3}.$$

These are linear equations in the variables t, v, w, and s that have an obvious solution $(t,v,w,s) = (1/3, 1/3, 1/3, 1/3)$. Hence they have infinitely many other solutions. Choose any of these solutions and let $\mathscr{Z}$ be the center defined by the function f that corresponds to that choice. Then for the given triangle ABC, $\mathscr{Z}$ is the centroid, as desired. □

Remarks. (4) The question that underlies this paper is whether two centers can coincide for a scalene triangle. The analogous question, for higher dimensional simplices, of how much regularity is implied by the coincidence of two or more centers has led to various interesting results in [18], [19], [10], [11], and [16].

(5, due to the referee) Let $\mathcal{O}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{I}$ be the circumcenter, centroid, orthocenter, and incenter of a non-equilateral triangle. Euler's theorem states that $\mathcal{O}$, $\mathcal{G}$, and $\mathcal{H}$ are collinear with $\mathcal{OG} : \mathcal{GH} = 1 : 2$. A theorem of Guinand in [13] shows that $\mathcal{I}$ ranges freely over the interior of the centroidal disk (with diameter $\mathcal{GH}$) punctured at the nine-point center $\mathcal{N}$. It follows that no two of the centers $\mathcal{O}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{I}$ coincide for a non-equilateral triangle, thus providing a proof, other than case by case chasing, of the very first statement made in the introduction.

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