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Some Triangle Centers Associated with the Excircles

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Abstract. We construct a few new triangle centers associated with the excircles of a triangle.

1. Introduction

Consider a triangle ABC with its excircles. We study a triad of extouch triangles and construct some new triangle centers associated with them. By the A-extouch triangle, we mean the triangle with vertices the points of tangency of the A-excircle with the sidelines of ABC. This is triangle $A_aB_aC_a$ in Figure 1. Similarly, the B- and C-extouch triangles are respectively $A_bB_bC_b$, and $A_cB_cC_c$. Consider also the incircles of these extouch triangles, with centers I_1, I_2, I_3 respectively, and points of tangency X of (I_1) with B_aC_a, Y of (I_2) with C_bA_b, and Z of (I_3) with A_cB_c.

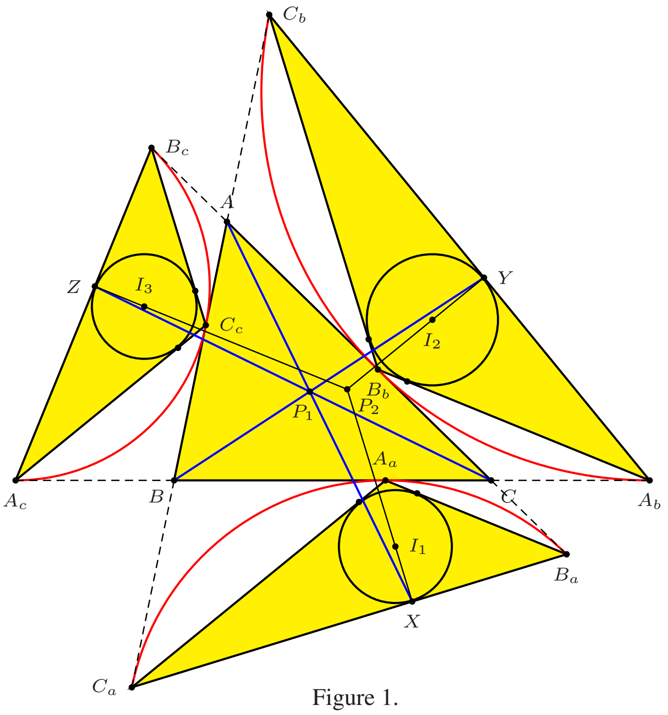

Figure 1.

In this paper, we adopt the usual notations of triangle geometry as in [3] and work with homogeneous barycentric coordinates with reference to triangle ABC.

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Theorem 1. (1) *The lines AX, BY, CZ are concurrent at*

$$P_1 = \left(\cos\frac{A}{2}\cos^2\frac{A}{4} : \cos\frac{B}{2}\cos^2\frac{B}{4} : \cos\frac{C}{2}\cos^2\frac{C}{4}\right).$$

(2) *The lines I_1X, I_2Y, I_3Z are concurrent at*

$$\begin{aligned} P_2 = \Bigg(a\left(1-\cos\frac{B}{2}-\cos\frac{C}{2}\right)+(b+c)\cos\frac{A}{2} \\ : b\left(1-\cos\frac{C}{2}-\cos\frac{A}{2}\right)+(c+a)\cos\frac{B}{2} \\ : c\left(1-\cos\frac{A}{2}-\cos\frac{B}{2}\right)+(a+b)\cos\frac{C}{2}\Bigg). \end{aligned}$$

2. Some preliminary results

Let s and R be the semiperimeter and circumradius respectively of triangle ABC. The following homogeneous barycentric coordinates are well known.

$$\begin{array}{lll} A_a = (0 : s-b : s-c), & B_a = (-(s-b) : 0 : s), & C_a = (-(s-c) : s : 0); \\ A_b = (0 : -(s-a) : s), & B_b = (s-a : 0 : s-c), & C_b = (s : -(s-c) : 0); \\ A_c = (0 : s : -(s-c)), & B_c = (s : 0 : -(s-a)), & C_c = (s-a : s-b : 0). \end{array}$$

The lengths of the sides of the A-extouch triangle are as follows:

$$B_aC_a = 2s\cdot\sin\frac{A}{2}, \quad C_aA_a = 2(s-c)\cos\frac{B}{2}, \quad A_aB_a = 2(s-b)\cos\frac{C}{2}. \tag{1}$$

Lemma 2.

$$\begin{aligned} s &= 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}, \\ s-a &= 4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, \\ s-b &= 4R\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}, \\ s-c &= 4R\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}. \end{aligned}$$

We omit the proof of this lemma. It follows easily from, for example, [1, §293].

3. Proof of Theorem 1

$$
\begin{aligned}
B_aX =&\frac{1}{2}(B_aC_a - A_aC_a + A_aB_a)\\
=&s\cdot\sin\frac{A}{2} - (s-c)\cos\frac{B}{2} + (s-b)\cos\frac{C}{2}\\
=&4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left(\cos\frac{A}{2} - \sin\frac{B}{2} + \sin\frac{C}{2}\right)\\
=&4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left(\sin\frac{B+C}{2} - \sin\frac{B}{2} + \sin\frac{C}{2}\right)\\
=&4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left(2\sin\frac{B+C}{4}\cos\frac{B+C}{4} - 2\sin\frac{B-C}{4}\cos\frac{B+C}{4}\right)\\
=&16R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{B+C}{4}\cdot\cos\frac{B}{4}\sin\frac{C}{4}.
\end{aligned}
$$

Similarly, $XC_a = 16R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{B+C}{4}\cdot\sin\frac{B}{4}\cos\frac{C}{4}$. The point X therefore divides B_aC_a in the ratio

$$B_aX : XC_a = \cos\frac{B}{4}\sin\frac{C}{4} : \sin\frac{B}{4}\cos\frac{C}{4}.$$

This allows us to compute its absolute barycentric coordinate in terms of B_a and C_a. Note that

$$B_a = \frac{\left(-\sin\frac{A}{2}\sin\frac{C}{2},\ 0,\ \cos\frac{A}{2}\cos\frac{C}{2}\right)}{\sin\frac{B}{2}}, \qquad C_a = \frac{\left(-\sin\frac{A}{2}\sin\frac{B}{2},\ \cos\frac{A}{2}\cos\frac{B}{2},\ 0\right)}{\sin\frac{C}{2}}.$$

From these we have

$$
\begin{aligned}
X =&\frac{\sin\frac{B}{4}\cos\frac{C}{4}\cdot B_a + \cos\frac{B}{4}\sin\frac{C}{4}\cdot C_a}{\sin\frac{B+C}{4}}\\
=&\frac{\sin\frac{B}{4}\cos\frac{C}{4}\cdot\frac{\left(-\sin\frac{A}{2}\sin\frac{C}{2},0,\cos\frac{A}{2}\cos\frac{C}{2}\right)}{\sin\frac{B}{2}} + \cos\frac{B}{4}\sin\frac{C}{4}\cdot\frac{\left(-\sin\frac{A}{2}\sin\frac{B}{2},\cos\frac{A}{2}\cos\frac{B}{2},0\right)}{\sin\frac{C}{2}}}{\sin\frac{B+C}{4}}\\
=&\frac{\cos\frac{C}{4}\cdot\frac{\left(-\sin\frac{A}{2}\sin\frac{C}{2},0,\cos\frac{A}{2}\cos\frac{C}{2}\right)}{2\cos\frac{B}{4}} + \cos\frac{B}{4}\cdot\frac{\left(-\sin\frac{A}{2}\sin\frac{B}{2},\cos\frac{A}{2}\cos\frac{B}{2},0\right)}{2\cos\frac{C}{4}}}{\sin\frac{B+C}{4}}\\
=&\frac{\cos^2\frac{C}{4}\left(-\sin\frac{A}{2}\sin\frac{C}{2},0,\cos\frac{A}{2}\cos\frac{C}{2}\right) + \cos^2\frac{B}{4}\left(-\sin\frac{A}{2}\sin\frac{B}{2},\cos\frac{A}{2}\cos\frac{B}{2},\ 0\right)}{2\cos\frac{B}{4}\cos\frac{C}{4}\sin\frac{B+C}{4}}\\
=&\frac{\left(-\sin\frac{A}{2}\left(\sin\frac{B}{2}\cos^2\frac{B}{4} + \sin\frac{C}{2}\cos^2\frac{C}{4}\right),\cos\frac{A}{2}\cos\frac{B}{2}\cos^2\frac{B}{4},\ \cos\frac{A}{2}\cos\frac{C}{2}\cos^2\frac{C}{4}\right)}{2\cos\frac{B}{4}\cos\frac{C}{4}\sin\frac{B+C}{4}}.
\end{aligned}
$$

From this we obtain the homogeneous barycentric coordinates of X, and those of Y and Z by cyclic permutations of A, B, C:

$$
\begin{aligned}
X &= \left(-\sin\frac{A}{2}\left(\sin\frac{B}{2}\cos^2\frac{B}{4}+\sin\frac{C}{2}\cos^2\frac{C}{4}\right)\;:\cos\frac{A}{2}\cos\frac{B}{2}\cos^2\frac{B}{4}\;:\cos\frac{A}{2}\cos\frac{C}{2}\cos^2\frac{C}{4}\right),\\
Y &= \left(\cos\frac{B}{2}\cos\frac{A}{2}\cos^2\frac{A}{4}\;:-\sin\frac{B}{2}\left(\sin\frac{C}{2}\cos^2\frac{C}{4}+\sin\frac{A}{2}\cos^2\frac{A}{4}\right)\;:\cos\frac{B}{2}\cos\frac{C}{2}\cos^2\frac{C}{4}\right),\\
Z &= \left(\cos\frac{C}{2}\cos\frac{A}{2}\cos^2\frac{A}{4}\;:\cos\frac{C}{2}\cos\frac{B}{2}\cos^2\frac{B}{4}:-\sin\frac{C}{2}\left(\sin\frac{A}{2}\cos^2\frac{A}{4}+\sin\frac{B}{2}\cos^2\frac{B}{4}\right)\right).
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
X &= \left(-\tan\frac{A}{2}\left(\sin\frac{B}{2}\cos^2\frac{B}{4}+\sin\frac{C}{2}\cos^2\frac{C}{4}\right)\;:\cos\frac{B}{2}\cos^2\frac{B}{4}\;:\cos\frac{C}{2}\cos^2\frac{C}{4}\right),\\
Y &= \left(\cos\frac{A}{2}\cos^2\frac{A}{4}\;:-\tan\frac{B}{2}\left(\sin\frac{C}{2}\cos^2\frac{C}{4}+\sin\frac{A}{2}\cos^2\frac{A}{4}\right)\;:\cos\frac{C}{2}\cos^2\frac{C}{4}\right),\\
Z &= \left(\cos\frac{A}{2}\cos^2\frac{A}{4}\;:\cos\frac{B}{2}\cos^2\frac{B}{4}:-\tan\frac{C}{2}\left(\sin\frac{A}{2}\cos^2\frac{A}{4}+\sin\frac{B}{2}\cos^2\frac{B}{4}\right)\right).
\end{aligned}
$$

It is clear that the lines AX, BY, CZ intersect at a point P_1 with coordinates

$$\left(\cos\frac{A}{2}\cos^2\frac{A}{4}\;:\cos\frac{B}{2}\cos^2\frac{B}{4}:\cos\frac{C}{2}\cos^2\frac{C}{4}\right).$$

This completes the proof of Theorem 1(1).

For (2), note that the line I_1X is parallel to the bisector of angle A. Its has barycentric equation

$$\begin{vmatrix} -\sin\frac{A}{2}\left(\sin\frac{B}{2}\cos^2\frac{B}{4}+\sin\frac{C}{2}\cos^2\frac{C}{4}\right) & \cos\frac{A}{2}\cos\frac{B}{2}\cos^2\frac{B}{4} & \cos\frac{A}{2}\cos\frac{C}{2}\cos^2\frac{C}{4} \\ -(b+c) & b & c \\ x & y & z \end{vmatrix} = 0.$$

A routine calculation, making use of the fact that the sum of the entries in the first row is $\sin\frac{C}{2}\cos^2\frac{B}{4}+\sin\frac{B}{2}\cos^2\frac{C}{4}$, gives

$$-(x+y+z)\left(b\cos\frac{C}{2}-c\cos\frac{B}{2}\right)+bz-cy=0.$$

Similarly, the lines I_2Y and I_3Z have equations

$$
\begin{aligned}
-(x+y+z)\left(c\cos\frac{A}{2}-a\cos\frac{C}{2}\right)+cx-az &=0,\\
-(x+y+z)\left(a\cos\frac{B}{2}-b\cos\frac{A}{2}\right)+ay-bx &=0.
\end{aligned}
$$

These three lines intersect at

$$P_2 = \left(a\left(1-\cos\frac{B}{2}-\cos\frac{C}{2}\right)+(b+c)\cos\frac{A}{2}\right.$$
$$: b\left(1-\cos\frac{C}{2}-\cos\frac{A}{2}\right)+(c+a)\cos\frac{B}{2}$$
$$\left.: c\left(1-\cos\frac{A}{2}-\cos\frac{B}{2}\right)+(a+b)\cos\frac{C}{2}\right).$$

This completes the proof of Theorem 1(2).

Remark. The barycentric coordinates of the incenter I_1 of the A-extouch triangle are

$$\left(-\sin\frac{A}{2}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right):\cos\frac{B}{2}\left(\sin\frac{C}{2}+\cos\frac{A}{2}\right):\cos\frac{C}{2}\left(\cos\frac{A}{2}+\sin\frac{B}{2}\right)\right).$$

4. Some collinearities

The homogeneous barycentric coordinates of P_1 can be rewritten as

$$\left(\cos^2\frac{A}{2}+\cos\frac{A}{2}\ :\cos^2\frac{B}{2}+\cos\frac{B}{2}\ :\cos^2\frac{C}{2}+\cos\frac{C}{2}\right).$$

From this it is clear that the point P_1 lies on the line joining the two points with coordinates $\left(\cos^2\frac{A}{2}:\cos^2\frac{B}{2}:\cos^2\frac{C}{2}\right)$ and $\left(\cos\frac{A}{2}:\cos\frac{B}{2}:\cos\frac{C}{2}\right)$. We briefly recall their definitions.

(i) The point $M=\left(\cos^2\frac{A}{2}:\cos^2\frac{B}{2}:\cos^2\frac{C}{2}\right)=(a(s-a):b(s-b):c(s-c))$ is the Mittenpunkt. It is the perspector of the excentral triangle and the medial triangle. It is the triangle center X_9 of [2].

(ii) The point $Q=\left(\cos\frac{A}{2}\ :\cos\frac{B}{2}\ :\cos\frac{C}{2}\right)$ appears as X_{188} in [2], and is named the second mid-arc point. Here is an explicit description. Consider the anticomplementary triangle $A'B'C'$ of ABC, with its incircle (I'). If the segments $I'A'$, $I'B'$, $I'C'$ intersect the incircle (I') at A'', B'', C'', then the lines AA'', BB'', CC'' are concurrent at Q. See Figure 2.

Proposition 3. (1) *The point P_1 lies on the line MQ.*
(2) *The point P_2 lies on the line joining the incenter to Q.*

Proof. We need only prove (2). This is clear from

$$P_2=\left(1-\cos\frac{A}{2}-\cos\frac{B}{2}-\cos\frac{C}{2}\right)I+\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)Q.$$

In fact,

$$P_2=I+\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)\overrightarrow{IQ}. \tag{2}$$

□

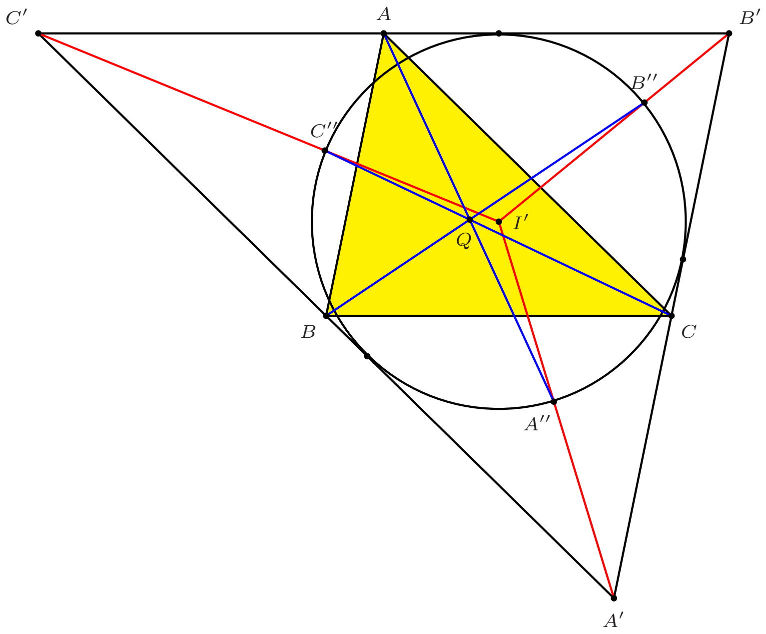

Figure 2.

5. The excircles of the extouch triangles

Consider the excircle of triangle $A_aB_aC_a$ tangent to the side B_aC_a at X'. It is clear that X' and X are symmetric with respect to the midpoint of B_aC_a. Since triangle AB_aC_a is isosceles, the lines AX' and AX are isogonal with respect to AB_a and AC_a. As such, they are isogonal with respect to AB and AC. Likewise, if we consider the excircle of $A_bB_bC_b$ tangent to C_bA_b at Y', and that of $A_cB_cC_c$ tangent to A_cB_c at Z', then the lines AX', BY', CZ', being respectively isogonal to AX, BY, CZ, intersect at the isogonal conjugate of P_1.

Proposition 4. *The barycentric coordinates of P_1^* are*

$$\left(\cos\frac{A}{2}\sin^2\frac{A}{4} : \cos\frac{B}{2}\sin^2\frac{B}{4} : \cos\frac{C}{2}\sin^2\frac{C}{4}\right).$$

Proof. This follows from

$$\begin{aligned}
P_1^* &= \left(\frac{\sin^2 A}{\cos\frac{A}{2}\cos^2\frac{A}{4}} : \frac{\sin^2 B}{\cos\frac{B}{2}\cos^2\frac{B}{4}} : \frac{\sin^2 C}{\cos\frac{C}{2}\cos^2\frac{C}{4}}\right)\\
&= \left(\frac{\sin^2\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{4}} : \frac{\sin^2\frac{B}{2}\cos\frac{B}{2}}{\cos^2\frac{B}{4}} : \frac{\sin^2\frac{C}{2}\cos\frac{C}{2}}{\cos^2\frac{C}{4}}\right)\\
&= \left(\cos\frac{A}{2}\sin^2\frac{A}{4} : \cos\frac{B}{2}\sin^2\frac{B}{4} : \cos\frac{C}{2}\sin^2\frac{C}{4}\right). :
\end{aligned}$$

□

Corollary 5. *The points P_1, P_1^* and Q are collinear.*

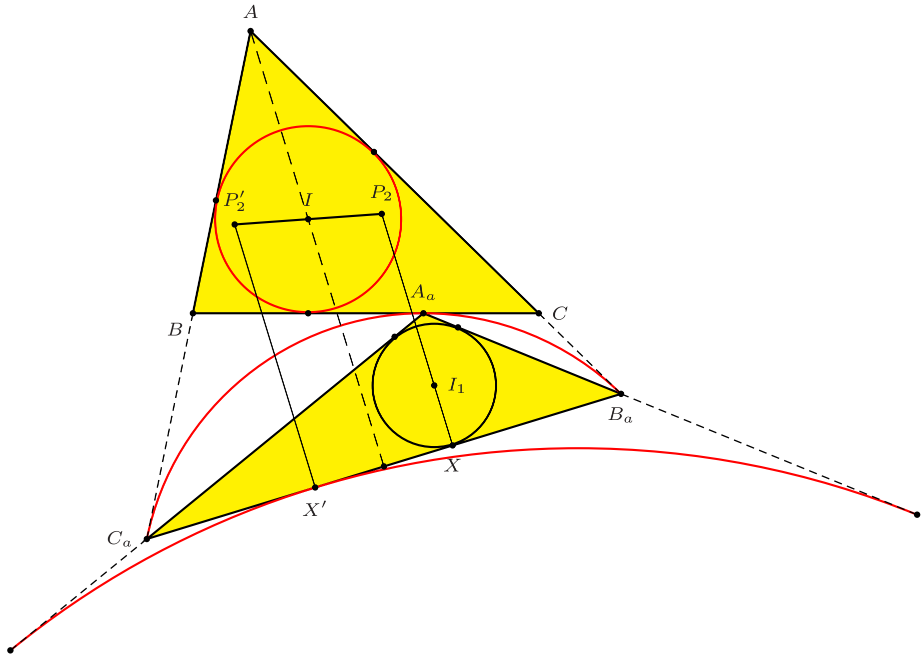

Figure 3.

Proposition 6. *The perpendiculars to B_aC_a at X', to C_bA_b at Y', and to A_cB_c at Z' are concurrent at the reflection of P_2 in I, which is the point*

$$\begin{aligned} P_2' =& a\left(1+\cos\frac{B}{2}+\cos\frac{C}{2}\right)-(b+c)\cos\frac{A}{2}\\ &:b\left(1+\cos\frac{C}{2}+\cos\frac{A}{2}\right)-(c+a)\cos\frac{B}{2}\\ &:c\left(1+\cos\frac{A}{2}+\cos\frac{B}{2}\right)-(a+b)\cos\frac{C}{2}. \end{aligned}$$

Proof. Let P_2' be the reflection of P_2 in I. Since X and X' are symmetric in the midpoint of B_aC_a, and P_2X is perpendicular to B_aC_a, it follows that $P_2'X'$ is also perpendicular to B_aC_a. The same reasoning shows that $P_2'Y'$ and $P_2'Z'$ are perpendicular to C_bA_b and A_cB_c respectively. It follows from (2) that

$$P_2' = I - \left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)\overrightarrow{IQ}.$$

From this, we easily obtain the homogeneous barycentric coordinates as given above. □

We conclude this paper with the construction of another triangle center. It is known that the perpendiculars from A_a to B_aC_a, B_b to C_bA_b, and C_c to A_cB_c

intersect at

$$P_3 = ((b+c)\cos A\ : (c+a)\cos B\ : (a+b)\cos C). \tag{3}$$

This is the triangle center X_{72} in [2].

If we let X_0, Y_0, Z_0 be these pedals, then it is also known that AX_0, BY_0, CZ_0 intersect at the Mittenpunkt X_9. Now, let X_1, Y_1, Z_1 be the reflections of X_0, Y_0, Z_0 in the midpoints of B_aC_a, C_bA_b, A_cB_c respectively. The lines AX_1, BY_1, CZ_1 clearly intersect at the reflection of X_{72} in I. This is the point

$$P_3' = ((b+c)\cos A - 2a\ : (c+a)\cos B - 2b\ : (a+b)\cos C - 2c)\,.$$

These coordinates are particularly simple since the sum of the coordinates of P_3 given in (3) is $a+b+c$.

The triangle centers P_1, P_1^*, P_2, P_2' and P_3' do not appear in [2].

References

[1] R. A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, 1960.
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