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Fixed Points and Fixed Lines of Ceva Collineations

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Abstract. In the plane of a triangle ABC, the U-Ceva collineation maps points to points and lines to lines. If U is a triangle center other than the incenter, then the U-Ceva collineation has three distinct fixed points F_1, F_2, F_3 and three distinct fixed lines F_2F_3, F_3F_1, F_1F_2, these being the trilinear polars of F_1, F_2, F_3. When U is the circumcenter, the fixed points are the symmedian point and the isogonal conjugates of the points in which the Euler line intersects the circumcircle.

1. Introduction

This note is a sequel to [3], in which the notion of a U-Ceva collineation is introduced. In this introduction, we briefly summarize the main results of [3].

We use homogeneous trilinear coordinates and denote the isogonal conjugate of a point X by X^{-1}. The X-Ceva conjugate of $U = u : v : w$ and $X = x : y : z$ is given by

$$X©U = u(-uyz + vzx + wxy) : v(uyz - vzx + wxy) : w(uyz + vzx - wxy),$$

and if $P = p : q : r$ is a point, then the equation $P = X©U$ is equivalent to

$$\begin{aligned} X &= (ru + pw)(pv + qu) : (pv + qu)(qw + rv) : (qw + rv)(ru + pw) \qquad (1) \\ &= \text{cevapoint}(P, U). \end{aligned}$$

If $\mathcal{L}_1$ is a line $l_1\alpha + m_1\beta + n_1\gamma = 0$ and $\mathcal{L}_2$ is a line $l_2\alpha + m_2\beta + n_2\gamma = 0$, then there exists a unique point U such that if $X \in \mathcal{L}_1$, then $X^{-1}©U \in \mathcal{L}_2$, and the mapping $X \to X^{-1}©U$ is surjective. This mapping is written as $\mathcal{C}_U(X) = X^{-1}©U$, and $\mathcal{C}_U$ is called the U-Ceva collineation. Explicitly,

$$\mathcal{C}_U(X) = u(-ux + vy + wz) : v(ux - vy + wz) : w(ux + vy - wz).$$

The inverse mapping is given by

$$\begin{aligned} \mathcal{C}_U^{-1}(X) &= wy + vz : uz + wx : vx + uy \\ &= (\text{cevapoint}(X, U))^{-1}. \end{aligned}$$

The collineation $\mathcal{C}_U$ maps the vertices A, B, C to the vertices of the anticevian triangle of U and maps U^{-1} to U. The collineation $\mathcal{C}_U^{-1}$ maps A, B, C to the vertices of the cevian triangle of U^{-1} and maps U to U^{-1}.

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2. Fixed points

The fixed points of the $\mathcal{C}_U$-collineation are also the fixed points of the inverse collineation, $\mathcal{C}_U^{-1}$. In this section, we seek all points X satisfying $\mathcal{C}_U^{-1}(X) = X$; *i.e.*, we wish to solve the equation

$$\mathcal{C}_U^{-1}(X) = \begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = MX$$

for the vector X. Writing $(M - tI)X = 0$, where I denotes the 3×3 identity matrix, we have the characteristic equation $\det(M - tI) = 0$ of M, which can be written

$$\begin{vmatrix} -t & w & v \\ w & -t & u \\ v & u & -t \end{vmatrix} = 0.$$

Expanding the determinant gives

$$t^3 - gt - h = 0, \tag{2}$$

where $g = u^2+v^2+w^2$ and $h = 2uvw$. Now suppose t is a root, *i.e.*, an eigenvalue of M. The equation $(M - tI)X = 0$ is equivalent to the system

$$\begin{aligned} -tx + wy + vz &= 0 \\ wx - ty + uz &= 0 \\ vx + uy - tz &= 0. \end{aligned}$$

For any z, the first two of the three equations can be written as

$$\begin{pmatrix} -t & w \\ w & -t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -vz \\ -uz \end{pmatrix},$$

and if $t^2 \neq w^2$, then

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -t & w \\ w & -t \end{pmatrix}^{-1} \begin{pmatrix} -vz \\ -uz \end{pmatrix} \\ &= \frac{1}{t^2 - w^2} \begin{pmatrix} tvz + uwz \\ tuz + vwz \end{pmatrix}, \end{aligned}$$

Thus, for each z,

$$x = \frac{1}{t^2 - w^2}(tvz + uwz) \quad \text{and} \quad y = \frac{1}{t^2 - w^2}(tuz + vwz),$$

so that

$$x : y = tv + uw : tu + vw \quad \text{and} \quad \frac{y}{z} = \frac{1}{t^2 - w^2}(tu + vw),$$

and $x : y : z$ is as shown in (6) below.

Continuing with the case $t^2 \neq w^2$, let $f(t)$ be the polynomial in (2), and let

$$r = \sqrt{(u^2 + v^2 + w^2)/3},$$

so that

$$f(-r) = -2uvw + \frac{2}{3}(u^2+v^2+w^2)r; \tag{3}$$
$$f(r) = -2uvw - \frac{2}{3}(u^2+v^2+w^2)r.$$

Clearly, $f(r) < 0$. To see that $f(-r) \geq 0$, we shall use the inequality of the geometric and arithmetic means, stated here for $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$:

$$(x_1x_2x_3)^{1/3} \leq \frac{x_1+x_2+x_3}{3}. \tag{4}$$

Taking $x_1 = u^2$, $x_2 = v^2$, $x_3 = w^2$ gives

$$27u^2v^2w^2 \leq \left(u^2+v^2+w^2\right)^3,$$

or equivalently,

$$3uvw \leq \left(u^2+v^2+w^2\right) r,$$

so that by (3), we have $f(-r) \geq 0$. We consider two cases: $f(-r) > 0$ and $f(r) = 0$. In the first case, there is a root t in the interval $(-\infty, -r)$. Since $f(0) < 0$, there is a root in $(-r, 0)$, and since $f(r) < 0$, there is a root in (r, ∞). For each of the three roots, or eigenvalues, there is an eigenvector, or point X, such that $\mathcal{C}_U^{-1}(X) = X$.

In the second case, that $f(r) = 0$, we have $(u^2+v^2+w^2)r = -3uvw$, so that $(u^2+v^2+w^2)^3 = 27u^2v^2w^2$, which implies that equality holds in (4). This is known to occur if and only if if $x_1 = x_2 = x_3$, or equivalently, $u^2 = v^2 = w^2$, which is to say that U is the incenter or one of the excenters; *i.e.*, that U is a member of the set

$$\{1:1:1, \quad -1:1:1, \quad 1:-1:1, \quad 1:1:-1\}. \tag{5}$$

We consider this case further in Examples 1 and 2 below, and summarize the rest of this section as a theorem.

Theorem 1. *Suppose U is not one of the four points in* (4)*, that t is a root of* (2)*, and that $t^2 \neq w^2$. Then the point*

$$X = tv+uw : tu+vw : t^2-w^2 \tag{6}$$

is a fixed point of $\mathcal{C}_U^{-1}$, hence also a fixed point of C_U. There are three distinct roots t, hence three distinct fixed points X.

3. Examples

As a first example, we address the possibility that the hypothesis $t^2 \neq w^2$ in Theorem 1 does not hold.

Example 1. $U = 1:1:1$. The characteristic polynomial is

$$\begin{vmatrix} -t & 1 & 1 \\ 1 & -t & 1 \\ 1 & 1 & -t \end{vmatrix} = (2-t)(t+1)^2.$$

We have two cases: $t = 2$ and $t = -1$. For $t = 2$, we easily find the fixed point $1 : 1 : 1$. For $t = -1$, the method of proof of Theorem 1 does not apply because $t^2 = w^2$. Instead, the system to be solved degenerates to the single equation $z = -x - y$. The solutions, all fixed points, are many; for example, let $f : g : h$ be any point, and let

$$x = g - h, \quad y = h - f, \quad z = f - g$$

(e.g., $x : y : z = b - c : c - a : a - b$, which is the triangle center[1] X_{512}.). Geometrically, $x : y : z$ are coefficients of the line joining $1 : 1 : 1$ and $f : g : h$.

Example 2. $U = -1 : 1 : 1$, the A-excenter. The characteristic polynomial is

$$\begin{vmatrix} -t & 1 & 1 \\ -1 & -t & 1 \\ -1 & 1 & -t \end{vmatrix} = -(t+1)\left(t^2 - t + 2\right).$$

For $t = -1$, we find that every point on the line $x + y + z = 0$ is a fixed point. If $t^2 - t + 2 = 0$, then $t = (1 \pm \sqrt{-7})/2$, and the (nonreal) fixed point is $1 : 1 : t - 1$. Similar results are obtained for $U \in \{1 : -1 : 1, 1 : 1 : -1\}$.

Example 3. $U = \cos A : \cos B : \cos C$. It can be checked using a computer algebra system that X_6, X_{2574}, and X_{2575} are fixed points. The first of these corresponds to the eigenvalue $t = 1$, as shown here:

$$\begin{aligned} x : y : z &= tv + uw : tu + vw : t^2 - w^2 \\ &= \cos B + \cos A \cos C : \cos A + \cos B \cos C : 1 - \cos^2 C \\ &= \sin A \sin C : \sin B \sin C : \sin C \sin C \\ &= \sin A : \sin B : \sin C \\ &= X_6. \end{aligned}$$

See also Example 6.

Example 4. $U = a(b^2 + c^2) : b(c^2 + a^2) : c(a^2 + b^2) = X_{39}$. The three roots of $t^3 - gt - h = 0$ are

$$-2abc, \quad abc - \sqrt{3a^2b^2c^2 + S(2,4)}, \quad abc + \sqrt{3a^2b^2c^2 + S(2,4)},$$

where

$$S(2,4) = a^2b^4 + a^4b^2 + a^2c^4 + a^4c^2 + b^2c^4 + b^4c^2.$$

The solution $t = -2abc$ easily leads to the fixed point

$$X_{512} = (b^2 - c^2)/a : (c^2 - a^2)/b : (a^2 - b^2)/c.$$

[1] We use the indexing of triangle centers in the *Encyclopedia of Triangle Centers* [3].

Example 5. For arbitrary real n, let $u = \cos nA$, $v = \cos nB$, $w = \cos nC$. A fixed point is $X = \sin nA : \sin nB : \sin nC$, as shown here:

$$\begin{aligned}\mathcal{C}_U^{-1}(X) = &\ \sin nB \cos nC + \sin nC \cos nB \\ &: \sin nC \cos nA + \sin nA \cos nC \\ &: \sin nA \cos nB + \sin nB \cos nA \\ = &\ \sin(nB + nC) : \sin(nC + nA) : \sin(nA + nB) \\ = &\ \sin nA : \sin nB : \sin nC.\end{aligned}$$

4. Images of lines

Let $\mathcal{L}$ be the line $l\alpha + m\beta + n\gamma = 0$ and let L the point[2] $l : m : n$. We shall determine coefficients of the line $\mathcal{C}_U^{-1}(\mathcal{L})$. Two points on $\mathcal{L}$ are

$$P = cm - bn : an - cl : bl - am \quad \text{and} \quad Q = m - n : n - l : l - m$$

Their images on $\mathcal{C}_U^{-1}(\mathcal{L})$ are given by

$$\begin{aligned} P' = \mathcal{C}_U^{-1}(P) &= \begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix} \begin{pmatrix} cm - bn \\ an - cl \\ bl - am \end{pmatrix}, \\ Q' = \mathcal{C}_U^{-1}(Q) &= \begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix} \begin{pmatrix} m - n \\ n - l \\ l - m \end{pmatrix}. \end{aligned}$$

We expand these products and use the resulting trilinears as rows 2 and 3 of the following determinant:

$$\begin{aligned} &\begin{vmatrix} \alpha & \beta & \gamma \\ w(an - cl) + v(bl - am) & w(cm - bn) + u(bl - am) & v(cm - bn) + u(an - cl) \\ w(n - l) + v(l - m) & w(m - n) + u(l - m) & v(m - n) + u(n - l) \end{vmatrix} \\ = &\ -\left((b - c)l + (c - a)m + (a - b)n\right) \\ &\cdot \left(u\,(-ul + vm + wn)\,\alpha + b\,(ul - vm + wn)\,\beta + c\,(ul + vm - wn)\,\gamma\right). \end{aligned}$$

If the first factor is not 0, then the required line $\mathcal{C}_U^{-1}(\mathcal{L})$ is given by

$$u\,(-ul + vm + wn)\,\alpha + v\,(ul - vm + wn)\,\beta + w\,(ul + vm - wn)\,\gamma = 0, \tag{7}$$

of which the coefficients are the trilinears of the point

$$L^{-1} \copyright U = u\,(-ul + vm + wn) : v\,(ul - vm + wn) : w\,(ul + vm - wn)\,.$$

Even if the first factor is 0, the points P' and Q' are easily checked to lie on the line (7).

[2]Geometrically, $\mathcal{L}$ is the trilinear polar of L^{-1}. However, the methods in this paper are algebraic rather than geometric, and the results extend beyond the boundaries of euclidean geometry. For example, in this paper, a, b, c are unrestricted positive real numbers; *i.e.*, they need not be sidelengths of a triangle.

The same method shows that the coefficients of the line $\mathcal{C}_U(\mathcal{L})$ are the trilinears of $(\text{cevapoint}(L,U))^{-1}$; that is, $\mathcal{C}_U(\mathcal{L})$ is the line

$$(wm+vn)\,\alpha+(un+wl)\,\beta+w\,(vl+um)\,\gamma=0.$$

5. Fixed lines

The line $\mathcal{L}$ is a fixed line of $\mathcal{C}_U$ (and of $\mathcal{C}_U^{-1}$) if $\mathcal{C}_U(\mathcal{L})=\mathcal{L}$, that is, if

$$(\text{cevapoint}(U,L))^{-1}=L,$$

or, equivalently,

$$\begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix}\begin{pmatrix} l \\ m \\ n \end{pmatrix}=\begin{pmatrix} l \\ m \\ n \end{pmatrix}.$$

This is the same equation as already solved (with L in place of X) in Section 2. For each of the three roots of (2), there is an eigenvector, or point L, and hence a line $\mathcal{L}$, such that $\mathcal{C}_U(\mathcal{L})=\mathcal{L}$, and we have the following theorem.

Theorem 2. *The mapping C_U has three distinct fixed lines, corresponding to the three distinct real roots of $f(t)$ in* (2). *For each root t, the corresponding fixed line $l\alpha+m\beta+n\gamma=0$ is given by*

$$l:m:n=tv+uw:tu+vw:t^2-w^2. \tag{8}$$

6. Iterations and convergence

In this section we examine sequences

$$X,\ \mathcal{C}_U^{-1}(X),\ \mathcal{C}_U^{-1}(\mathcal{C}_U^{-1}(X)),\ldots \tag{9}$$

of iterates. If X is a fixed point of $\mathcal{C}_U^{-1}$, then the sequence is simply $X,X,X,\ldots$; otherwise, with exceptions to be recognized, the sequence converges to a fixed point. We begin with the case that X lies on a fixed line, so that all the points in (9) lie on that same line. Let the two fixed points on the fixed line be

$$F_1=f_1:g_1:h_1 \quad\text{and}\quad F_2=f_2:g_2:h_2.$$

Then for X on the line F_1F_2, we have

$$X=f_1+tf_2:g_1+tg_2:h_1+th_2$$

for some function t homogeneous in a,b,c, and we wish to show that (9) converges to F_1 or F_2. As a first step,

$$\mathcal{C}_U^{-1}(X)=\begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix}\begin{pmatrix} f_1+tf_2 \\ g_1+tg_2 \\ h_1+th_2 \end{pmatrix}=\begin{pmatrix} wg_1+vh_1+t(wg_2+vh_2) \\ wf_1+uh_1+t(wf_2+uh_2) \\ vf_1+ug_1+t(vf_2+ug_2) \end{pmatrix}.$$

For $i = 1, 2$, because $f_i : g_i : h_i$ is fixed by $\mathcal{C}_U^{-1}$, there exists a homogeneous function t_i such that

$$\begin{aligned} wg_i + vh_i &= t_i f_i, \\ wf_i + vh_i &= t_i g_i, \\ vf_i + ug_i &= t_i h_i, \end{aligned}$$

so that

$$\mathcal{C}_U^{-1}(X) = \begin{pmatrix} t_1 f_1 + t_2 t f_2 \\ t_1 g_1 + t_2 t g_2 \\ t_1 h_1 + t_2 t h_2 \end{pmatrix} = t_1 \begin{pmatrix} f_1 + \frac{t_2}{t_1} t f_2 \\ g_1 + \frac{t_2}{t_1} t g_2 \\ h_1 + \frac{t_2}{t_1} t h_2 \end{pmatrix}.$$

Applying $\mathcal{C}_U^{-1}$ again thus gives

$$\mathcal{C}_U^{-2}(X) = \begin{pmatrix} f_1 + \frac{t_4}{t_3}\frac{t_2}{t_1} t f_2 \\ g_1 + \frac{t_4}{t_3}\frac{t_2}{t_1} t g_2 \\ h_1 + \frac{t_4}{t_3}\frac{t_2}{t_1} t h_2 \end{pmatrix},$$

where t_3 and t_4 satisfy

$$\begin{pmatrix} wg_1 + vh_1 + \frac{t_2}{t_1} t(wg_2 + vh_2) \\ wf_1 + uh_1 + \frac{t_2}{t_1} t(wf_2 + uh_2) \\ vf_1 + ug_1 + \frac{t_2}{t_1} t(vf_2 + ug_2) \end{pmatrix} = \begin{pmatrix} t_3 f_1 + t_4 \frac{t_2}{t_1} t f_2 \\ t_3 g_1 + t_4 \frac{t_2}{t_1} t f_2 \\ t_3 h_1 + t_4 \frac{t_2}{t_1} t f_2 \end{pmatrix} = t_3 \begin{pmatrix} f_1 + \frac{t_4}{t_3}\frac{t_2}{t_1} t f_2 \\ g_1 + \frac{t_4}{t_3}\frac{t_2}{t_1} t g_2 \\ h_1 + \frac{t_4}{t_3}\frac{t_2}{t_1} t h_2 \end{pmatrix}.$$

Now

$$\begin{aligned} t_1 &= \frac{wg_1 + vh_1}{f_1} = \frac{wf_1 + uh_1}{g_1} = \frac{vf_1 + ug_1}{h_1}, \\ t_2 &= \frac{wg_2 + vh_2}{f_2} = \frac{wf_2 + uh_2}{g_2} = \frac{vf_2 + ug_2}{h_2}, \\ t_3 &= \frac{wg_1 + vh_1}{f_1} = \frac{wf_1 + uh_1}{g_1} = \frac{vf_1 + ug_1}{h_1} = t_1, \\ t_4 &= \frac{w(\frac{t_2}{t_1})g_2 + v(\frac{t_2}{t_1})h_2}{(\frac{t_2}{t_1})f_2} = \frac{w(\frac{t_2}{t_1})f_2 + u(\frac{t_2}{t_1})h_2}{(\frac{t_2}{t_1})g_2} = \frac{v(\frac{t_2}{t_1})f_2 + u(\frac{t_2}{t_1})g_2}{(\frac{t_2}{t_1})h_2} = t_2. \end{aligned}$$

Consequently,

$$\mathcal{C}_U^{-2}(X) = \begin{pmatrix} f_1 + (\frac{t_2}{t_1})^2 t f_2 \\ g_1 + (\frac{t_2}{t_1})^2 t g_2 \\ h_1 + (\frac{t_2}{t_1})^2 t h_2 \end{pmatrix},$$

and, by induction,

$$\mathcal{C}_U^{-n}(X) = \begin{pmatrix} f_1 + (\frac{t_2}{t_1})^n t f_2 \\ g_1 + (\frac{t_2}{t_1})^n t g_2 \\ h_1 + (\frac{t_2}{t_1})^n t h_2 \end{pmatrix}. \tag{10}$$

Regarding the quotient $\frac{t_2}{t_1}$ in (10), if $\frac{t_2}{t_1} = 1$ then $\mathcal{C}_U^{-n}(X)$ is invariant of n, which is to say that X is a fixed point. If $\frac{t_2}{t_1} = -1$, then $\mathcal{C}_U^{-2}(X) = X$, which is to say that X is a fixed point of the collineation $\mathcal{C}_U^{-2}$. If $\left|\frac{t_2}{t_1}\right| \neq 1$, we call the line F_1F_2 a *regular fixed line*, and in this case, by (10), $\lim_{n\to\infty} \mathcal{C}_U^{-n}(X)$ is F_1 or F_2, according as $\left|\frac{t_2}{t_1}\right| < 1$ or $\left|\frac{t_2}{t_1}\right| > 1$. We summarize these findings as Lemma 3.

Lemma 3. *If X lies on a regular fixed line of $\mathcal{C}_U^{-1}$ (or equivalently, a regular fixed line of $\mathcal{C}_U$), then the sequence of points $\mathcal{C}_U^{-n}(X)$ (or equivalently, the points $\mathcal{C}_U^{n}(X)$) converges to a fixed point of $\mathcal{C}_U^{-1}$ (and of $\mathcal{C}_U$).*

Next, suppose that P is an arbitrary point in the plane of ABC. We shall show that $\mathcal{C}_U^{-n}(P)$ converges to a fixed point. Let F_1, F_2, F_3 be distinct fixed points. Define

$$P_2 = PF_2 \cap F_1F_3, \quad P_3 = PF_3 \cap F_1F_2 \quad P^{(0)} = \mathcal{C}_U^{-1}(P);$$
$$P_2^{(n)} = \mathcal{C}^{-n}(P_2) \text{ and } P_3^{(n)} = \mathcal{C}^{-n}(P_3) \text{ for } n = 1, 2, 3, \ldots$$

The collineation $\mathcal{C}_U^{-1}$ maps the line F_2P to the line $F_2P^{(0)}$, which is also the line $F_2P_2^{(1)}$ because F_2, P, P_2 are collinear; likewise, $\mathcal{C}_U^{-1}$ maps the line F_3P to the line $F_3P_3^{(1)}$. Consequently,

$$P^{(0)} = F_2P_2^{(1)} \cap F_3P_3^{(1)},$$

and by induction,

$$\mathcal{C}_U^{-n}(P) = F_2P_2^{(n)} \cap F_3P_3^{(n)}. \tag{10}$$

By Lemma 3,

$$\lim_{n\to\infty} \mathcal{C}_U^{-n}(P_2) \quad \text{and} \quad \lim_{n\to\infty} \mathcal{C}_U^{-n}(P_3)$$

are fixed points, so that by (10),

$$\lim_{n\to\infty} \mathcal{C}_U^{-n}(P)$$

must also be a fixed point. This completes a proof of the following theorem.

Theorem 4. *Suppose that the fixed lines of $\mathcal{C}_U^{-1}$ (or, equivalently, of $\mathcal{C}_U$) are regular. Then for every point X, the sequence of points $\mathcal{C}_U^{-n}(X)$ (or equivalently, the sequence $\mathcal{C}_U^{n}(X)$) converges to a fixed point of $\mathcal{C}_U^{-1}$(and of $\mathcal{C}_U$).*

Example 6. Extending Example 3, the three fixed lines, X_6X_{2574}, X_6X_{2575}, $X_{2574}X_{2575}$ are regular. The points X_{2574} and X_{2575} are the isogonal conjugates of the points X_{1113} and X_{1114} in which the Euler line intersects the circumcircle. Thus, the line $X_{2574}X_{2575}$ is the line at infinity. Because X_{1113} and X_{1114} are antipodal points on the circumcircle, the lines X_6X_{2574} are X_6X_{2575} are perpendicular (proof indicated at (x) below).

While visiting the author in February, 2007, Peter Moses analyzed the configuration in Example 6. His findings are given here.

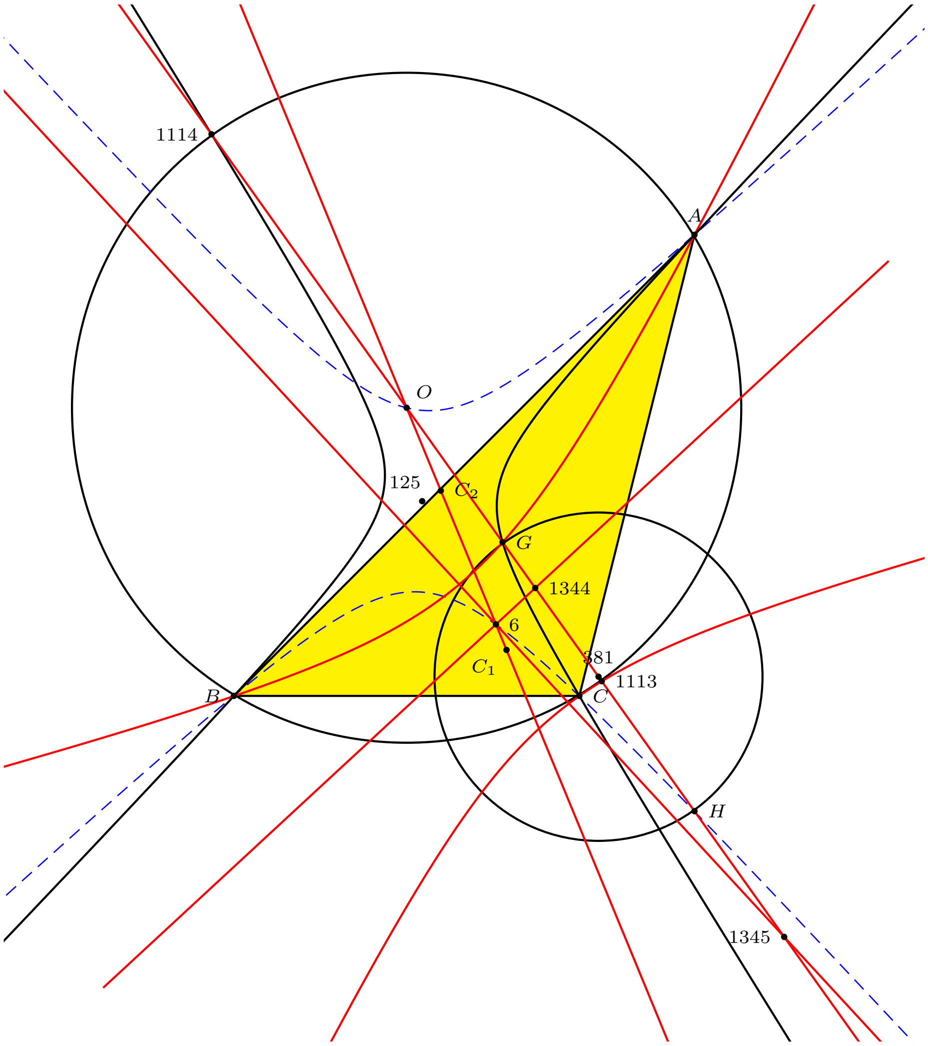

Figure 1.

(i) A point on line X_6X_{2574} is X_{1344}; a point on X_6X_{2575} is X_{1345}.

(ii) Segment GH (in Figure 1) is the diameter of the orthocentroidal circle, with center X_{381}. The points X_{1344} and X_{1345} are the internal and external centers of similitude of the orthocentroidal circle and the circumcircle.

(iii) Line GH, the Euler line, passes through the points

$$O,\ X_{1113},\ X_{1114},\ X_{1344},\ X_{1345}.$$

(iv) X_{125} is the center of the Jerabek hyperbola, which is the isogonal conjugate of the Euler line. (As isogonal conjugacy is a function, one may speak of its image when applied to lines as well as individual points).

(v) The line through X_{125} parallel to line X_6X_{1344} is the Simson line of X_{1114}, and the line through X_{125} parallel to line X_6X_{1345} is the Simson line of X_{1113}.

(vi) Hyperbola $ABCGX_{1113}$, with center C_1, is the isogonal conjugate of the $\mathcal{C}_U$-fixed line X_6X_{2574}, and hyperbola $ABCGX_{1114}$, with center C_2, is the isogonal conjugate of the $\mathcal{C}_U$-fixed line X_6X_{2575}.

(vii) C_1 is the barycentric square of X_{2575}, and C_2 is the barycentric square of X_{2574}.

(viii) The perspectors of the hyperbolas $ABCGX_{1113}$ and $ABCGX_{1114}$ are X_{2575} and X_{2574}, respectively. The fact that these perspectors are at infinity implies that the two conics, $ABCGX_{1113}$ and $ABCGX_{1114}$, are indeed hyperbolas.

(ix) The midpoint of the points C_1 and C_2 is the point $X_3X_6 \cap X_2X_{647}$.

(x) Line X_6X_{2574} is parallel to the Simson line of X_{1114}, and line X_6X_{2575} is parallel to the Simson line of X_{1113}. The two Simson lines are perpendicular ([1, p. 207]), so that the $\mathcal{C}_U$-fixed lines X_6X_{2574} and X_6X_{2575} are perpendicular.

(xi) The circle that passes through the points X_6, X_{1344}, and X_{1345} also passes through the point X_{2453}, which is the reflection of X_6 in the Euler line. This circle is a member of the coaxal family of the circumcircle, the nine-point circle, and the orthocentroidal circle.

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