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Steinhaus' Problem on Partition of a Triangle

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Abstract. H. Steinhaus has asked whether inside each acute triangle there is a point from which perpendiculars to the sides divide the triangle into three parts of equal areas. We present two solutions of Steinhaus' problem.

The n-dimensional case of Theorem 1 below was proved in [6], see also [2] and [4, Theorem 2.1, p. 152]. For an earlier mass-partition version of Theorem 1, for bounded convex masses in $\mathbb{R}^n$ and $r_1 = r_2 = ... = r_{n+1}$, see [7].

Theorem 1 (Kuratowski-Steinhaus)**.** *Let* $\boldsymbol{T} \subseteq \mathbb{R}^2$ *be a bounded measurable set, and let* $|\boldsymbol{T}|$ *be the measure of* $\boldsymbol{T}$. *Let* α_1, α_2, α_3 *be the angles determined by three rays emanating from a point, and let* $\alpha_1 < \pi$, $\alpha_2 < \pi$, $\alpha_3 < \pi$. *Let* r_1, r_2, r_3 *be nonnegative numbers such that* $r_1 + r_2 + r_3 = |\boldsymbol{T}|$. *Then there exists a translation* $\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $|\lambda(\boldsymbol{T}) \cap \alpha_1| = r_1$, $|\lambda(\boldsymbol{T}) \cap \alpha_2| = r_2$, $|\lambda(\boldsymbol{T}) \cap \alpha_3| = r_3$.

H. Steinhaus asked ([10], [11]) whether *inside* each acute triangle there is a point from which perpendiculars to the sides divide the triangle into three parts with equal areas. Long and elementary solutions of Steinhaus' problem appeared in [8, pp. 101–104], [9, pp. 103–105], [12, pp. 133–138] and [13]. For some acute triangles with rational coordinates of vertices, the point solving Steinhaus' problem is not constructible with ruler and compass alone, see [15]. Following article [14], we will present two solutions of Steinhaus' problem.

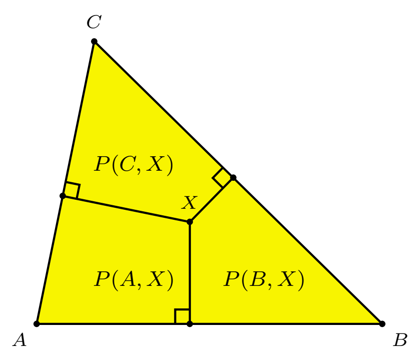

Figure 1

For $X \in \triangle ABC$, we denote by $P(A, X)$, $P(B, X)$, $P(C, X)$ the areas of the quadrangles containing vertices A, B, C respectively (see Figure 1). The areas

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$P(A, X)$, $P(B, X)$, $P(C, X)$ are continuous functions of X in the triangle ABC. The function

$$f(X) = \min\{P(A, X), P(B, X), P(C, X)\}$$

is also continuous. By Weierstrass' theorem f attains a maximum in triangle ABC, *i.e.*, there exists $X_0 \in \triangle ABC$ such that $f(X) \leq f(X_0)$ for all $X \in \triangle ABC$.

Lemma 2. *For a point X lying on a side of an acute triangle, the area at the opposite vertex is greater than one of the remaining two areas.*

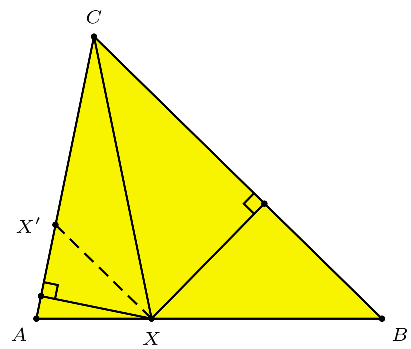

Figure 2

Proof. Without loss of generality, we may assume that $X \in \overline{AB}$ and $|AX| \leq |BX|$, see Figure 2. Straight line XX' parallel to straight line BC cuts the triangle AXX' greater than $P(A, X)$ (as the angle ACB is acute), but not greater than the triangle CXX' because $|AX'| < \frac{|AC|}{2} < |X'C|$. Hence $P(A, X) < |\triangle AXX'| \leq |\triangle CXX'| < P(C, X)$. □

Theorem 3. *If a triangle ABC is acute and f attains a maximum at X_0, then $P(A, X_0) = P(B, X_0) = P(C, X_0) = \frac{|\triangle ABC|}{3}$.*

Proof. $f(A) = f(B) = f(C) = 0$, and 0 is not a maximum of f. Therefore X_0 is not a vertex of the triangle ABC. Let us assume that $f(X_0) = P(A, X_0)$. By Lemma 2, $X_0 \notin \overline{BC}$. Suppose, on the contrary, that some of the other areas, let's say $P(C, X_0)$, is greater than $P(A, X_0)$.

Case 1: $X_0 \notin \overline{AC}$. When shifting X_0 from the segment $\overline{AB}$ by appropriately small ε and perpendicularly to the segment $\overline{AB}$ (see Figure 3), we receive $P(C, X)$ further greater than $f(X_0)$ and at the same time $P(A, X) > P(A, X_0)$ and $P(B, X) > P(B, X_0)$. Hence $f(X) > f(X_0)$, a contradiction.

Case 2: $X_0 \in \overline{AC} \setminus \{A, C\}$. By Lemma 2,

$$\begin{aligned} P(B, X_0) &> \min\{P(A, X_0), P(C, X_0)\} \\ &\geq \min\{P(A, X_0), P(B, X_0), P(C, X_0)\} \\ &= f(X_0). \end{aligned}$$

When shifting X_0 from the segment $\overline{AC}$ by appropriately small ε and perpendicularly to the segment $\overline{AC}$ (see Figure 4), we receive $P(B, X)$ further greater than

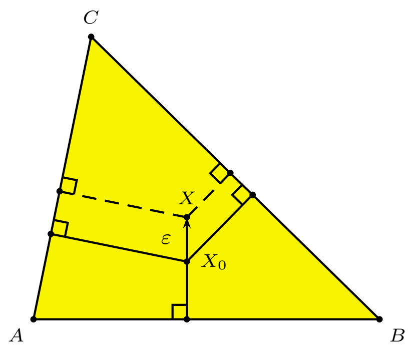

Figure 3

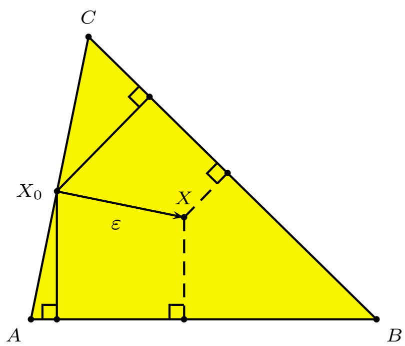

Figure 4

$f(X_0)$ and at the same time $P(A,X) > P(A,X_0)$ and $P(C,X) > P(C,X_0)$. Hence $f(X) > f(X_0)$, a contradiction. □

For each acute triangle ABC there is a unique $X_0 \in \triangle ABC$ such that $P(A,X_0) = P(B,X_0) = P(C,X_0) = \frac{|\triangle ABC|}{3}$. Indeed, if $X \neq X_0$ then X lies in some of the quadrangles determined by X_0. Let us say that X lies in the quadrangle with vertex A (see Figure 5). Then $P(A,X) < P(A,X_0) = \frac{|\triangle ABC|}{3}$.

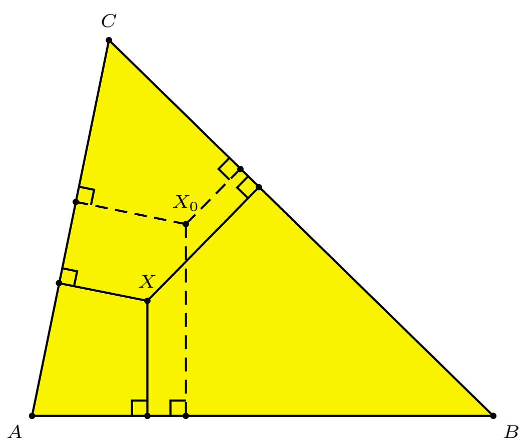

Figure 5.

The sets $R_A = \{X \in \triangle ABC : P(A,X) = f(X)\}$, $R_B = \{X \in \triangle ABC : P(B,X) = f(X)\}$ and $R_C = \{X \in \triangle ABC : P(C,X) = f(X)\}$ are closed and cover the triangle ABC. Assume that the triangle ABC is acute. By Lemma 2, $R_A \cap \overline{BC} = \emptyset$, $R_B \cap \overline{AC} = \emptyset$, and $R_C \cap \overline{AB} = \emptyset$. The theorem proved in [5] guarantees that $R_A \cap R_B \cap R_C \neq \emptyset$, see also [4, item D4, p. 101] and [1, item 2.23, p. 162]. Any point belonging to $R_A \cap R_B \cap R_C$ lies inside the triangle ABC and determines the partition of the triangle ABC into three parts with equal areas.

The above proof remains valid for all right triangles, because the hypothesis of Lemma 2 holds for all right triangles. For each triangle the following statements are true.

(1) There is a unique point in the plane which determines the partition of the triangle into three equal areas.
(2) The point of partition into three equal areas lies inside the triangle if and only if the hypothesis of Lemma 2 holds for the triangle.
(3) The point of partition into three equal areas lies inside the triangle if and only if the maximum of f on the boundary of the triangle is smaller than the maximum of f on the whole triangle. For each acute or right triangle ABC, the maximum of f on the boundary does not exceed $\frac{|\triangle ABC|}{4}$.
(4) The point of partition into three equal areas lies inside the triangle, if the triangle has two angles in the interval $\left(\arctan \frac{1}{\sqrt{2}}, \frac{\pi}{2}\right]$. This condition holds for each acute or right triangle.
(5) If the point of partition into three equal areas lies inside the triangle, then it is a partition into quadrangles.

Assume now $C > \frac{\pi}{2}$. The point of partition into three equal areas lies inside the triangle if and only if

$$\sqrt{(1+\tan^2 A)\tan B} + \sqrt{(1+\tan^2 B)\tan A} > \sqrt{3(\tan A + \tan B)}.$$

If, on the other hand,

$$\sqrt{(1+\tan^2 A)\tan B} + \sqrt{(1+\tan^2 B)\tan A} = \sqrt{3(\tan A + \tan B)},$$

then the unique $X_0 \in \overline{AB}$ such that

$$|AX_0| = \sqrt{\frac{(1+\tan^2 A)\tan B}{3(\tan A + \tan B)}}|AB|, \qquad |BX_0| = \sqrt{\frac{(1+\tan^2 B)\tan A}{3(\tan A + \tan B)}}|AB|$$

determines the partition of the triangle ABC into three equal areas. It is a partition into a triangle with vertex A, and a triangle with vertex B, and a quadrangle. Finally, when

$$\sqrt{(1+\tan^2 A)\tan B} + \sqrt{(1+\tan^2 B)\tan A} < \sqrt{3(\tan A + \tan B)}, \qquad (*)$$

there is a straight line a perpendicular to the segment $\overline{AC}$ which cuts from the triangle ABC a figure with the area $\frac{|\triangle ABC|}{3}$ (see Figure 6). There is a straight line b perpendicular to the segment $\overline{BC}$ which cuts from the triangle ABC a figure with the area $\frac{|\triangle ABC|}{3}$. By $(*)$, the intersection point of the straight lines a and b lies outside the triangle ABC. This point determines the partition of the triangle ABC into three equal areas.

J.-P. Ehrmann [3] has subsequently found a constructive solution of a generalization of Steinhaus' problem of partitioning a given triangle into three quadrangles with prescribed proportions.

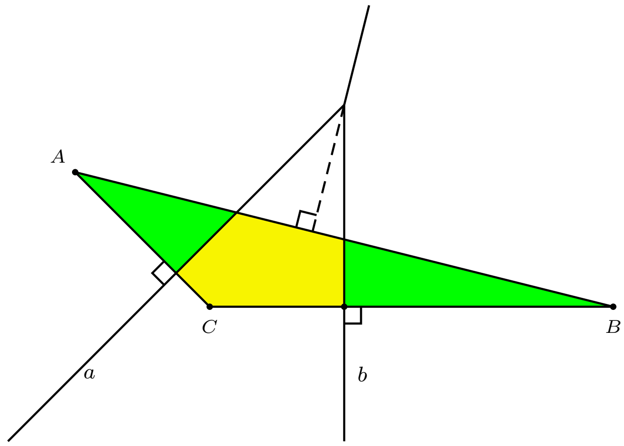

Figure 6.

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