Constructive Solution of a Generalization of Steinhaus’ Problem on Partition of a Triangle

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Abstract. We present a constructive solution to a generalization of Hugo Steinhaus’ problem of partitioning a given triangle, by dropping perpendiculars from an interior point, into three quadrilaterals whose areas are in prescribed proportions.

1. Generalized Steinhaus problem

Given an acute angled triangle $ABC$, Steinhaus’ problem asks a point $P$ in its interior with pedals $P_a, P_b, P_c$ on $BC, CA, AB$ such that the quadrilaterals $AP_bPP_c, BP_cPP_a,$ and $CP_aPP_b$ have equal areas. See [3] and the bibliographic information therein. A. Tyszka [2] has also shown that Steinhaus’ problem is in general not soluble by ruler-and-compass. We present a simple constructive solution (using conics) of a generalization of Steinhaus’ problem. In this note, the area of a polygon $P$ will be denoted by $\Delta(P)$. In particular, $\Delta = \Delta(ABC)$. Thus, given three positive real numbers $u, v, w$, we look for the point(s) $P$ such that

(1) $P$ is inside $ABC$ and $P_a, P_b, P_c$ lie respectively in the segments $BC, CA, AB$.

(2) $\Delta(AP_bPP_c) : \Delta(BP_cPP_a) : \Delta(CP_aPP_b) = u : v : w$.

We do not require the triangle to be acute-angled.

Lemma 1. Consider a point $P$ inside the angular sector bounded by the half-lines $AB$ and $AC$, with projections $P_b$ and $P_c$ on $AC$ and $AB$ respectively. For a positive real number $k$, $\Delta(AP_bPP_c) = k \cdot \Delta(ABC)$ if and only if $P$ lies on the rectangular hyperbola with center $A$, focal axis the internal bisector $AI$, and semi-major axis $\sqrt{kbc}$.

Proof. We take $A$ for pole and the bisector $AI$ for polar axis, let $(\rho, \theta)$ be the polar coordinates of $P$. As $AP_b = \rho \cos \left(\frac{\pi}{2} - \theta\right)$ and $PP_b = \rho \sin \left(\frac{\pi}{2} - \theta\right)$, we have $\Delta(AP_bPP_c) = \frac{1}{2}\rho^2 \sin(A - 2\theta)$. Similarly, $\Delta(AP_cPP_a) = \frac{1}{2}\rho^2 \sin(A + 2\theta)$. Hence the quadrilateral $AP_bPP_c$ has area $\frac{1}{2}\rho^2 \sin A \cos 2\theta$. Therefore,

$$\Delta(AP_bPP_c) = k \cdot \Delta(ABC) \iff \rho^2 \cos 2\theta = \frac{2k \cdot \Delta(ABC)}{\sin A} = kbc.$$
Theorem 2. Let \( U \) be the point with barycentric coordinates \((u : v : w)\) and \( M_1, M_2, M_3 \) be the antipodes on the circumcircle \( \Gamma \) of \( ABC \) of the points whose Simson lines pass through \( U \) and \( P \) the incenter of the triangle \( M_1M_2M_3 \). If \( P \) verifies (1), then \( P \) is the unique solution of our problem. Otherwise, the generalized Steinhaus problem has no solution.

Remarks. (a) Of course, if \( ABC \) is acute angled, and \( P \) inside \( ABC \), then (1) will be verified.

(b) As \( U \) lies inside the Steiner deltoid, there exist three real Simson lines through \( U \); so \( M_1, M_2, M_3 \) are real and distinct.

(c) Let \( h_A \) be the rectangular hyperbola with center \( A \), focal axis \( AI \), and semi-major axis \( \sqrt{\frac{u}{u + v + w}} \cdot bc \), and define rectangular hyperbolas \( h_B \) and \( h_C \) analogously.

If \( P \) verifies (1), it will verify (2) if and only if \( P \in h_A \cap h_B \). In this case, \( P \in h_C \), and the solutions of our problem are the common points of \( h_A, h_B, h_C \) verifying (1).

(d) The four common points \( P_1, P_2, P_3, P_4 \) (real or imaginary) of the rectangular hyperbolae \( h_A, h_B, h_C \) form an orthocentric system. As \( h_A, h_B, h_C \) are centered respectively at \( A, B, C \), any conic through \( P_1, P_2, P_3, P_4 \) is a rectangular hyperbola with center on \( \Gamma \). As the vertices of the diagonal triangle of this orthocentric system are the centers of the degenerate conics through \( P_1, P_2, P_3, P_4 \), they lie on \( \Gamma \).

(e) We will see later that \( P_1, P_2, P_3, P_4 \) are always real.

2. Proof of Theorem 2

If \( P \) has homogeneous barycentric coordinates \((x : y : z)\) with reference to triangle \( ABC \), then

\[
(x + y + z)^2 \Delta(APP_b) = y \left( z + \frac{b^2 + c^2 - a^2}{2b^2} y \right) \Delta,
\]

\[
(x + y + z)^2 \Delta(PP_c) = z \left( y + \frac{b^2 + c^2 - a^2}{2c^2} z \right) \Delta,
\]

where \( \Delta = \Delta(ABC) \). Hence the barycentric equation of \( h_A \) is

\[
h_A(x, y, z) := \frac{b^2 + c^2 - a^2}{2} \left( \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) + 2 y z - \frac{u}{u + v + w} (x + y + z)^2 = 0.
\]

We get \( h_B \) and \( h_C \) by cyclically permuting \( a, b, c; u, v, w; x, y, z \).

If \( M = (x : y : z) \) is a vertex of the diagonal triangle of \( P_1P_2P_3P_4 \), it has the same polar line (the opposite side) with respect to the three conics \( h_A, h_B, h_C \). Hence,

\[
\frac{\partial h_B}{\partial y} \frac{\partial h_C}{\partial z} - \frac{\partial h_B}{\partial z} \frac{\partial h_C}{\partial y} = \frac{\partial h_C}{\partial x} \frac{\partial h_A}{\partial z} - \frac{\partial h_C}{\partial z} \frac{\partial h_A}{\partial x} = \frac{\partial h_A}{\partial y} \frac{\partial h_B}{\partial x} - \frac{\partial h_A}{\partial x} \frac{\partial h_B}{\partial y} = 0.
\]

Let \( N \) be the reflection of \( M \) in the circumcenter \( O \); \( N_a, N_b, N_c \) the pedal triangle of \( N \). Clearly, \( N_a, N_b, N_c \) are the reflections of the vertices of the pedal triangle
of $M$ in the midpoints of the corresponding sides of $ABC$. Now, $N_b$ and $N_c$ have coordinates
\[(b^2 + c^2 - a^2)y + 2b^2z : 0 : (a^2 + b^2 - c^2)y + 2b^2x\]
and
\[(b^2 + c^2 - a^2)z + 2c^2y : (c^2 + a^2 - b^2)z + 2c^2x : 0\]
respectively. A straightforward computation shows that
\[
\det[N_b, N_c, U] = b^2c^2(u + v + w) \left( \frac{\partial h_B}{\partial y} \frac{\partial h_C}{\partial z} - \frac{\partial h_B}{\partial z} \frac{\partial h_C}{\partial y} \right) = 0.
\]
Similarly, $\det[N_c, N_a, U] = \det[N_a, N_b, U] = 0$. It follows that $N$ lies on the circumcircle (we knew that already by Remark (d)), and the Simson line of $N$ passes through $U$.

Hence, $M_1M_2M_3$ is the diagonal triangle of the orthocentric system $P_1P_2P_3P_4$, which means that $P_1P_2P_3P_4$ are real and are the incenter and the three excenters of $M_1M_2M_3$.

As the three excenters of a triangle lie outside his circumcircle, the incenter of $M_1M_2M_3$ is the only common point of $h_A, h_B, h_C$ inside $\Gamma$. This completes the proof of Theorem 2.

3. Constructions

In [1], the author has given a construction of the points on the circumcircle whose Simson line pass through a given point. Let $U^-$ and $U^+$ be the complement and the anticomplement of $U$, i.e., the images of $U$ under the homotheties $h(G, -\frac{1}{2})$ and $h(G, -2)$ respectively. Since
\[
(\text{Reflection in } O) \circ (\text{Translation by } \overrightarrow{HU}) = \text{Reflection in } U^-,
\]
if $h_0$ is the reflection in $U^-$ of the rectangular circumhyperbola through $U$, and $M_4$ the antipode of $U^+$ on $h_0$, then $M_1, M_2, M_3, M_4$ are the four common points of $h_0$ and the circumcircle.

In the case $u = v = w = 1$, $h_0$ is the reflection in the centroid $G$ of the Kiepert hyperbola of $ABC$. It intersects the circumcircle $\Gamma$ at $M_1, M_2, M_3$ and the Steiner point of $ABC$. See Figure 1.
Figure 1.

References


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