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The Soddy Circles

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Abstract. Given three circles externally tangent to each other, we investigate the construction of the two so called Soddy circles, that are tangent to the given three circles. From this construction we get easily the formulas of the radii and the barycentric coordinates of Soddy centers relative to the triangle ABC that has vertices the centers of the three given circles.

1. Construction of Soddy circles

In the general Apollonius problem it is known that, given three arbitrary circles with noncollinear centers, there are at most 8 circles tangent to each of them. In the special case when three given circles are tangent externally to each other, there are only two such circles. These are called the inner and outer Soddy circles respectively of the given circles. Let the mutually externally tangent circles be $\mathscr{C}_a(A, r_1)$, $\mathscr{C}_b(B, r_2)$, $\mathscr{C}_c(C, r_3)$, and A_1, B_1, C_1 be their tangency points (see Figure 1).

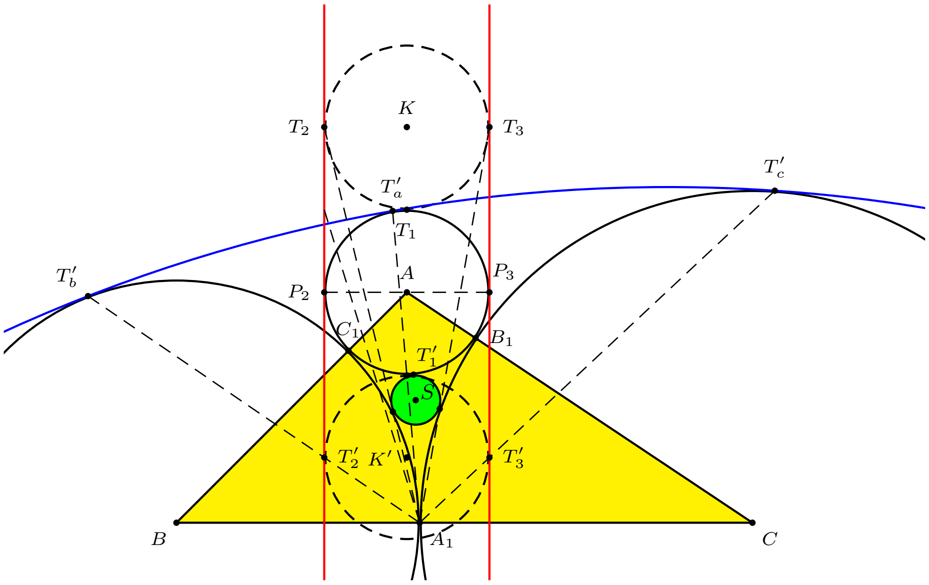

Figure 1.

Consider the inversion τ with pole A_1 that maps $\mathscr{C}_a$ to $\mathscr{C}_a$. This also maps the circles $\mathscr{C}_b$, $\mathscr{C}_c$ to the two lines perpendicular to BC and tangent to $\mathscr{C}_a$ at the points P_2, P_3 where P_2P_3 is parallel from A to BC. The only circles tangent to $\mathscr{C}_a$ and to the above lines are the circles $K(T_1)$, $K'(T_1')$ where T_1, T_1' are lying on $\mathscr{C}_a$ and

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the A-altitude of ABC. These circles are the images, in the above inversion, of the Soddy circles we are trying to construct. Since the circle $K(T_1)$ must be the inverse of the inner Soddy circle, the lines A_1T_1, A_1T_2, A_1T_3, $(P_2T_2 = P_3T_3 = P_2P_3)$ meet $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ at the points T_a, T_b, T_c respectively, that are the tangency points of the inner Soddy circle. Hence the lines BT_b and CT_c give the center S of the inner Soddy circle. Similarly the lines A_1T_1', A_1T_2', A_1T_3', $(P_2T_2' = P_3T_3' = P_2P_3)$, meet $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ at the points T_a', T_b', T_c' respectively, that are the tangency points of the outer Soddy circle. Triangles $T_aT_bT_c$, $T_a'T_b'T_c'$ are the inner and outer Soddy triangles. A construction by the so called Soddy hyperbolas can be found in [5, §12.4.2].

2. The radii of Soddy circles

If the sidelengths of ABC are a, b, c, and $s = \frac{1}{2}(a+b+c)$, then

$$\begin{array}{lll} a = r_2 + r_3, & b = r_3 + r_1, & c = r_1 + r_2; \\ r_1 = s - a, & r_2 = s - b, & r_3 = s - c. \end{array}$$

If $\triangle$ is the area of ABC, then $\triangle = \sqrt{r_1r_2r_3(r_1+r_2+r_3)}$. The A-altitude of ABC is $AD = h_a = \frac{2\triangle}{a}$, and the inradius is $r = \frac{\triangle}{r_1+r_2+r_3}$.

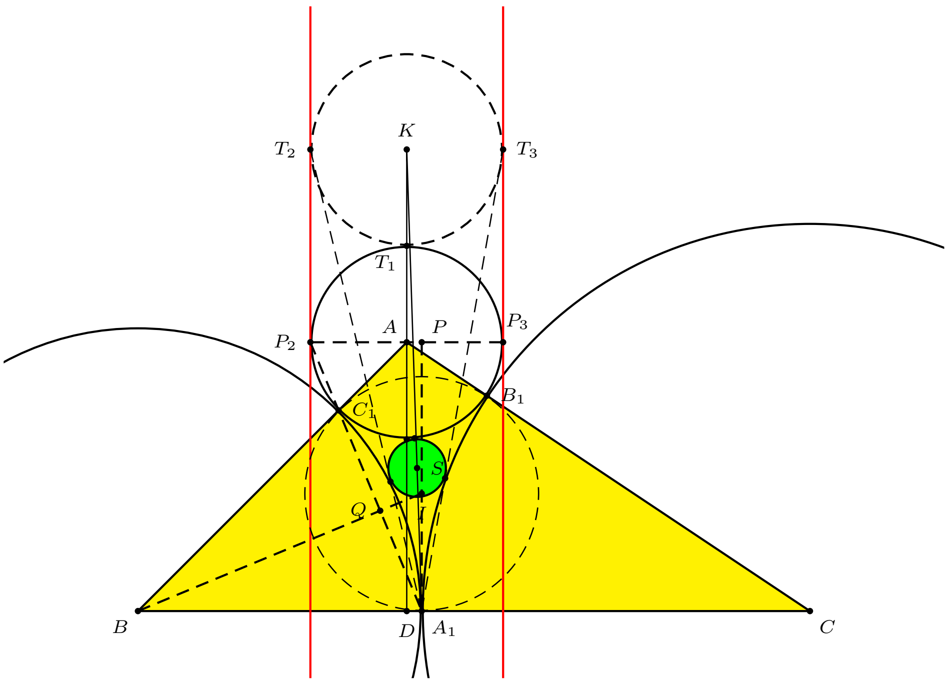

Figure 2.

The points A_1, B_1, C_1 are the points of tangency of the incircle $I(r)$ of ABC with the sidelines. If A_1P is perpendicular to P_2P_3 and IB meets A_1C_1 at Q, then the inversion τ maps C_1 to P_2, and the quadrilateral IQP_2P is cyclic (see Figure 2). The power of the inversion is

$$d^2 = A_1C_1 \cdot A_1P_2 = 2A_1Q \cdot A_1P_2 = 2A_1I \cdot A_1P = 2rh_a = \frac{4r_1r_2r_3}{r_2+r_3}. \quad (1)$$

2.1. *Inner Soddy circle.* Since the inner Soddy circle is the inverse of the circle $K(r_1)$, its radius is given by

$$x = \frac{d^2}{A_1K^2 - r_1^2} \cdot r_1. \quad (2)$$

In triangle A_1AK, $A_1K^2 - A_1A^2 = 2AK \cdot T_1D = 4r_1(r_1 + h_a)$. Hence,

$$A_1K^2 - r_1^2 = A_1A^2 - r_1^2 + 4r_1(r_1+h_a) = d^2 + 4r_1(r_1+h_a),$$

and from (1), (2),

$$x = \frac{r_1r_2r_3}{r_2r_3 + r_3r_1 + r_1r_2 + 2\triangle}. \quad (3)$$

Here is an alternative expression for x. If r_a, r_b, r_c are the exradii of triangle ABC, and R its circumradius, it is well known that

$$r_a + r_b + r_c = 4R + r.$$

See, for example, [4, §2.4.1]. Now also that $r_1r_a = r_2r_b = r_3r_c = \triangle$. Therefore,

$$\begin{aligned} x =& \frac{r_1r_2r_3}{r_2r_3 + r_3r_1 + r_1r_2 + 2\triangle} \\ =& \frac{\triangle}{\frac{\triangle}{r_1} + \frac{\triangle}{r_2} + \frac{\triangle}{r_3} + 2 \cdot \frac{\triangle^2}{r_1r_2r_3}} \\ =& \frac{\triangle}{r_a + r_b + r_c + 2(r_1+r_2+r_3)} \\ =& \frac{\triangle}{4R + r + 2s}. \end{aligned} \quad (4)$$

As a special case, if $r_1 \to \infty$, then the circle $\mathscr{C}_a$ tends to a common tangent of $\mathscr{C}_b$, $\mathscr{C}_c$, and

$$\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{r_2}} + \frac{1}{\sqrt{r_3}}. \quad (5)$$

In this case the outer Soddy circle degenerates into the common tangent of $\mathscr{C}_b$ and $\mathscr{C}_c$.

2.2. *Outer Soddy circle.* If $\mathscr{C}_a$ is the smallest of the three circles $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ and is greater than the circle of (5), *i.e.*, $\frac{1}{\sqrt{r_1}} < \frac{1}{\sqrt{r_2}} + \frac{1}{\sqrt{r_3}}$, then the outer Soddy circle is internally tangent to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$. Otherwise, the outer Soddy circle is externally tangent to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$.

Since the outer Soddy circle is the inverse of the circle $K'(r_1)$, its radius is given by

$$x' = \frac{d^2}{A_1K'^2 - r_1^2} \cdot r_1. \quad (6)$$

This is a signed radius and is negative when A_1 is inside the circle $K'(r_1)$ or when the outer Soddy circle is tangent internally to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$. In triangle A_1AK', $A_1A^2 - A_1K'^2 = 2AK' \cdot T_1'D = 4r_1(h_a - r_1)$, and from (6),

$$x' = \frac{r_1r_2r_3}{r_1r_2 + r_2r_3 + r_3r_1 - 2\triangle}. \tag{7}$$

Analogous to (4) we also have

$$x' = \frac{\triangle}{4R + r - 2s}. \tag{8}$$

Hence this radius is negative, equivalently, the outer Soddy circle is tangent internally to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$, when $4R + r < 2s$. From (4) and (8), we have

$$\frac{1}{x} - \frac{1}{x'} = \frac{2s}{\triangle} = \frac{4}{r}.$$

If $4R + r = 2s$, then $x = \frac{r}{4}$.

3. The barycentric coordinates of Soddy centers

3.1. *The Inner Soddy center.* If d_1 is the distance of the inner Soddy circle center S from BC, then since A_1 is the center of similitute of the inner Soddy circle and the circle $K(r_1)$ we have $\frac{d_1}{KD} = \frac{x}{r_1}$, or

$$d_1 = \frac{x(2r_1 + h_a)}{r_1} = 2x\left(1 + \frac{h_a}{2r_1}\right) = 2x\left(1 + \frac{\triangle}{a(s-a)}\right).$$

Similarly we obtain the distances d_2, d_3 from S to the sides CA and AB respectively. Hence the homogeneous barycentric coordinates of S are

$$(ad_1 : bd_2 : cd_3) = \left(a + \frac{\triangle}{s-a} :\ b + \frac{\triangle}{s-b} :\ c + \frac{\triangle}{s-c}\right).$$

The inner Soddy center S appears in [3] as the triangle center X_{176}, also called the equal detour point. It is obvious that for the Inner Soddy center S, the "detour" of triangle SBC is

$$SB + SC - BC = (x + r_2) + (x + r_3) - (r_2 + r_3) = 2x.$$

Similarly the triangles SCA and SAB also have detours $2x$. Hence the three incircles of triangles SBC, SCA, SAB are tangent to each other and their three tangency point A_2, B_2, C_2 are the points T_a, T_b, T_c on the inner Soddy circle [1] since $SA_2 = SB_2 = SC_2 = x$. See Figure 3.

Working with absolute barycentric coordinates, we have

$$\begin{aligned} S =& \frac{\left(a + \frac{\triangle}{s-a}\right)A + \left(b + \frac{\triangle}{s-b}\right)B + \left(c + \frac{\triangle}{s-c}\right)C}{a + \frac{\triangle}{s-a} + b + \frac{\triangle}{s-b} + c + \frac{\triangle}{s-c}} \\ =& \frac{(a+b+c)I + \triangle\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right)G_{\mathrm{e}}}{\frac{\triangle}{x}}, \end{aligned} \tag{9}$$

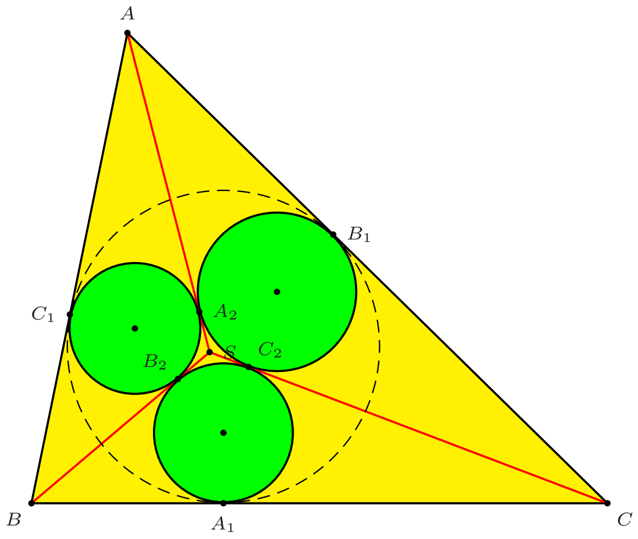

Figure 3.

where $G_e = \left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c}\right)$ is the Gergonne point. Hence, the inner Soddy center S lies on the line connecting the incenter I and G_e. This explains whey IG_e is called the Soddy line. Indeed, S divides IG_e in the ratio

$$IS : SG_e = r_a + r_b + r_c : a + b + c = 4R + r : 2s.$$

3.2. *The outer Soddy center.* If d_1' is the distance of the outer Soddy circle center S' from BC, then since A_1 is the center of similitute of the outer Soddy circle and the circle $K'(r_1)$, a similar calculation referring to Figure 1 shows that

$$d_1' = -2x\left(1 - \frac{\triangle}{a(s-a)}\right).$$

Similarly, we have the distances d_2' and d_3' from S' to CA and AB respectively. The homogeneous barycentric coordinates of S' are

$$(ad_1' : bd_2' : cd_3') = \left(a - \frac{\triangle}{s-a} : b - \frac{\triangle}{s-b} : c - \frac{\triangle}{s-c}\right).$$

This is the triangle center X_{175} of [3], called the isoperimetric point. It is obvious that if the outer Soddy circle is tangent internally to $\mathscr{C}_a, \mathscr{C}_b, \mathscr{C}_c$ or $4R+r < 2s$, then the perimeter of triangle $S'BC$ is

$$S'B + S'C + BC = (x' - r_2) + (x' - r_3) + (r_2 + r_3) = 2x'.$$

Similarly the perimeters of triangles $S'CA$ and $S'AB$ are also $2x'$. Therefore the S'-excircles of triangles $S'BC$, $S'CA$, $S'AB$ are tangent to each other at the tangency points T_a', T_b', T_c' of the outer Soddy circle with $\mathscr{C}_a, \mathscr{C}_b, \mathscr{C}_c$.

If the outer Soddy circle is tangent externally to $\mathscr{C}_a, \mathscr{C}_b, \mathscr{C}_c$, equivalently, $4R + r > 2s$, then the triangles $S'BC$, $S'CA$, $S'AB$ have equal detours $2x'$ because for triangle $S'BC$,

$$S'B + S'C - BC = (x' + r_2) + (x' + r_3) - (r_2 + r_3) = 2x',$$

and similarly for the other two triangles. In this case, S' is second equal detour point. Analogous to (9), we have

$$S' = \frac{(a+b+c)I - \triangle\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right)G_{\mathrm{e}}}{\frac{\triangle}{x'}}. \tag{10}$$

A comparison of (9) and (10) shows that S and S' are harmonic conjugates with respect to IG_{e}.

4. The barycentric equations of Soddy circles

We find the barycentric equation of the inner Soddy circle in the form

$$a^2yz + b^2zx + c^2xy - (x+y+z)(p_1x + p_2y + p_3z) = 0,$$

where p_1, p_2, p_3 are the powers of A, B, C with respect to the circle. See [5, Proposition 7.2.3]. It is easy to see that

$$\begin{aligned} p_1 =& r_1(r_1 + 2x) = (s-a)(s-a+2x), \\ p_2 =& r_2(r_2 + 2x) = (s-b)(s-b+2x), \\ p_3 =& r_3(r_3 + 2x) = (s-c)(s-c+2x). \end{aligned}$$

Similarly, the barycentric equation of the outer Soddy circle is

$$a^2yz + b^2zx + c^2xy - (x+y+z)(q_1x + q_2y + q_3z) = 0,$$

where

$$\begin{aligned} q_1 =& (s-a)(s-a+2x'), \\ q_2 =& (s-b)(s-b+2x'), \\ q_3 =& (s-c)(s-c+2x'), \end{aligned}$$

where x' is the *signed* radius of the circle given by (8), treated as negative when $2s > 4R + r$.

5. The Soddy triangles and the Eppstein points

The incenter I of ABC is the radical center of the circles $\mathscr{C}_a, \mathscr{C}_b, \mathscr{C}_c$. The inversion with respect to the incircle leaves each of $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ invariant and swaps the inner and outer Soddy circles. In particular, it interchanges the points of tangency T_a and T'_a; similarly, T_b and T'_b, T_c and T'_c. The Soddy triangles $T_aT_bT_c$ and $T'_aT'_bT'_c$ are clearly perspective at the incenter I. They are also perspective with ABC, at S and S' respectively. Since $AT_a : T_aS = r_1 : x$, we have, $T_a = \frac{xA + r_1S}{x + r_1}$. In homogeneous barycentric coordinates,

$$T_a = \left(a + \frac{2\triangle}{r_1} : b + \frac{\triangle}{r_2} : c + \frac{\triangle}{r_3}\right).$$

Since the intouch point A_1 has coordinates $\left(0 : \frac{1}{r_2} : \frac{1}{r_3}\right)$, the line T_aA_1 clearly contains the point

$$E = \left(a + \frac{2\triangle}{r_1} : b + \frac{2\triangle}{r_2} : c + \frac{2\triangle}{r_3}\right).$$

Similarly, the lines T_bB_1 and T_cC_1 also contain the same point E, which is therefore the perspector of the triangles $T_aT_bT_c$ and the intouch triangle. This is the Eppstein pont X_{481} in [3]. See also [2]. It is clear that E also lies on the Soddy line. See Figure 4.

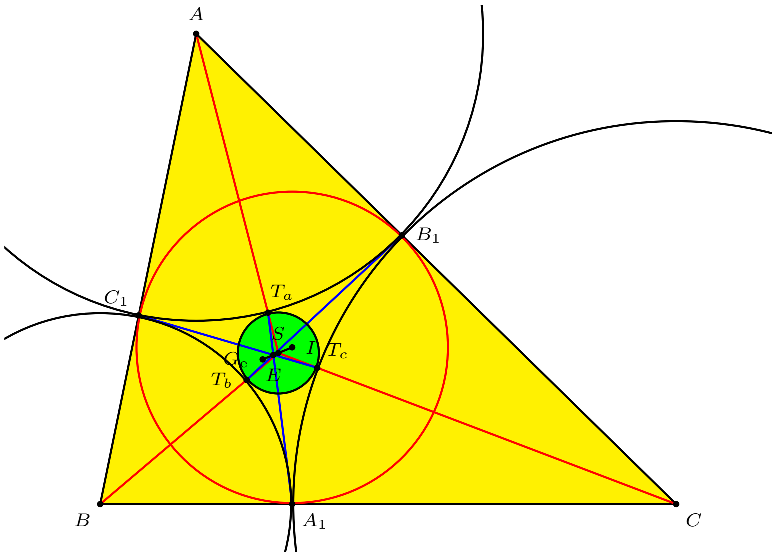

Figure 4.

The triangle $T'_aT'_bT'_c$ is also perspective with the intouch triangle, at a point

$$E' = \left(a - \frac{2\triangle}{r_1} : b - \frac{2\triangle}{r_2} : c - \frac{2\triangle}{r_3}\right),$$

on the Soddy line, dividing with E the segment IG_e harmonically. This is the second Eppstein point X_{482} of [3].

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