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Another Verification of Fagnano's Theorem

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Abstract. We present a trigonometrical proof of Fagnano's theorem which states that, among all inscribed triangles in a given acute-angled triangle, the feet of its altitudes are the vertices of the one with the least perimeter.

1. Introduction

At the outset, and to avoid ambiguity, we fix the following terminology. Let ABC be any triangle. The feet of its altitudes are the vertices of what we call its *orthic* triangle, and, if X, Y, and Z, respectively, are interior points of the sides AB, BC, and CA, respectively, we call the triangle XYZ an *inscribed triangle* of ABC.

In 1775, Fagnano proved the following theorem.

Theorem 1. *Suppose ABC is an acute-angled triangle. Of all inscribed triangles in ABC, its orthic triangle has the smallest perimeter.*

Not surprisingly, over the years this beautiful result has attracted the attentions of many mathematicians, and there are several proofs known of it [1]. Fagnano himself apparently used differential calculus to prove it, though, by modern standards, it seems to me that this is far from being a routine exercise. Perhaps the most appealing proofs of the theorem are those based on the Reflection Principle, and two of these, in particular, due independently to L. Fejér and H. A. Schwarz, have made their appearance in several books aimed at general audiences [2], [3], [4], [6]. A proof based on vector calculus appeared recently [5]. The purpose of this note is to offer one based on trigonometry.

Theorem 2. *Let ABC be any triangle, with $a = |BC|, b = |CA|, c = |AB|$, and area Δ. If XYZ is inscribed in ABC, then*

$$|XY| + |YZ| + |ZX| \geq \frac{8\Delta^2}{abc}. \tag{1}$$

Equality holds in (1) *if and only if ABC is acute-angled; and then only if XYZ is its orthic triangle. If ABC is right-angled* (*respectively, obtuse-angled*), *and C*

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is the right-angle (respectively, the obtuse-angle), then an inequality stronger than (1) holds, viz.,

$$|XY| + |YZ| + |ZX| > 2h_c, \tag{2}$$

where h_c denotes the length of the altitude from C; and, in either case, this estimate is best possible.

2. Proof of Theorem 2

Let XYZ be a triangle inscribed in ABC. Let $x = |BX|, y = |CY|, z = |AZ|$. Then $0 < x < a$, $0 < y < b$, $0 < z < c$. By applying the Cosine Rule in the triangle ZBX we have

$$\begin{aligned} |ZX|^2 &= (c-z)^2 + x^2 - 2x(c-z)\cos B \\ &= (c-z)^2 + x^2 + 2x(c-z)\cos(A+C) \\ &= (x\cos A + (c-z)\cos C)^2 + (x\sin A - (c-z)\sin C)^2. \end{aligned}$$

Hence,

$$|ZX| \geq |x\cos A + (c-z)\cos C|\,,$$

with equality if and only if $x\sin A = (c-z)\sin C$, *i.e.*, if and only if

$$ax + cz = c^2, \tag{3}$$

by the Sine Rule. Similarly,

$$|XY| \geq |y\cos B + (a-x)\cos A|\,,$$

with equality if and only if

$$ax + by = a^2. \tag{4}$$

And

$$|YZ| \geq |z\cos C + (b-y)\cos B|\,,$$

with equality if and only if

$$by + cz = b^2. \tag{5}$$

Thus, by the triangle inequality for real numbers,

$$\begin{aligned} &|XY| + |YZ| + |ZX| \\ &\geq |y\cos B + (a-x)\cos A| + |z\cos C + (b-y)\cos B| + |x\cos A + (c-z)\cos C| \\ &\geq |y\cos B + (a-x)\cos A + z\cos C + (b-y)\cos B + x\cos A + (c-z)\cos C| \\ &= |a\cos A + b\cos B + c\cos C| \\ &= \frac{|a^2(b^2+c^2-a^2) + b^2(c^2+a^2-b^2) + c^2(a^2+b^2-c^2)|}{2abc} \\ &= \frac{8\Delta^2}{abc}. \end{aligned}$$

This proves (1). Moreover, there is equality here if and only if equations (3), (4), and (5) hold, and the expressions

$$u = x\cos A + (c - z)\cos C,$$
$$v = y\cos B + (a - x)\cos A,$$
$$w = z\cos C + (b - y)\cos B,$$

are either all non-negative or all non-positive. Now it is easy to verify that the system of equations (3), (4), and (5), has a unique solution given by

$$x = c\cos B,\ \ y = a\cos C,\ \ z = b\cos A,$$

in which case

$$u = b\cos B,\ v = c\cos C,\ w = a\cos A.$$

Thus, in this case, at most one of u, v, w can be non-positive. But, if one of u, v, w is zero, then one of x, y, z must be zero, which is not possible. It follows that

$$|XY| + |YZ| + |ZX| > \frac{8\Delta^2}{abc},$$

unless ABC is acute-angled, and XYZ is its orthic triangle. If ABC is acute-angled, then $\frac{8\Delta^2}{abc}$ is the perimeter of its orthic triangle, in which case we recover Fagnano's theorem, equality being attained in (1) when and only when XYZ is the orthic triangle.

Turning now to the case when ABC is not acute-angled, suppose first that C is a right-angle. Then

$$|XY| + |YZ| + |ZX| > \frac{8\Delta^2}{abc} = \frac{4\Delta}{c} = 2h_c,$$

and so (2) holds in this case. Next, if C is an obtuse-angle, denote by D and E, respectively, the points of intersection of the side AB and the lines through C that are perpendicular to the sides BC and CA, respectively. Then Z is an interior point of one of the line segments $[B, D]$ and $[E, A]$. Suppose, for definiteness, that Z is an interior point of $[B, D]$. If Y' is the point of intersection of $[X, Y]$ and $[C, D]$, then

$$\begin{aligned} |XY| + |YZ| + |ZX| &= |XY'| + |Y'Y| + |YZ| + |ZX| \\ &> |XY'| + |Y'Z| + |ZX| \\ &> 2h_c, \end{aligned}$$

since the triangle $XY'Z$ is inscribed in the right-angled triangle BCD. A similar argument works if Z is an interior point of $[E, A]$. Hence, (2) also holds if C is obtuse.

That (2) is stronger than (1), for a non acute-angled triangle, follows from the fact that, in any triangle ABC,

$$\frac{4\Delta^2}{abc} = \frac{2\Delta\sin C}{c} = a\sin B\sin C \le a\sin B = h_c.$$

It remains to prove that inequality (2) cannot be improved when the angle C is right or obtuse. To see this, let Z be the foot of the perpendicular from C to AB, and $0 < \varepsilon < 1$. Choose Y on CA so that $|CY| = \varepsilon b$, and X on BC so that XY is parallel to AB. Then, as $\varepsilon \to 0^+$, both X and Y converge to C, and so

$$\lim_{\varepsilon \to 0^+} (|XY| + |YZ| + |ZX|) = |CC| + |CZ| + |ZC| = 2|CZ| = 2h_c.$$

This finishes the proof.

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