Forum Geometricorum
Volume 7 (2007) 211–229.

FORUM GEOM
ISSN 1534-1178

How Pivotal Isocubics Intersect the Circumcircle

Bernard Gibert

Abstract. Given the pivotal isocubic $\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$, we seek its common points with the circumcircle and we also study the tangents at these points.

1. Introduction

A pivotal cubic $\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$ with pole Ω, pivot P, is the locus of point M such that P, M and its Ω-isoconjugate M^* are collinear. It is also the locus of point M such that P^* (the isopivot or secondary pivot), M and the cevian quotient P/M are collinear. See [2] for more information. [1] The isocubic $\mathcal{K}$ meets the circumcircle $(\mathcal{O})$ of the reference triangle ABC at its vertices and three other points Q_1, Q_2, Q_3, one of them being always real. This paper is devoted to a study of these points and special emphasis on their tangents.

2. Isogonal pivotal cubics

We first consider the case where the pivotal isocubic $\mathcal{K} = \mathrm{p}\mathcal{K}(X_6, P)$ is isogonal with pole the Lemoine point K.

2.1. *Circular isogonal cubics.* When the pivot P lies at infinity, $\mathcal{K}$ contains the two circular points at infinity. Hence it is a circular cubic of the class **CL035** in [3], and has only one real intersection with $(\mathcal{O})$. This is the isogonal conjugate P^* of the pivot.

The tangent at P is the real asymptote PP^* of the cubic and the isotropic tangents meet at the singular focus F of the circular cubic. F is the antipode of P^* on $(\mathcal{O})$.

The pair P and P^* are the foci of an inscribed conic, which is a parabola with focal axis PP^*. When P traverses the line at infinity, this axis envelopes the deltoid $\mathcal{H}_3$ tritangent to $(\mathcal{O})$ at the vertices of the circumtangential triangle. The contact of the deltoid with this axis is the reflection in P^* of the second intersection of PP^* with the circumcircle. See Figure 1 with the Neuberg cubic **K001** and the Brocard cubic **K021**. For example, with the Neuberg cubic, $P^* = X_{74}$, the second point on the axis is X_{476}, the contact is the reflection of X_{476} in X_{74}.

Publication Date: December 10, 2007. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this paper.

[1] Most of the cubics cited here are now available on the web-site http://perso.orange.fr/bernard.gibert/index.html, where they are referenced under a catalogue number of the form **Knnn**.

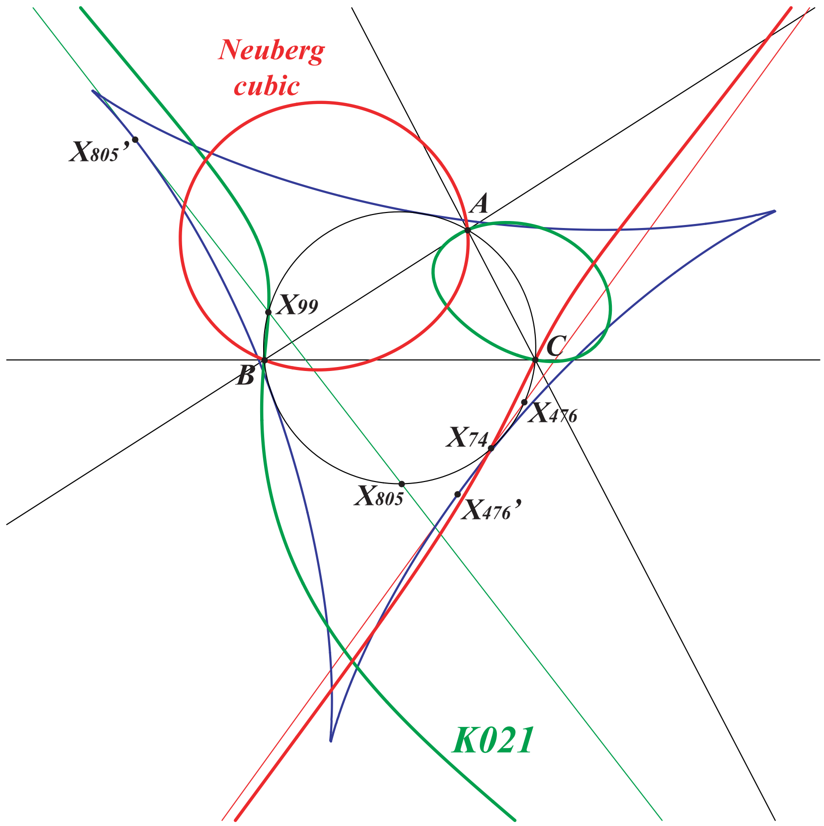

Figure 1. Isogonal circular cubic with pivot at infinity

2.2. *Isogonal cubics with pivot on the circumcircle.* When P lies on $(\mathcal{O})$, the remaining two intersections Q_1, Q_2 are antipodes on $(\mathcal{O})$. They lie on the perpendicular at O to the line PP^* or the parallel at O to the Simson line of P. The isocubic $\mathcal{K}$ has three real asymptotes:
(i) One is the parallel at P/P^* (cevian quotient) to the line PP^*.
(ii) The two others are perpendicular and can be obtained as follows. Reflect P in Q_1, Q_2 to get S_1, S_2 and draw the parallels at S_1^*, S_2^* to the lines PQ_1, PQ_2. These asymptotes meet at X on the line OP. Note that the tangent to the cubic at Q_1, Q_2 are the lines $Q_1S_1^*$, $Q_2S_2^*$. See Figure 2.

2.3. *The general case.* In both cases above, the orthocenter of the triangle formed by the points Q_1, Q_2, Q_3 is the pivot P of the cubic, although this triangle is not a proper triangle in the former case and a right triangle in the latter case. More generally, we have the following

Theorem 1. *For any point P, the isogonal cubic $\mathcal{K} = \mathrm{p}\mathcal{K}(X_6, P)$ meets the circumcircle at A, B, C and three other points Q_1, Q_2, Q_3 such that P is the orthocenter of the triangle $Q_1Q_2Q_3$.*

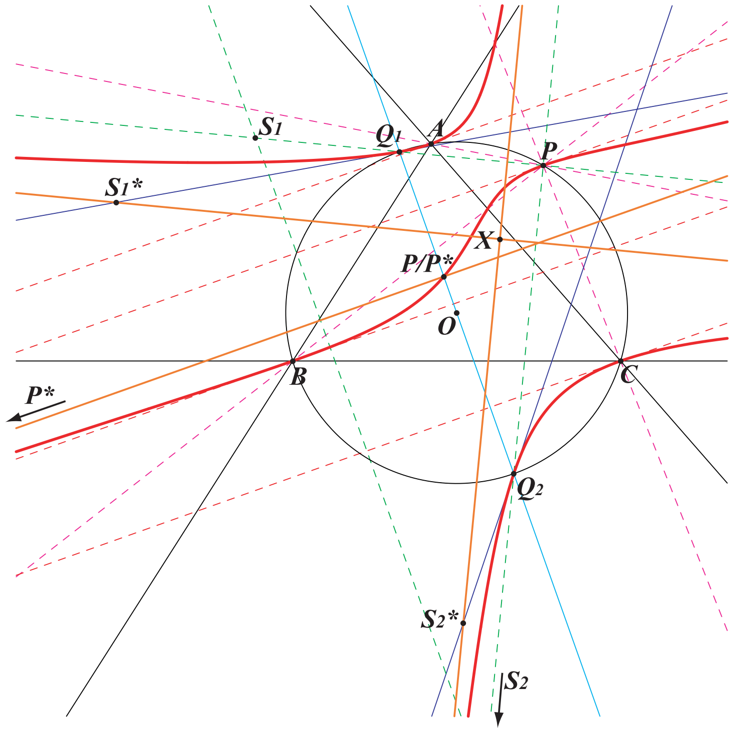

Figure 2. Isogonal cubic with pivot on the circumcircle

Proof. The lines $Q_1Q_1^*$, $Q_2Q_2^*$, $Q_3Q_3^*$ pass through the pivot P and are parallel to the asymptotes of the cubic. Since they are the axes of three inscribed parabolas, they must be tangent to the deltoid $\mathcal{H}_3$, the anticomplement of the Steiner deltoid. This deltoid is a bicircular quartic of class 3. Hence, for a given P, there are only three tangents (at least one of which is real) to the deltoid passing through P.

According to a known result, Q_1 must be the antipode on $(\mathcal{O})$ of Q_1', the isogonal conjugate of the infinite point of the line Q_2Q_3. The Simson lines of Q_1', Q_2, Q_3 are concurrent. Hence, the axes are also concurrent at P. But the Simson line of Q_1 is parallel to Q_2Q_3. Hence $Q_1Q_1^*$ is an altitude of $Q_1Q_2Q_3$. This completes the proof. See Figure 3. □

Remark. These points Q_1, Q_2, Q_3 are not necessarily all real nor distinct. In [1], H. M. Cundy and C. F. Parry have shown that this depends of the position of P with respect to $\mathcal{H}_3$. More precisely, these points are all real if and only if P lies strictly inside $\mathcal{H}_3$. One only is real when P lies outside $\mathcal{H}_3$. This leaves a special case when P lies on $\mathcal{H}_3$. See §2.4.

Recall that the contacts of the deltoid $\mathcal{H}_3$ with the line $PQ_1Q_1^*$ is the reflection in Q_1 of the second intersection of the circumcircle and the line $PQ_1Q_1^*$. Consequently, every conic passing through P, Q_1, Q_2, Q_3 is a rectangular hyperbola and all these hyperbolas form a pencil $\mathcal{F}$ of rectangular hyperbolas.

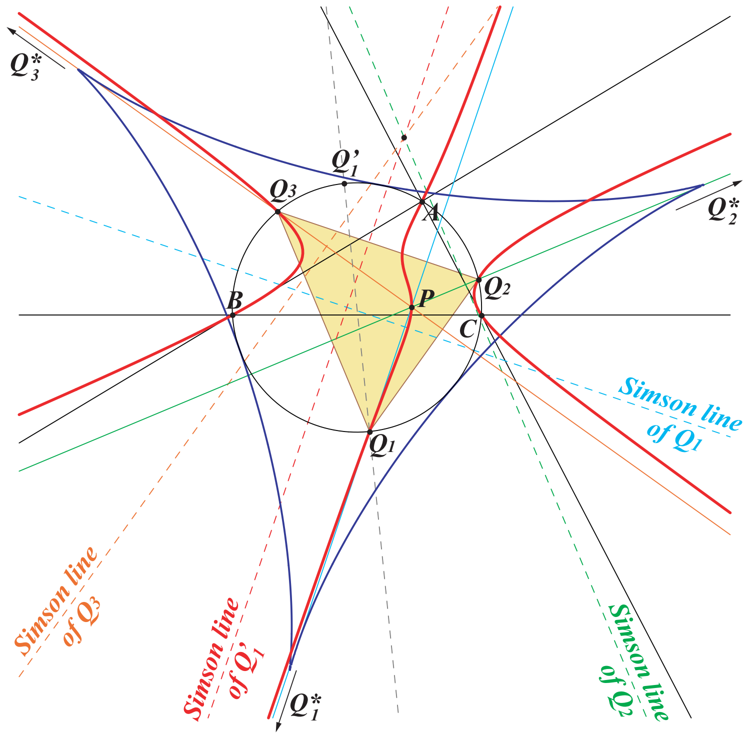

Figure 3. The deltoid $\mathcal{H}_3$ and the points Q_1, Q_2, Q_3

Let $\mathcal{D}$ be the diagonal rectangular hyperbola which contains the four in/excenters of ABC, P^*, and P/P^*. Its center is $\Omega_\mathcal{D}$. Note that the tangent at P^* to $\mathcal{D}$ contains P and the tangent at P/P^* to $\mathcal{D}$ contains P. In other words, the polar line of P in $\mathcal{D}$ is the line through P^* and P/P^*.

The pencil $\mathcal{F}$ contains the hyperbola $\mathcal{H}$ passing through P, P^*, P/P^* and $\Omega_\mathcal{D}$ having the same asymptotic directions as $\mathcal{D}$. The center of $\mathcal{H}$ is the midpoint of P and $\Omega_\mathcal{D}$. This gives an easy conic construction of the points Q_1, Q_2, Q_3 when P is given. See Figure 4. The pencil $\mathcal{F}$ contains another very simple rectangular hyperbola $\mathcal{H}'$, which is the homothetic of the polar conic of P in $\mathcal{K}$ under $\mathsf{h}\left(P, \frac{1}{2}\right)$. Since this polar conic is the diagonal conic passing through the in/excenters and P, $\mathcal{H}'$ contains P and the four midpoints of the segments joining P to the in/excenters.

Corollary 2. *The isocubic $\mathcal{K}$ contains the projections R_1, R_2, R_3 of P^* on the sidelines of $Q_1Q_2Q_3$. These three points lie on the bicevian conic $\mathcal{C}(G, P)$.* [2]

Proof. Let R_1 be the third point of $\mathcal{K}$ on the line Q_2Q_3. The following table shows the collinearity relations of nine points on $\mathcal{K}$ and proves that P^*, R_1 and Q_1^* are collinear.

---

[2]This is the conic through the vertices of the cevian triangles of G and P. This is the P-Ceva conjugate of the line at infinity.

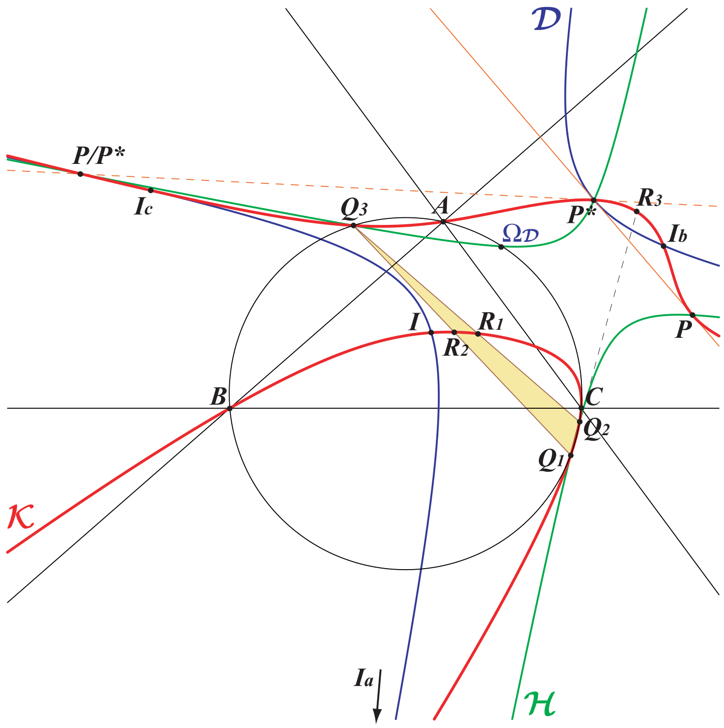

Figure 4. The hyperbolas $\mathcal{H}$ and $\mathcal{D}$

P	P	P^*	$\leftarrow P^*$ is the tangential of P
Q_2	Q_3	R_1	$\leftarrow$ definition of R_1
Q_2^*	Q_3^*	Q_1^*	$\leftarrow$ these three points lie at infinity

This shows that, for $i = 1, 2, 3$, the points P^*, R_i and Q_i^* are collinear and, since P, Q_i and Q_i^* are also collinear, the lines PQ_i and P^*R_i are parallel. It follows from Theorem 1 that R_i is the projection of P^* onto the line R_jR_k.

Recall that P^* is the secondary pivot of $\mathcal{K}$ hence, for any point M on $\mathcal{K}$, the points P^*, M and P/M (cevian quotient) are three collinear points on $\mathcal{K}$. Consequently, $R_i = P/Q_i^*$ and, since Q_i^* lies at infinity, R_i is a point on $\mathcal{C}(G, P)$. □

Corollary 3. *The lines* $Q_iR_i^*$, $i = 1, 2, 3$, *pass through the cevian quotient* P/P^*.

Proof. This is obvious from the following table.

P^*	P^*	P	$\leftarrow P/P^*$ is the tangential of P^*
P	Q_1^*	Q_1	$\leftarrow Q_1Q_1^*$ must contain the pivot P
P	R_1	R_1^*	$\leftarrow R_1R_1^*$ must contain the pivot P

Recall that P^* is the tangential of P (first column). The second column is the corollary above. □

Corollary 4. *Let S_1, S_2, S_3 be the reflections of P in Q_1, Q_2, Q_3 respectively. The asymptotes of $\mathcal{K}$ are the parallel at S_i^* to the lines PQ_i or P^*R_i.*

Proof. These points S_i lie on the polar conic of the pivot P since they are the harmonic conjugate of P with respect to Q_i and Q_i^*. The construction of the asymptotes derives from [2, §1.4.4]. □

Theorem 5. *The inconic $\mathcal{I}(P)$ concentric with $\mathcal{C}(G, P)$* [3] *is also inscribed in the triangle $Q_1Q_2Q_3$ and in the triangle formed by the Simson lines of Q_1, Q_2, Q_3.*

Proof. Since the triangles ABC and $Q_1Q_2Q_3$ are inscribed in the circumcircle, there must be a conic inscribed in both triangles. The rest is mere calculation. □

In [4, §29, p.88], A. Haarbleicher remarks that the triangle ABC and the reflection of $Q_1Q_2Q_3$ in O circumscribe the same parabola. These two parabolas are obviously symmetric about O. Their directrices are the line through H and the reflection P' of P in O in the former case, and its reflection in O in the latter case. The foci are the isogonal conjugates of the infinite points of these directrices and its reflection about O.

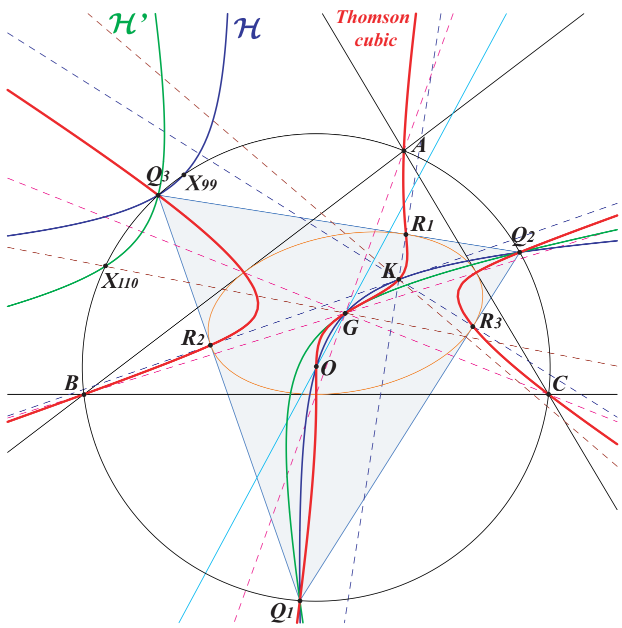

Figure 5. Thomson cubic

[3]This center is the complement of the complement of P, *i.e.*, the homothetic of P under $\mathsf{h}(G, \frac{1}{4})$. Note that these two conics $\mathcal{I}(P)$ and $\mathcal{C}(G, P)$ are bitangent at two points on the line GP. When $P = G$, they coincide since they both are the Steiner in-ellipse.

For example, Figures 5 and 6 show the case $P = G$. Note, in particular,
– $\mathcal{K}$ is the Thomson cubic,
– $\mathcal{D}$ is the Steiner (or Don Wallace) hyperbola,
– $\mathcal{H}$ contains X_2, X_3, X_6, X_{110}, X_{154}, X_{354}, X_{392}, X_{1201}, X_{2574}, X_{2575},
– $\mathcal{H}'$ contains X_2, X_{99}, X_{376}, X_{551},
– the inconic $\mathcal{I}(P)$ and the bicevian conic $\mathcal{C}(G, P)$ are the Steiner in-ellipse,
– the two parabolas are the Kiepert parabola and its reflection in O.

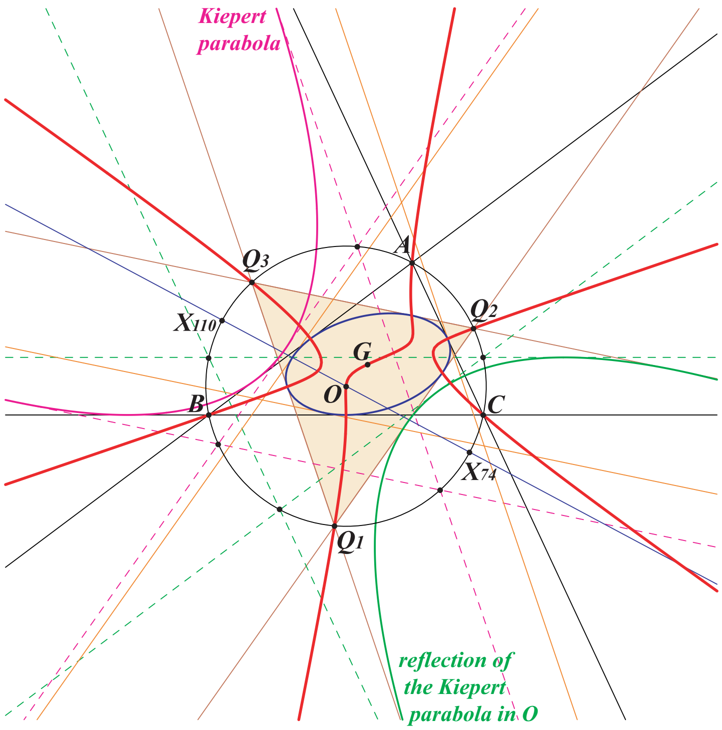

Figure 6. The Thomson cubic and the two parabolas

More generally, any $\mathrm{p}\mathcal{K}(X_6, P)$ with pivot P on the Euler line is obviously associated to the same two parabolas. In other words, any cubic of the Euler pencil meets the circumcircle at three (not always real) points Q_1, Q_2, Q_3 such that the reflection of the Kiepert parabola in O is inscribed in the triangle $Q_1Q_2Q_3$ and in the circumcevian triangle of O.

In particular, taking $P = O$, we obtain the McCay cubic and this shows that the reflection of the Kiepert parabola in O is inscribed in the circumnormal triangle.

Another interesting case is $\mathrm{p}\mathcal{K}(X_6, X_{145})$ in Figure 7 since the incircle is inscribed in the triangle $Q_1Q_2Q_3$.

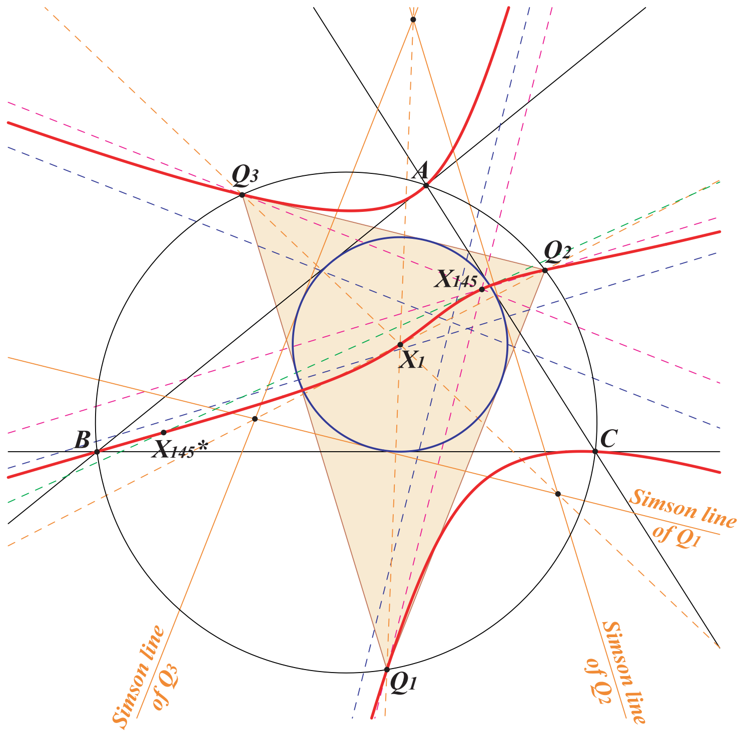

Figure 7. $\mathrm{p}\mathcal{K}(X_6, X_{145})$

2.4. *Isogonal pivotal cubics tangent to the circumcircle.* In this section, we take P on $\mathcal{H}_3$ so that $\mathcal{K}$ has a multiple point at infinity.

Here is a special case. $\mathcal{H}_3$ is tangent to the six bisectors of ABC. If we take the bisector AI, the contact P is the reflection of A in the second intersection A_i of AI with the circumcircle. The corresponding cubic $\mathcal{K}$ is the union of the bisector AI and the conic passing through B, C, the excenters I_b and I_c, A_i, the antipode of A on the circumcircle.

Let us now take M on the circle $\mathcal{C}_H$ with center H, radius $2R$ and let us denote by $\mathcal{T}_M$ the tangent at M to $\mathcal{C}_H$. The orthopole P of $\mathcal{T}_M$ with respect to the antimedial triangle is a point on $\mathcal{H}_3$.

The corresponding cubic $\mathcal{K}$ meets $(\mathcal{O})$ at P_1 (double) and P_3. The common tangent at P_1 to $\mathcal{K}$ and $(\mathcal{O})$ is parallel to $\mathcal{T}_M$. Note that P_1 lies on the Simson line $\mathcal{S}_P$ of P with respect to the antimedial triangle.

The perpendicular at P_1 to $\mathcal{S}_P$ meets $(\mathcal{O})$ again at P_3 which is the antipode on $(\mathcal{O})$ of the second intersection Q_3 of $\mathcal{S}_P$ and $(\mathcal{O})$. The Simson line of P_3 is parallel to $\mathcal{T}_M$.

It follows that $\mathcal{K}$ has a triple common point with $(\mathcal{O})$ if and only if P_1 and Q_3 are antipodes on $(\mathcal{O})$ i.e. if and only if $\mathcal{S}_P$ passes through O. This gives the following theorem.

Theorem 6. *There are exactly three isogonal pivotal cubics which are tridents.*

Their pivots are the cusps of the deltoid $\mathcal{H}_3$. The triple contacts with $(\mathcal{O})$ are the vertices of the circumnormal triangle.[4] These points are obviously inflexion points and the inflexional tangent is parallel to a sideline of the Morley triangle. See Figure 8.

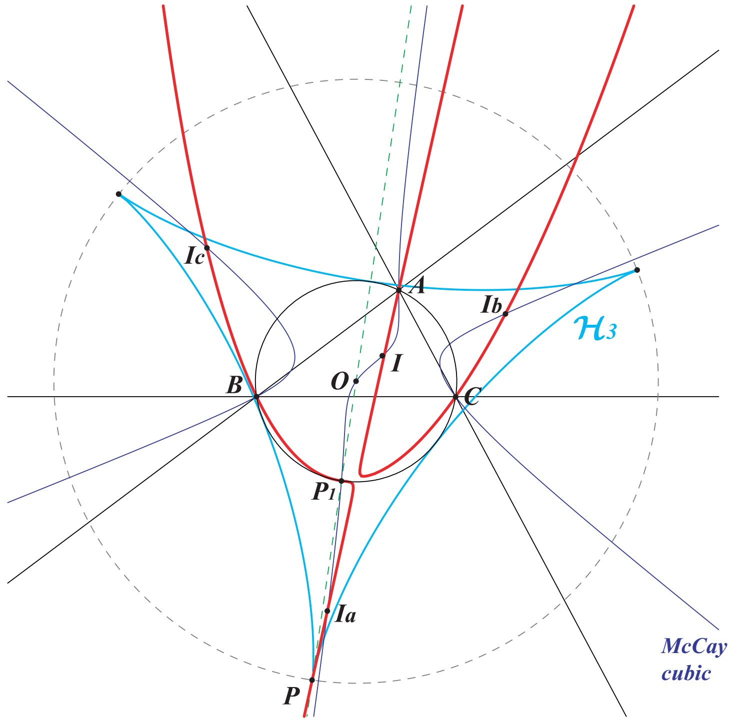

Figure 8. Isogonal pivotal trident

2.5. *Tangents at* Q_1, Q_2, Q_3. We know that the tangents at A, B, C to any pivotal cubic concur at P^*. This is not necessarily true for those at Q_1, Q_2, Q_3.

Theorem 7. *The tangents at* Q_1, Q_2, Q_3 *to the isogonal cubic* $\mathrm{p}\mathcal{K}(X_6, P)$ *concur if and only if* P *lies on the quintic* **Q063** *with equation*

$$\sum_{\text{cyclic}} a^2y^2z^2\left(S_C(x+y) - S_B(x+z)\right) = 0.$$

Q063 is a circular quintic with singular focus X_{376}, the reflection of G in O. It has three real asymptotes parallel to those of the Thomson cubic and concurrent at G.

[4]These three points are the common points of the circumcircle and the McCay cubic apart A, B, C.

A, B, C are nodes and the fifth points on the sidelines of ABC are the vertices A', B', C' of the pedal triangle of X_{20}, the de Longchamps point. The tangents at these points pass through X_{20} and meet the corresponding bisectors at six points on the curve. See Figure 9.

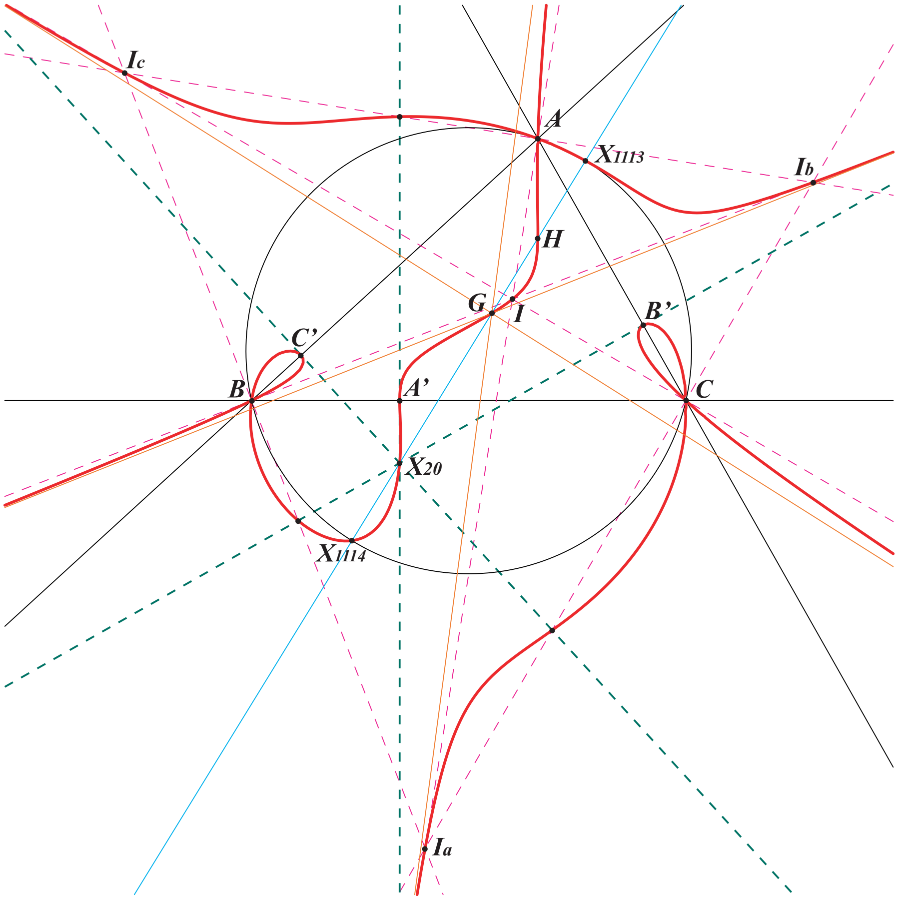

Figure 9. The quintic **Q063**

Q063 contains I, the excenters, G, H, X_{20}, X_{1113}, X_{1114}. Hence, for the Thomson cubic, the orthocubic, and the Darboux cubic, the tangents at Q_1, Q_2, Q_3 concur. The intersection of these tangents are X_{25} for the orthocubic, and X_{1498} for the Darboux cubic. For the Thomson cubic, this is an unknown point[5] in the current edition of ETC on the line GX_{1350}.

[5]This has first barycentric coordinate
$$a^2(3S_A^2 + 2a^2S_A + 5b^2c^2).$$

3. Non-isogonal pivotal cubics

We now consider a non-isogonal pivotal cubic $\mathcal{K}$ with pole $\omega \neq K$ and pivot π.

We recall that π^* is the $\omega-$isoconjugate of π and that π/π^* is the cevian quotient of π and π^*, these three points lying on the cubic.

3.1. *Circular cubics.* In this special case, two of the points, say Q_2 and Q_3, are the circular points at infinity. This gives already five common points of the cubic on the circumcircle and the sixth point Q_1 must be real.

The isoconjugation with pole ω swaps the pivot π and the isopivot π^* which must be the inverse (in the circumcircle of ABC) of the isogonal conjugate of π. In this case, the cubic contains the point T, isogonal conjugate of the complement of π. This gives the following

Theorem 8. *A non isogonal circular pivotal cubic $\mathcal{K}$ meets the circumcircle at A, B, C, the circular points at infinity and another (real) point Q_1 which is the second intersection of the line through T and π/π^* with the circle passing through π, π^* and π/π^*.*

Example: The Droussent cubic **K008**. This is the only circular isotomic pivotal cubic. See Figure 10.

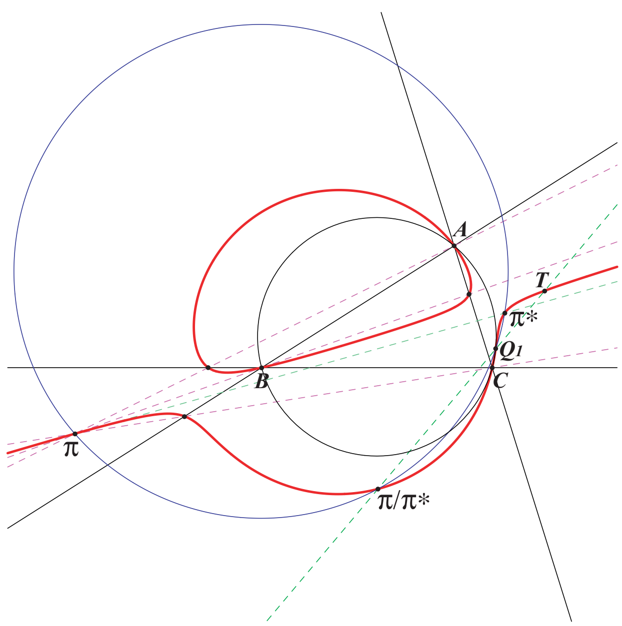

Figure 10. The Droussent cubic **K008**

The points π, π^*, T, Q_1 are X_{316}, X_{67}, X_{671}, X_{2373} respectively. The point π/π^* is not mentioned in the current edition of [6].

Note that when $\pi = H$, there are infinitely many circular pivotal cubics with pivot H, with isopivot π^* at infinity. These cubics are the isogonal circular pivotal cubics with respect to the orthic triangle. They have their singular focus F on the nine point circle and their pole ω on the orthic axis. The isoconjugate H^* of H is the point at infinity of the cubic. The intersection with their real asymptote is X, the antipode of F on the nine point circle and, in this case, $X = \pi/\pi^*$. This asymptote envelopes the Steiner deltoid $\mathcal{H}_3$. The sixth point Q_1 on the circumcircle is the orthoassociate of X, i.e. the inverse of X in the polar circle.

Example: The Neuberg orthic cubic **K050**. This is the Neuberg cubic of the orthic triangle. See [3].

3.2. *General theorems for non circular cubics.*

Theorem 9. *$\mathcal{K}$ meets the circumcircle at A, B, C and three other points Q_1, Q_2, Q_3 (one at least is real) lying on a same conic passing through π, π^* and π/π^*.*

Note that this conic meets the circumcircle again at the isogonal conjugate of the infinite point of the trilinear polar of the isoconjugate of ω under the isoconjugation with fixed point π.

With $\omega = p : q : r$ and $\pi = u : v : w$, this conic has equation

$$\sum_{\text{cyclic}} p^2v^2w^2(c^2y+b^2z)(wy-vz)+qru^2x(vw(c^2v-b^2w)x+u(b^2w^2y-c^2v^2z)) = 0,$$

and the point on the circumcircle is :

$$\frac{a^2}{u^2(rv^2-qw^2)} : \frac{b^2}{v^2(pw^2-ru^2)} : \frac{c^2}{w^2(qu^2-pv^2)}$$

Theorem 10. *The conic inscribed in triangles ABC and $Q_1Q_2Q_3$ is that with perspector the cevian product of π and $tg\omega$, the isotomic of the isogonal of ω.*

3.3. *Relation with isogonal pivotal cubics.*

Theorem 11. *$\mathcal{K}$ meets the circumcircle at the same points as the isogonal pivotal cubic with pivot $P = u : v : w$ if and only if its pole ω lie on the cubic $\mathcal{K}_{\text{pole}}$ with equation*

$$\sum_{\text{cyclic}} (v+w)(c^4y-b^4z)\,\frac{x^2}{a^2} - \left(\sum_{\text{cyclic}} (b^2-c^2)u\right) xyz = 0$$

$$\iff \sum_{\text{cyclic}} a^2u(c^2y-b^2z)(-a^4yz+b^4zx+c^4xy) = 0.$$

In other words, for any point ω on $\mathcal{K}_{\text{pole}}$, there is a pivotal cubic with pole ω meeting the circumcircle at the same points as the isogonal pivotal cubic with pivot $P = u : v : w$.

$\mathcal{K}_{\text{pole}}$ is a circum-cubic passing through K, the vertices of the cevian triangle of gc$P$, the isogonal conjugate of the complement of P. The tangents at A, B, C are the cevians of X_{32}.

The second equation above clearly shows that all these cubics belong to a same net of circum-cubics passing through K having the same tangents at A, B, C.

This net can be generated by three decomposed cubics, one of them being the union of the symmedian AK and the circum-conic with perspector the A-harmonic associate of X_{32}.

For example, with $P = H$, $\mathcal{K}_{\text{pole}}$ is a nodal cubic with node K and nodal tangents parallel to the asymptotes of the Jerabek hyperbola. It contains X_6, X_{66}, X_{193}, X_{393}, X_{571}, X_{608}, X_{1974}, X_{2911} which are the poles of cubics meeting the circumcircle at the same points as the orthocubic **K006**.

Theorem 12. *$\mathcal{K}$ meets the circumcircle at the same points as the isogonal pivotal cubic with pivot $P = u : v : w$ if and only if its pivot π lie on the cubic $\mathcal{K}_{\text{pivot}}$ with equation*

$$\sum_{\text{cyclic}} (v+w)(c^4 y - b^4 z)\; x^2 + \left(\sum_{\text{cyclic}} (b^2 - c^2) u \right) xyz = 0.$$

In other words, for any point π on $\mathcal{K}_{\text{pivot}}$, there is a pivotal cubic with pivot π meeting the circumcircle at the same points as the isogonal pivotal cubic with pivot $P = u : v : w$.

$\mathcal{K}_{\text{pivot}}$ is a circum-cubic tangent at A, B, C to the symmedians. It passes through P, the points on the circumcircle and on the isogonal pivotal cubic with pivot P, the infinite points of the isogonal pivotal cubic with pivot the complement of P, the vertices of the cevian triangle of tc$P$, the isotomic conjugate of the complement of P.

Following the example above, with $P = H$, $\mathcal{K}_{\text{pivot}}$ is also a nodal cubic with node H and nodal tangents parallel to the asymptotes of the Jerabek hyperbola. It contains X_3, X_4, X_8, X_{76}, X_{847} which are the pivots of cubics meeting the circumcircle at the same points as the orthocubic, three of them being $\text{p}\mathcal{K}(X_{193}, X_{76})$, $\text{p}\mathcal{K}(X_{571}, X_3)$ and $\text{p}\mathcal{K}(X_{2911}, X_8)$.

Remark. Adding up the equations of $\mathcal{K}_{\text{pole}}$ and $\mathcal{K}_{\text{pivot}}$ shows that these two cubics generate a pencil containing the $\text{p}\mathcal{K}$ with pole the X_{32}-isoconjugate of cP, pivot the X_{39}-isoconjugate of cP and isopivot X_{251}.

For example, with $P = X_{69}$, this cubic is $\text{p}\mathcal{K}(X_6, X_{141})$. The nine common points of all the cubics of the pencil are A, B, C, K, X_{1169} and the four foci of the inscribed ellipse with center X_{141}, perspector X_{76}.

3.4. *Pivotal $\mathcal{K}_{\text{pole}}$ and $\mathcal{K}_{\text{pivot}}$.* The equations of $\mathcal{K}_{\text{pole}}$ and $\mathcal{K}_{\text{pivot}}$ clearly show that these two cubics are pivotal cubics if and only if P lies on the line GK. This gives the two following corollaries.

Corollary 13. *When P lies on the line GK, $\mathcal{K}_{\text{pole}}$ is a pivotal cubic and contains K, X_{25}, X_{32}. Its pivot is gc$P$ (on the circum-conic through G and K) and its isopivot is X_{32}. Its pole is the barycentric product of X_{32} and gc$P$. It lies on the circum-conic through X_{32} and X_{251}.*

All these cubics belong to a same pencil of pivotal cubics. Furthermore, $\mathcal{K}_{\text{pole}}$ contains the cevian quotients of the pivot gc$P$ and K, X_{25}, X_{32}. Each of these points is the third point of the cubic on the corresponding sideline of the triangle with vertices K, X_{25}, X_{32}. In particular, X_{25} gives the point gt$P$.

Table 1 shows a selection of these cubics.

P	$\mathcal{K}_{\rm pole}$ contains K, X_{25}, X_{32} and	cubic
X_2	X_{31}, X_{41}, X_{184}, X_{604}, X_{2199}	**K346**
X_{69}	X_2, X_3, X_{66}, X_{206}, X_{1676}, X_{1677}	**K177**
X_{81}	X_{1169}, X_{1333}, X_{2194}, X_{2206}	
X_{86}	X_{58}, X_{1171}	
X_{193}	X_{1974}, X_{3053}	
X_{298}	X_{15}, X_{2981}	
X_{323}	X_{50}, X_{1495}	
X_{325}	X_{511}, X_{2987}	
X_{385}	X_{1691}, X_{1976}	
X_{394}	X_{154}, X_{577}	
X_{491}	X_{372}, X_{589}	
X_{492}	X_{371}, X_{588}	
X_{524}	X_{111}, X_{187}	
X_{1270}	X_{493}, X_{1151}	
X_{1271}	X_{494}, X_{1152}	
X_{1654}	X_{42}, X_{1918}, X_{2200}	
X_{1992}	X_{1383}, X_{1384}	
X_{1994}	X_{51}, X_{2965}	
X_{2895}	X_{37}, X_{213}, X_{228}, X_{1030}	
atX_{1916}	X_{237}, X_{384}, X_{385}, X_{694}, X_{733}, X_{904}, X_{1911}, X_{2076}, X_{3051}	

Table 1. $\mathcal{K}_{\rm pole}$ with P on the line GK.

Remark. atX_{1916} is the anticomplement of the isotomic conjugate of X_{1916}.

Corollary 14. *When P lies on the line GK, $\mathcal{K}_{\rm pivot}$ contains P, G, H, K. Its pole is gc$P$ (on the cubic) and its pivot is tc$P$ on the Kiepert hyperbola.*

All these cubics also belong to a same pencil of pivotal cubics.

Table 2 shows a selection of these cubics.

We remark that $\mathcal{K}_{\rm pole}$ is the isogonal of the isotomic transform of $\mathcal{K}_{\rm pivot}$ but this correspondence is not generally true for the pivot π and the pole ω. To be more precise, for π on $\mathcal{K}_{\rm pivot}$, the pole ω on $\mathcal{K}_{\rm pole}$ is the Ceva-conjugate of gc$P$ and gtπ.

From the two corollaries above, we see that, given an isogonal pivotal cubic $\mathcal{K}$ with pivot on the line GK, we can always find two cubics with poles X_{25}, X_{32} and three cubics with pivots G, H, K sharing the same points on the circumcircle as $\mathcal{K}$. Obviously, there are other such cubics but their pole and pivot both depend of P. In particular, we have $\mathrm{p}\mathcal{K}$(gc$P$, tc$P$) and $\mathrm{p}\mathcal{K}(O\times$ gc$P$, gc$P)$.

We illustrate this with $P = G$ (and gc$P = K$) in which case $\mathcal{K}_{\rm pivot}$ is the Thomson cubic **K002** and $\mathcal{K}_{\rm pole}$ is **K346**. For π and ω chosen accordingly on these

P	$\mathcal{K}_{\mathrm{pivot}}$ contains X_2, X_4, X_6 and	cubic
X_2	X_1, X_3, X_9, X_{57}, X_{223}, X_{282}, X_{1073}, X_{1249}	**K002**
X_6	X_{83}, X_{251}, X_{1176}	
X_{69}	X_{22}, X_{69}, X_{76}, X_{1670}, X_{1671}	**K141**
X_{81}	X_{21}, X_{58}, X_{81}, X_{572}, X_{961}, X_{1169}, X_{1220}, X_{1798}, X_{2298}	**K379**
X_{86}	X_{86}, X_{1126}, X_{1171}	
X_{193}	X_{25}, X_{193}, X_{371}, X_{372}, X_{2362}	**K233**
X_{298}	X_{298}, X_{2981}	
X_{323}	X_{30}, X_{323}, X_{2986}	
X_{325}	X_{325}, X_{2065}, X_{2987}	
X_{385}	X_{98}, X_{237}, X_{248}, X_{385}, X_{1687}, X_{1688}, X_{1976}	**K380**
X_{394}	X_{20}, X_{394}, X_{801}	
X_{491}	X_{491}, X_{589}	
X_{492}	X_{492}, X_{588}	
X_{524}	X_{23}, X_{111}, X_{524}, X_{671}, X_{895}	**K273**
X_{1270}	X_{493}, X_{1270}	
X_{1271}	X_{494}, X_{1271}	
X_{1611}	X_{439}, X_{1611}	
X_{1654}	X_{10}, X_{42}, X_{71}, X_{199}, X_{1654}	
X_{1992}	X_{598}, X_{1383}, X_{1992}, X_{1995}	**K283**
X_{1993}	X_{54}, X_{275}, X_{1993}	
X_{1994}	X_5, X_{1166}, X_{1994}	
X_{2287}	X_{1817}, X_{2287}	
X_{2895}	X_{37}, X_{72}, X_{321}, X_{2895}, X_{2915}	
X_{3051}	X_{384}, X_{3051}	
atX_{1916}	X_{39}, X_{256}, X_{291}, X_{511}, X_{694}, X_{1432}, X_{1916}	**K354**

Table 2. $\mathcal{K}_{\mathrm{pivot}}$ with P on the line GK

cubics, we obtain a family of pivotal cubics meeting the circumcircle at the same points as the Thomson cubic. See Table 3 and Figure 11.

With $P = X_{69}$ (isotomic conjugate of H), we obtain several interesting cubics related to the centroid G=gc$P$, the circumcenter O=gt$P$. $\mathcal{K}_{\mathrm{pole}}$ is **K177**, $\mathcal{K}_{\mathrm{pivot}}$ is **K141** and the cubics $\mathrm{p}\mathcal{K}(X_2, X_{76})$ = **K141**, $\mathrm{p}\mathcal{K}(X_3, X_2)$ = **K168**, $\mathrm{p}\mathcal{K}(X_6, X_{69})$ = **K169**, $\mathrm{p}\mathcal{K}(X_{32}, X_{22})$ = **K174**, $\mathrm{p}\mathcal{K}(X_{206}, X_6)$ have the same common points on the circumcircle.

π	ω (X_i or SEARCH)	cubic or X_i on the cubic
X_1	X_{41}	$X_1, X_6, X_9, X_{55}, X_{259}$
X_2	X_6	**K002**
X_3	X_{32}	**K172**
X_4	0.1732184721703	$X_4, X_6, X_{20}, X_{25}, X_{154}, X_{1249}$
X_6	X_{184}	**K167**
X_9	X_{31}	$X_1, X_6, X_9, X_{56}, X_{84}, X_{165}, X_{198}, X_{365}$
X_{57}	X_{2199}	$X_6, X_{40}, X_{56}, X_{57}, X_{198}, X_{223}$
X_{223}	X_{604}	$X_6, X_{57}, X_{223}, X_{266}, X_{1035}, X_{1436}$
X_{282}	0.3666241407629	$X_6, X_{282}, X_{1035}, X_{1436}, X_{1490}$
X_{1073}	0.6990940852287	$X_6, X_{64}, X_{1033}, X_{1073}, X_{1498}$
X_{1249}	X_{25}	$X_4, X_6, X_{64}, X_{1033}, X_{1249}$

Table 3. Thomson cubic **K002** and some related cubics

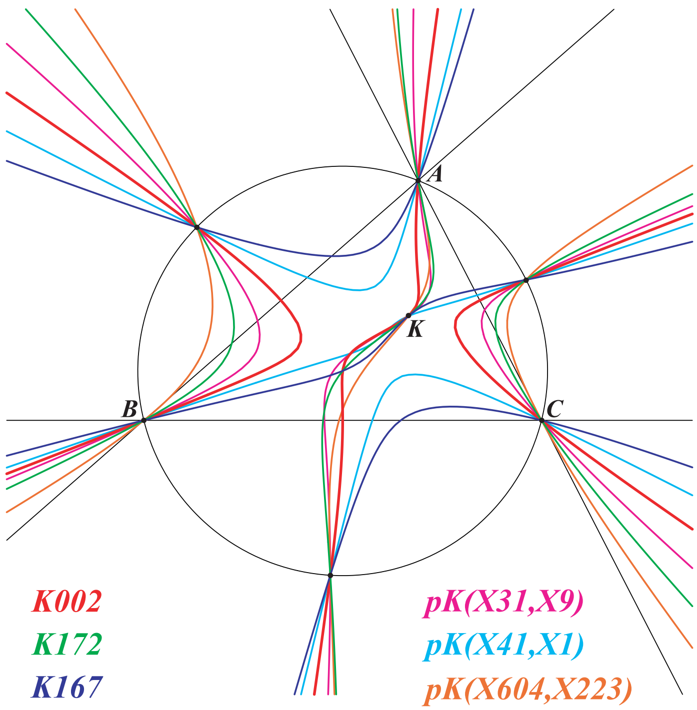

Figure 11. Thomson cubic **K002** and some related cubics

4. Non isogonal pivotal cubics and concurrent tangents

We now generalize Theorem 7 for any pivotal cubic with pole $\Omega = p : q : r$ and pivot $P = u : v : w$, meeting the circumcircle at A, B, C and three other points Q_1, Q_2, Q_3. We obtain the two following theorems.

Theorem 15. *For a given pole Ω, the tangents at Q_1, Q_2, Q_3 to the pivotal cubic with pole Ω are concurrent if and only if its pivot P lies on the quintic $\mathcal{Q}(\Omega)$.*

Remark. $\mathcal{Q}(\Omega)$ contains the following points:
– A, B, C which are nodes,
– the square roots of Ω,
– tgΩ, the Ω-isoconjugate of K,
– the vertices of the cevian triangle of $Z = \left(\frac{(c^4pq+b^4rp-a^4qr)p}{a^2} : \cdots : \cdots\right)$, the isoconjugate of the crossconjugate of K and tgΩ in the isoconjugation with fixed point tgΩ,
– the common points of the circumcircle and the trilinear polar Δ_1 of tgΩ,
– the common points of the circumcircle and the line Δ_2 passing through tgΩ and the cross-conjugate of K and tgΩ.

Theorem 16. *For a given pivot P, the tangents at Q_1, Q_2, Q_3 to the pivotal cubic with pivot P are concurrent if and only if its pole Ω lies on the quintic $\mathcal{Q}'(P)$.*

Remark. $\mathcal{Q}'(P)$ contains the following points:
– the barycentric product $P \times K$,
– A, B, C which are nodes, the tangents being the cevian lines of X_{32} and the sidelines of the anticevian triangle of $P \times K$,
– the barycentric square P^2 of P and the vertices of its cevian triangle, the tangent at P^2 passing through $P \times K$.

5. Equilateral triangles

The McCay cubic meets the circumcircle at A, B, C and three other points N_a, N_b, N_c which are the vertices of an equilateral triangle. In this section, we characterize all the pivotal cubics $\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$ having the same property.

We know that the isogonal conjugates of three such points N_a, N_b, N_c are the infinite points of an equilateral cubic (a $\mathcal{K}_{60}$, see [2]) and that the isogonal transform of $\mathcal{K}$ is another pivotal cubic $\mathcal{K}' = \mathrm{p}\mathcal{K}(\Omega', P')$ with pole Ω' the X_{32}-isocongate of Ω, with pivot P' the barycentric product of P and the isogonal conjugate of Ω. Hence $\mathcal{K}$ meets the circumcircle at the vertices of an equilateral triangle if and only if $\mathcal{K}'$ is a $\mathrm{p}\mathcal{K}_{60}$.

Following [2, §6.2], we obtain the following theorem.

Theorem 17. *For a given pole Ω or a given pivot P, there is one and only one pivotal cubic $\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$ meeting the circumcircle at the vertices of an equilateral triangle.*

With $\Omega = K$ (or $P = O$) we obviously obtain the McCay cubic and the equilateral triangle is the circumnormal triangle. More generally, a $\mathrm{p}\mathcal{K}$ meets the circumcircle at the vertices of circumnormal triangle if and only if its pole Ω lies on the circum-cubic **K378** passing through K, the vertices of the cevian triangle of the Kosnita point X_{54}, the isogonal conjugates of X_{324}, X_{343}. The tangents at A, B, C are the cevians of X_{32}. The cubic is tangent at K to the Brocard axis and K is a flex on the cubic. See [3] and Figure 12.

The locus of pivots of these same cubics is **K361**. See [3] and Figure 13.

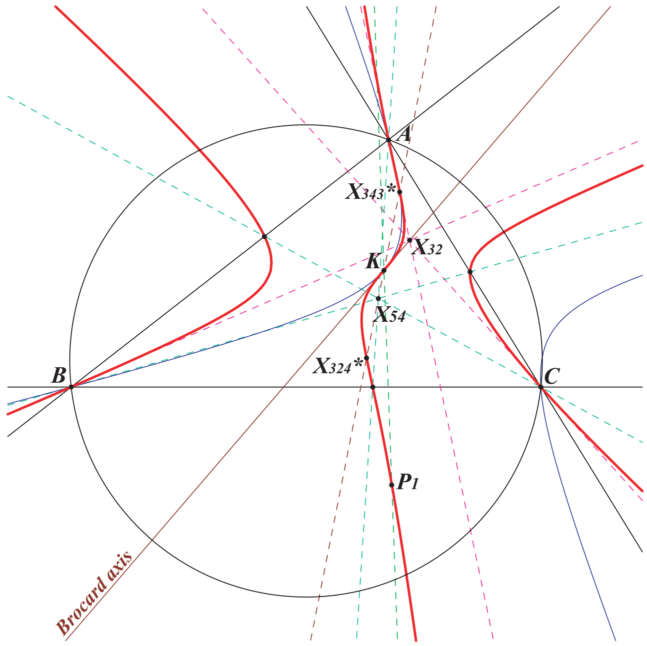

Figure 12. **K378**, the locus of poles of circumnormal p$\mathcal{K}$s

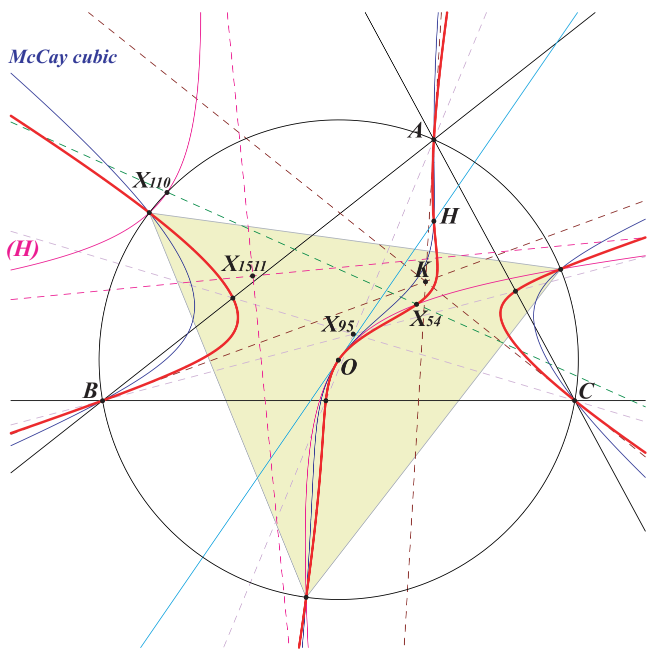

Figure 13. **K361**, the locus of pivots of circumnormal p$\mathcal{K}$s

References

[1] H. M. Cundy and C. F. Parry, Some cubic curves associated with a triangle, *Journal of Geometry*, 53 (1995) 41–66.
[2] J.-P. Ehrmann and B. Gibert, *Special Isocubics in the Triangle Plane*, available at http://perso.orange.fr/bernard.gibert/
[3] B. Gibert, *Cubics in the Triangle Plane*, available at http://perso.orange.fr/bernard.gibert/
[4] A. Haarbleicher, Cubiques auto-inverses isogonales par rapport à un triangle, *Annales de la faculté des sciences de Toulouse, 4ème série*, 4 (1940) 65–96.
[5] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.
[6] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

Bernard Gibert: 10 rue Cussinel, 42100 - St Etienne, France
E-mail address: bg42@orange.fr