FORUM GEOMETRICORUM

A Journal on Classical Euclidean Geometry and Related Areas

published by

Department of Mathematical Sciences
Florida Atlantic University

Volume 7
2007

http://forumgeom.fau.edu

ISSN 1534-1178

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Forum Geometricorum
Volume 7 (2007) 1–9.

FORUM GEOM
ISSN 1534-1178

Euler's Triangle Determination Problem

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Abstract. We give a simple proof of Euler's remarkable theorem that for a nondegenerate triangle, the set of points eligible to be the incenter is precisely the orthocentroidal disc, punctured at the nine-point center. The problem is handled algebraically with complex coordinates. In particular, we show how the vertices of the triangle may be determined from the roots of a complex cubic whose coefficients are functions of the classical centers.

1. Introduction

Consider the determination of a triangle from its centers.[1] What relations must be satisfied by points O, H, I so that a unique triangle will have these points as circumcenter, orthocenter, and incenter? In Euler's groundbreaking article [3], *Solutio facilis problematum quorundam geometricorum difficillimorum*, this intriguing question is answered synthetically, but without any comment on the geometric meaning of the solution.

Euler proved the existence of the required triangle by treating the lengths of the sides as zeros of a real cubic, the coefficients being functions of OI, OH, HI. He gave the following algebraic restriction on the distances to ensure that the cubic has three real zeros:

$$OI^2 < OH^2 - 2 \cdot HI^2 < 2 \cdot OI^2.$$

Though Euler did not remark on the geometric implications, his restriction was later proven equivalent to the simpler inequality

$$GI^2 + IH^2 < GH^2,$$

where G is the point that divides OH in the ratio $1:2$ (G is the centroid). This result was presented in a beautiful 1984 paper [4] by A. P. Guinand. Its geometric meaning is immediate: I must lie inside the circle on diameter GH. It also turns out that I cannot coincide with the midpoint of OH, which we denote by N (the nine-point center). The remarkable fact is that *all and only* points inside the circle and different from N are eligible to be the incenter. This region is often called the *orthocentroidal disc*, and we follow this convention.[2] Guinand considered the

Publication Date: January 8, 2007. Communicating Editor: Paul Yiu.
Dedicated to the tercentenary of Leonhard Euler.
[1]The phrase "determination of a triangle" is borrowed from [7].
[2]Conway discusses several properties of the orthocentroidal disc in [1].

cosines of the angles as zeros of a real cubic. He showed that this cubic has three real zeros with positive inverse cosines summing to π. Thus the angles are known, and the scale may be determined subsequently from OH. The problem received fresh consideration in 2002, when B. Scimemi [7] showed how to solve it using properties of the Kiepert focus, and again in 2005, when G. C. Smith [8] used statics to derive the solution.

The approach presented here uses complex coordinates. We show that the vertices of the required triangle may be computed from the roots of a certain complex cubic whose coefficients depend only upon the classical centers. This leads to a relatively simple proof.

2. Necessity of Guinand's Locus

Given a nonequilateral triangle, we show first that the incenter must lie within the orthocentroidal disc and must differ from the nine-point center. The equilateral triangle is uninteresting, since all the centers coincide.

Let $\triangle ABC$ be nonequilateral. As usual, we write O, H, I, G, N, R, r for the circumcenter, orthocenter, incenter, centroid, nine-point center, circumradius and inradius. Two formulas will feature very prominently in our discussion:

$$OI^2 = R(R - 2r) \quad \text{and} \qquad NI = \tfrac{1}{2}(R - 2r).$$

The first is due to Euler and the second to Feuerbach.[3] They jointly imply

$$OI > 2 \cdot NI,$$

provided the triangle is nonequilateral. Now given a segment PQ and a number $\lambda > 1$, the Apollonius Circle Theorem states that

(1) the equation $PX = \lambda \cdot QX$ describes a circle whose center lies on PQ, with P inside and Q outside;
(2) the inequality $PX > \lambda \cdot QX$ describes the interior of this circle (see [6]).

Thus the inequality $OI > 2 \cdot NI$ places I inside the circle $OX = 2 \cdot NX$, the center of which lies on the Euler line ON. Since G and H lie on the Euler line and satisfy the equation of the circle, GH is a diameter, and this circle turns out to be the orthocentroidal circle. Finally, the formulas of Euler and Feuerbach show that if $I = N$, then $O = I$. This means that the incircle and the circumcircle are *concentric*, forcing $\triangle ABC$ to be equilateral. Thus N is ineligible to be the incenter.

3. Complex Coordinates

Our aim now is to express the classical centers of $\triangle ABC$ as functions of A, B, C, regarded as complex numbers.[4] We are free to put $O = 0$, so that

$$|A| = |B| = |C| = R.$$

[3] Proofs of both theorems appear in [2].

[4] See [5] for a more extensive discussion of this approach.

The centroid is given by $3G = A + B + C$. The theory of the Euler line shows that $3G = 2O + H$, and since $O = 0$, we have

$$H = A + B + C.$$

Finally, it is clear that $2N = O + H = H$.

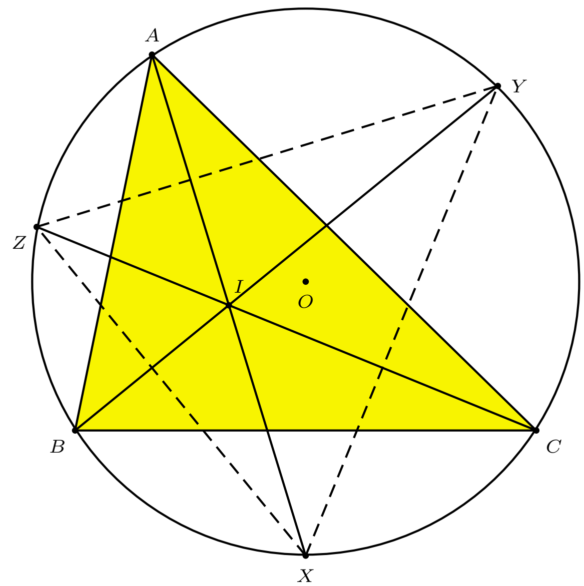

Figure 1.

To deal with the incenter, let X, Y, Z be the points at which the extended angle bisectors meet the circumcircle (Figure 1). It is not difficult to see that $AX \perp YZ$, $BY \perp ZX$ and $CZ \perp XY$. For instance, one angle between AX and YZ is the average of the minor arc from A to Z and the minor arc from X to Y. The first arc measures $\widehat{C}$, and the second, $\widehat{A} + \widehat{B}$. Thus the angle between AX and YZ is $\pi/2$. Evidently the angle bisectors of $\triangle ABC$ coincide with the altitudes of $\triangle XYZ$, and I is the orthocenter of $\triangle XYZ$. Since this triangle has circumcenter 0, its orthocenter is

$$I = X + Y + Z.$$

We now introduce complex square roots α, β, γ so that

$$\alpha^2 = A, \qquad \beta^2 = B, \qquad \gamma^2 = C.$$

There are two choices for each of α, β, γ. Observe that

$$|\beta\gamma| = R \qquad \text{and} \qquad \arg(\beta\gamma) = \tfrac{1}{2}(\arg B + \arg C),$$

so that $\pm\beta\gamma$ are the mid-arc points between B and C. It follows that $X = \pm\beta\gamma$, depending on our choice of signs. For reasons to be clarified later, we would like to arrange it so that

$$X = -\beta\gamma, \qquad Y = -\gamma\alpha, \qquad Z = -\alpha\beta.$$

These hold if α, β, γ are chosen so as to make $\triangle\alpha\beta\gamma$ *acute*, as we now show.

Let Γ denote the circle $|z| = \sqrt{R}$, on which α, β, γ must lie. Temporarily let α_1, α_2 be the two square roots of A, and β_1 a square root of B. Finally, let γ_1 be the square root of C on the side of $\alpha_1\alpha_2$ containing β_1 (Figure 2). Now $\triangle\alpha_i\beta_j\gamma_k$ is acute if and only if any two vertices are separated by the diameter of Γ through the remaining vertex. Otherwise one of its angles would be inscribed in a minor arc, rendering it obtuse. It follows that of all eight triangles $\triangle\alpha_i\beta_j\gamma_k$, only $\triangle\alpha_1\beta_2\gamma_1$ and $\triangle\alpha_2\beta_1\gamma_2$ are acute.

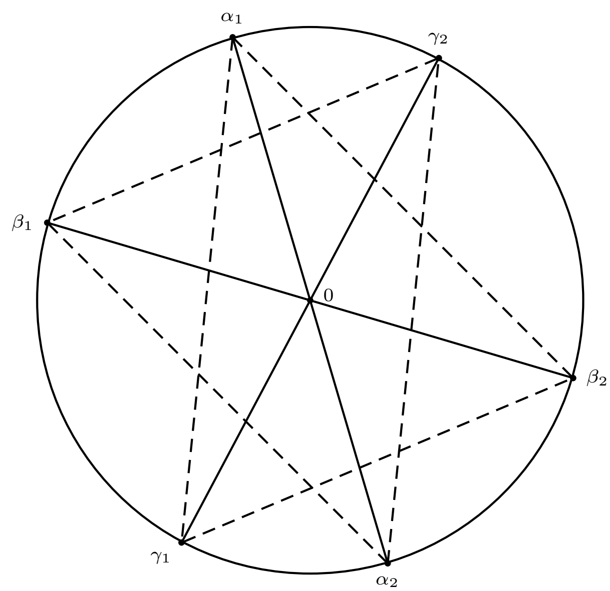

Figure 2.

Now let (α, β, γ) be either $(\alpha_1, \beta_2, \gamma_1)$ or $(\alpha_2, \beta_1, \gamma_2)$, so that $\triangle\alpha\beta\gamma$ is acute. Consider the stretch-rotation $z \mapsto \beta z$. This carries the diameter of Γ with endpoints $\pm\alpha$ to the diameter of $|z| = R$ with endpoints $\pm\alpha\beta$, one of which is Z. Now β and γ are separated by the diameter with endpoints $\pm\alpha$, and therefore B and $\beta\gamma$ are separated by the diameter with endpoints $\pm Z$. Thus to prove $X = -\beta\gamma$, we must only show that X and B are on the *same* side of the diameter with endpoints $\pm Z$. This will follow if the arc from Z to X passing through B is *minor* (Figure 3); but of course its measure is

$$\angle ZOB + \angle BOX = 2\angle ZCB + 2\angle BAX = \widehat{C} + \widehat{A} < \pi.$$

Hence $X = -\beta\gamma$. Similar arguments show that $Y = -\gamma\alpha$ and $Z = -\alpha\beta$.

To summarize, the incenter of $\triangle ABC$ may be expressed as

$$I = -(\beta\gamma + \gamma\alpha + \alpha\beta),$$

where α, β, γ are complex square roots of A, B, C for which $\triangle\alpha\beta\gamma$ is acute. Note that this expression is indifferent to the choice between $(\alpha_1, \beta_2, \gamma_1)$ and $(\alpha_2, \beta_1, \gamma_2)$, since each of these triples is the negative of the other.

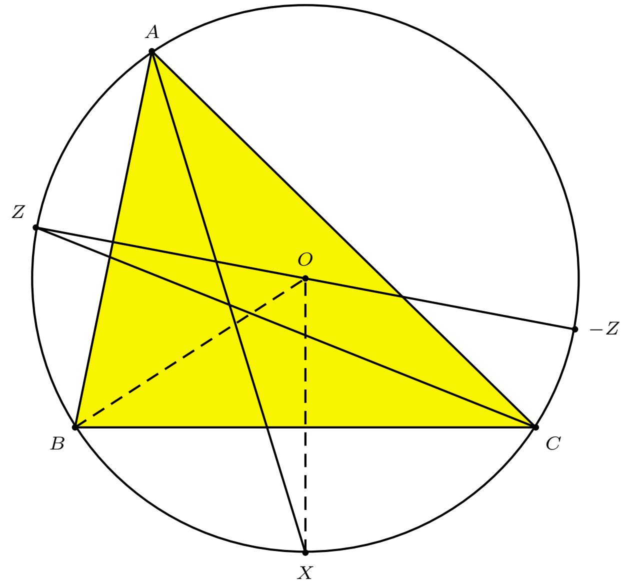

Figure 3.

4. Sufficiency of Guinand's Locus

Place O and H in the complex plane so that O lies at the origin. Define N and G as the points which divide OH internally in the ratios $1 : 1$ and $1 : 2$, respectively. Suppose that I is a point different from N selected from within the circle on diameter GH. Since $H - 2I = 2(N - I)$ is nonzero, we are free to scale coordinates so that $H - 2I = 1$. Let $u = |I|$. Guinand's inequality $OI > 2 \cdot NI$, which we write in complex coordinates as

$$|I| > 2|N - I|$$

now acquires the very simple form $u > 1$.

Consider the cubic equation

$$z^3 - z^2 - Iz + u^2 I = 0.$$

By the Fundamental Theorem of Algebra, this has three complex zeros α, β, γ. These turn out to be square roots of the required vertices. From the standard relations between zeros and coefficients, one has the important equations:

$$\alpha + \beta + \gamma = 1, \qquad \beta\gamma + \gamma\alpha + \alpha\beta = -I, \qquad \alpha\beta\gamma = -u^2 I.$$

Let us first show that the zeros lie on a circle centered at the origin. In fact,

$$|\alpha| = |\beta| = |\gamma| = u.$$

If z is a zero of the cubic, then $z^2(z - 1) = I(z - u^2)$. Taking moduli, we get

$$|z|^2|z - 1| = u|z - u^2|.$$

Squaring both sides and applying the rule $|w|^2 = w\bar{w}$, we find that

$$|z|^4(z - 1)(\bar{z} - 1) = u^2(z - u^2)(\bar{z} - u^2),$$

$$(|z|^6 - u^6) - (|z|^4 - u^4)(z + \bar{z}) + |z|^2(|z|^2 - u^2) = 0.$$

Assume for contradiction that a certain zero z has modulus $\neq u$. Then we may divide the last equation by the nonzero number $|z|^2 - u^2$, getting

$$|z|^4 + u^2|z|^2 + u^4 - (|z|^2 + u^2)(z + \bar{z}) + |z|^2 = 0,$$

or after a slight rearrangement,

$$(|z|^2 + u^2)(|z|^2 - (z + \bar{z})) + u^4 + |z|^2 = 0.$$

An elementary inequality of complex algebra says that

$$-1 \leq |z|^2 - (z + \bar{z}).$$

From this inequality and the above equation, we find that

$$(|z|^2 + u^2)(-1) + u^4 + |z|^2 \leq 0,$$

or after simplifying,

$$u^4 - u^2 \leq 0.$$

As this result is inconsistent with the hypothesis $u > 1$, we have proven that all the zeros of the cubic equation have modulus u.

Now *define* A, B, C by

$$A = \alpha^2, \qquad B = \beta^2, \qquad C = \gamma^2.$$

Clearly $|A| = |B| = |C| = u^2$. Since three points of a circle cannot be collinear, $\triangle ABC$ will be nondegenerate so long as A, B, C are distinct. Thus suppose for contradiction that $A = B$. It follows that $\alpha = \pm\beta$. If $\alpha = -\beta$, then $\gamma = \alpha + \beta + \gamma = 1$, yielding the falsehood $u = |\gamma| = 1$. The only remaining alternative is $\alpha = \beta$. In this case, $2\alpha + \gamma = 1$ and $\alpha(2\gamma + \alpha) = -I$, so that

$$|\alpha||2\gamma + \alpha| = |I|, \qquad \text{or} \qquad |2\gamma + \alpha| = 1.$$

Since $2\alpha + \gamma = 1$, one has $|2 - 3\alpha| = |2\gamma + \alpha| = 1$. Squaring this last result gives

$$4 - 6(\alpha + \bar{\alpha}) + 9|\alpha|^2 = 1, \qquad \text{or} \qquad 2(\alpha + \bar{\alpha}) = 1 + 3u^2.$$

Since $|\alpha + \bar{\alpha}| = 2|\mathrm{Re}(\alpha)| \leq 2|\alpha|$, we have $1 + 3u^2 \leq 4u$. Therefore the value of u is bounded between the zeros of the quadratic

$$3u^2 - 4u + 1 = (3u - 1)(u - 1),$$

yielding the falsehood $\frac{1}{3} \leq u \leq 1$. By this kind of reasoning, one shows that any two of A, B, C are distinct, and hence that $\triangle ABC$ is nondegenerate.

As in §3, since $\triangle ABC$ has circumcenter 0, its orthocenter is

$$\begin{aligned} A + B + C &= \alpha^2 + \beta^2 + \gamma^2 \\ &= (\alpha + \beta + \gamma)^2 - 2(\beta\gamma + \gamma\alpha + \alpha\beta) \\ &= 1 + 2I \\ &= H. \end{aligned}$$

Here we see the rationale for having chosen $I = -(\beta\gamma + \gamma\alpha + \alpha\beta)$.

Lastly we must show that the incenter of $\triangle ABC$ lies at I. It has already appeared that $I = -(\beta\gamma + \gamma\alpha + \alpha\beta)$. As in §3, exactly two of the eight possible triangles formed from square roots of A, B, C are acute, and these are mutual images under the map $z \mapsto -z$. Moreover, the incenter of $\triangle ABC$ is necessarily the value of the expression $-(z_2z_3 + z_3z_1 + z_1z_2)$ whenever $\triangle z_1z_2z_3$ is one of these two acute triangles. Thus to identify the incenter with I, we must only show that $\triangle\alpha\beta\gamma$ is acute.

Angle $\widehat{\alpha}$ is acute if and only if

$$|\beta - \gamma|^2 < |\alpha - \beta|^2 + |\alpha - \gamma|^2.$$

On applying the rule $|w|^2 = w\bar{w}$, this becomes

$$2u^2 - (\beta\bar{\gamma} + \bar{\beta}\gamma) < 4u^2 - (\alpha\bar{\beta} + \bar{\alpha}\beta + \alpha\bar{\gamma} + \bar{\alpha}\gamma),$$

$$\alpha\bar{\beta} + \bar{\alpha}\beta + \alpha\bar{\gamma} + \bar{\alpha}\gamma + \beta\bar{\gamma} + \bar{\beta}\gamma < 2(u^2 + \beta\bar{\gamma} + \bar{\beta}\gamma).$$

Here the left-hand side may be simplified considerably as

$$(\alpha + \beta + \gamma)(\bar{\alpha} + \bar{\beta} + \bar{\gamma}) - |\alpha|^2 - |\beta|^2 - |\gamma|^2 = 1 - 3u^2.$$

In a similar way, the right-hand side simplifies as

$$\begin{aligned} & 2u^2 + 2(\beta + \gamma)(\bar{\beta} + \bar{\gamma}) - 2|\beta|^2 - 2|\gamma|^2 \\ = \; & 2(1 - \alpha)(1 - \bar{\alpha}) - 2u^2 \\ = \; & 2(1 + |\alpha|^2 - \alpha - \bar{\alpha} - u^2) \\ = \; & 2 - 2(\alpha + \bar{\alpha}). \end{aligned}$$

To complete the proof that $\widehat{\alpha}$ is acute, it remains only to show that

$$2(\alpha + \bar{\alpha}) < 1 + 3u^2.$$

However, $2(\alpha + \bar{\alpha}) \leq 4|\alpha| = 4u$, and we have already seen that

$$4u < 1 + 3u^2,$$

since the opposite inequality yields the falsehood $\frac{1}{3} \leq u \leq 1$. Similar arguments establish that $\widehat{\beta}$ and $\widehat{\gamma}$ are acute.

To summarize, we have produced a nondegenerate triangle $\triangle ABC$ which has classical centers at the given points O, H, I. We now return to original notation and write $R = u^2$ for the circumradius of $\triangle ABC$.

5. Uniqueness

Suppose some other triangle $\triangle DEF$ has O, H, I as its classical centers. The formulas of Euler and Feuerbach presented in §2 have a simple but important consequence: If a triangle has O, N, I as circumcenter, nine-point center, and incenter, then its *circumdiameter* is OI^2/NI. This means that $\triangle ABC$ and $\triangle DEF$ share not only the same circumcenter, but also the same circumradius. It follows that $|D| = |E| = |F| = R$.

Since $\triangle DEF$ has circumcenter 0, its orthocenter H is equal to $D + E + F$. Choose square roots δ, ϵ, ζ of D, E, F so that the incenter I will satisfy

$$I = -(\epsilon\zeta + \zeta\delta + \delta\epsilon).$$

Then

$$\begin{aligned}(\delta+\epsilon+\zeta)^2 &= \delta^2+\epsilon^2+\zeta^2+2(\epsilon\zeta+\zeta\delta+\delta\epsilon)\\ &= D+E+F-2I\\ &= H-2I\\ &= 1.\end{aligned}$$

Since the map $z \mapsto -z$ leaves I invariant, but reverses the sign of $\delta+\epsilon+\zeta$, we may change the signs of δ, ϵ, ζ if necessary to make it so that

$$\delta+\epsilon+\zeta = 1.$$

Observe next that $|\delta\epsilon\zeta| = u^3 = |u^2 I|$. Thus we may write

$$\delta\epsilon\zeta = -\theta u^2 I, \qquad \text{where} \qquad |\theta| = 1.$$

The elementary symmetric functions of δ, ϵ, ζ are now

$$\delta+\epsilon+\zeta = 1, \qquad \epsilon\zeta+\zeta\delta+\delta\epsilon = -I, \qquad \delta\epsilon\zeta = -\theta u^2 I.$$

It follows that δ, ϵ, ζ are the roots of the cubic equation

$$z^3 - z^2 - Iz + \theta u^2 I = 0.$$

As in §4, we rearrange and take moduli of both sides to obtain

$$|z|^2|z-1| = u|z-\theta u|.$$

Squaring both sides of this result, we get

$$|z|^4(|z|^2 - z - \bar{z} + 1) = u^2(|z|^2 - u^2 z\bar{\theta} - u^2\bar{z}\theta + u^4).$$

Since all zeros of the cubic have modulus u, we may replace every occurrence of $|z|^2$ by u^2. This dramatically simplifies the equation, reducing it to

$$z + \bar{z} = z\bar{\theta} + \bar{z}\theta.$$

Substituting δ, ϵ, ζ here successively for z and adding the results, one finds that

$$2 = \bar{\theta} + \theta,$$

since

$$\delta+\epsilon+\zeta = \bar{\delta}+\bar{\epsilon}+\bar{\zeta} = 1.$$

It follows easily that $\theta = 1$. Evidently δ, ϵ, ζ are determined from the same cubic as α, β, γ. Therefore (D, E, F) is a permutation of (A, B, C), and the solution of the determination problem is unique.

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Forum Geometricorum
Volume 7 (2007) 11–17.

FORUM GEOM
ISSN 1534-1178

On a Porism Associated with the Euler and Droz-Farny Lines

Christopher J. Bradley, David Monk, and Geoff C. Smith

Abstract. The envelope of the Droz-Farny lines of a triangle is determined to be the inconic with foci at the circumcenter and orthocenter by using purely Euclidean means. The poristic triangles sharing this inconic and circumcircle have a common circumcenter, centroid and orthocenter.

1. Introduction

The triangle ABC has orthocenter H and circumcircle Σ. Suppose that a pair of perpendicular lines through H are drawn, then they meet the sides BC, CA, AB in pairs of points. The midpoints X, Y, Z of these pairs of points are known to be collinear on the Droz-Farny line [2]. The envelope of the Droz-Farny line is the inconic with foci at O and H, known recently as the Macbeath inconic, but once known as the Euler inconic [6]. We support the latter terminology because of its strong connection with the Euler line [3]. According to Goormaghtigh writing in [6] this envelope was first determined by Neuberg, and Goormaghtigh gives an extensive list of early articles related to the Droz-Farny line problem. We will not repeat the details since [6] is widely available through the archive service JSTOR.

We give a short determination of the Droz-Farny envelope using purely Euclidean means. Taken in conjunction with Ayme's recent proof [1] of the existence of the Droz-Farny line, this yields a completely Euclidean derivation of the envelope.

This envelope is the inconic of a porism consisting of triangles with a common Euler line and circumcircle. The sides of triangles in this porism arise as Droz-Farny lines of any one of the triangles in the porism. Conversely, if the orthocenter is interior to Σ, all Droz-Farny lines will arise as triangle sides.

2. The Droz-Farny envelope

Theorem. *Each Droz-Farny line of triangle ABC is the perpendicular bisector of a line segment joining the orthocenter H to a point on the circumcircle.*

Proof. Figure 1 may be useful. Let perpendicular lines l and l' through H meet BC, CA, AB at P and P', Q and Q', R and R' respectively and let X, Y, Z be the midpoints of PP', QQ', RR'.

The collinearity of X, Y, Z is the Droz-Farny theorem. Let K be the foot of the perpendicular from H to XYZ and produce HK to L with $HK = KL$. Now the circle HPP' has center X and $XH = XL$ so L lies on this circle. Let M, M' be the feet of the perpendiculars from L to l, l'. Note that $LMHM'$ is a rectangle

Publication Date: January 16, 2007. Communicating Editor: Paul Yiu.

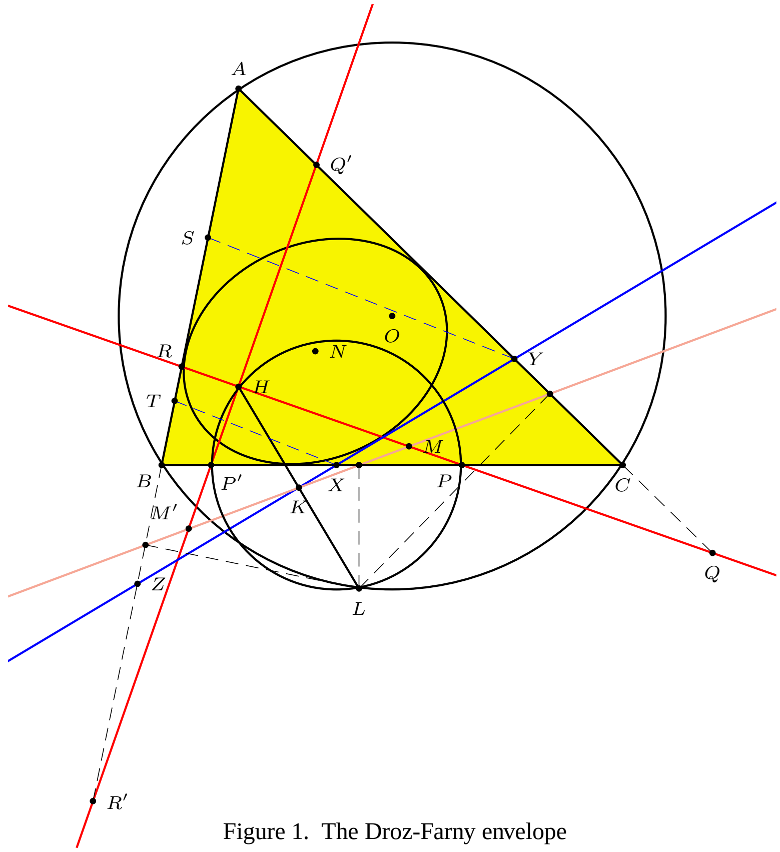

Figure 1. The Droz-Farny envelope

so K is on MM'. Then the foot of the perpendicular from L to the line PP' (*i.e.* BC) lies on MM' by the Wallace-Simson line property applied to the circumcircle of $PP'H$. Equally well, both perpendiculars dropped from L to AB and CA have feet on MM'. Hence L lies on circle ABC with MM' as its Wallace-Simson line. Therefore XYZ is a perpendicular bisector of a line segment joining H to a point on the circumcircle. □

Note that K lies on the nine-point circle of ABC. An expert in the theory of conics will recognize that the nine-point circle is the auxiliary circle of the Euler inconic of ABC with foci at the circumcenter and orthocenter, and for such a reader this article is substantially complete. The points X, Y, Z are collinear and the line XYZ is tangent to the conic inscribed in triangle ABC and having O, H as foci. The direction of the Droz-Farny line is a continuous function of the direction of the mutual perpendiculars; the argument of the Droz-Farny line against a reference axis increases monotonically as the perpendiculars rotate (say) anticlockwise through θ, with the position of the Droz-Farny line repeating itself as θ increases by $\frac{\pi}{2}$. By the intermediate value theorem, the envelope of the Droz-Farny lines is the whole Euler inconic.

We present a detailed discussion of this situation in §3 for the lay reader.

Incidentally, the fact that XY is a variable tangent to a conic of which BC, CA are fixed tangents mean that the correspondence $X \sim Y$ is a projectivity between the two lines. There is a neat way of setting up this map: let the perpendicular bisectors of AH, BH meet AB at S and T respectively. Then SY and TX are parallel. With a change of notation denote the lines BC, CA, AB, XYZ by a, b, c, d respectively; let e, f be the perpendicular bisectors of AH, BH. All these lines are tangents to the conic in question. Consider the Brianchon hexagon of lines a, b, c, d, e, f. The intersections ae, fb are at infinity so their join is the line at infinity. We have $ec = S$, $bd = Y$, $cf = T$, $da = X$. By Brianchon's theorem SY is parallel to XT.

3. The porism

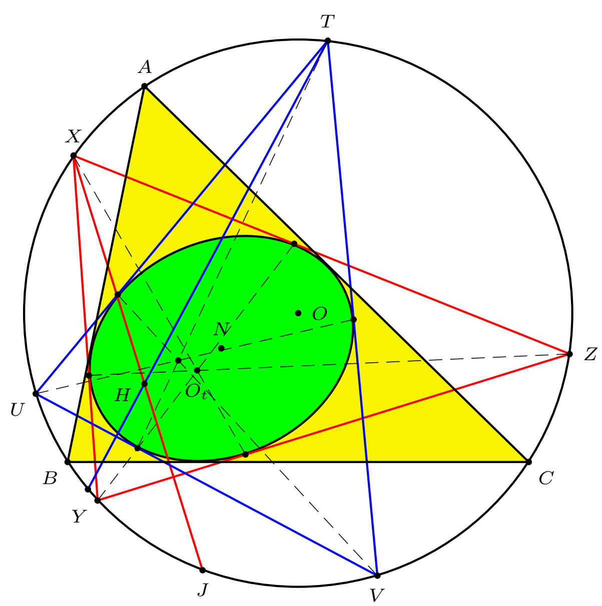

Figure 2. A porism associated with the Euler line

In a triangle with side lengths a, b and c, circumradius R and circumcenter O, the orthocenter H always lies in the interior of a circle center O and radius $3R$ since, as Euler showed, $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$.

We begin afresh. Suppose that we draw a circle Σ with center O and radius R in which is inscribed a non-right angled triangle ABC which has an orthocenter H, so $OH < 3R$ and H is not on Σ.

This H will serve as the orthocenter of infinitely many other triangles XYZ inscribed in the circle and a porism is obtained. We construct these triangles by choosing a point J on the circle. Next we draw the perpendicular bisector of HJ, and need this line to meet Σ again at Y and Z with XYZ anticlockwise. We can certainly arrange that the line and Σ meet by choosing X sufficiently close to A,

B or C. When this happens it follows from elementary considerations that triangle XYZ has orthocenter H, and is the only such triangle with circumcircle Σ and vertex X. In the event that H is inside the circumcircle (which happens precisely when triangle ABC is acute), then every point X on the circumcircle arises as a vertex of a triangle XYZ in the porism.

The construction may be repeated to create as many triangles ABC, TUV, PQR as we please, all inscribed in the circle and all having orthocenter H, as illustrated in Figure 2. Notice that the triangles in this porism have the same circumradius, circumcenter and orthocenter, so the sum of the squares of the side lengths of each triangle in the porism is the same.

We will show that all these triangles circumscribe a conic, with one axis of length R directed along the common Euler line, and with eccentricity $\frac{OH}{R}$. It follows that this inconic is an ellipse if H is chosen inside the circle, but a hyperbola if H is chosen outside.

Thus a porism arises which we call an *Euler line porism* since each triangle in the porism has the same circumcenter, centroid, nine-point center, orthocentroidal center, orthocenter *etc*. A triangle circumscribing a conic gives rise to a *Brianchon point* at the meet of the three Cevians which join each vertex to its opposite contact point.

We will show that the Brianchon point of a triangle in this porism is the isotomic conjugate O_t of the common circumcenter O.

In Figure 2 we pinpoint O_t for the triangle XYZ. The computer graphics system CABRI gives strong evidence for the conjecture that the locus of O_t, as one runs through the triangles of the porism, is a a subset of a conic.

It is possible to choose a point H at distance greater than $3R$ from O so there is no triangle inscribed in the circle which has orthocenter H and then there is no point J on the circle such that the perpendicular bisector of HJ cuts the circle.

The acute triangle case. See Figure 3. The construction is as follows. Draw AH, BH and CH to meet Σ at D, E and F. Draw DO, EO and FO to meet the sides at L, M, N. Let AO meet Σ at D^* and BC at L^*. Also let DO meet Σ at A^*. The points M^*, N^*, E^*, F^*, B^* and C^* are not shown but are similarly defined. Here A' is the midpoint of BC and the line through A' perpendicular to BC is shown.

3.1. *Proof of the porism.* Consider the ellipse defined as the locus of points P such that $HP + OP = R$, where R the circumradius of Σ. The triangle HLD is isosceles, so $HL + OL = LD + OL = R$; therefore L lies on the ellipse.

Now $\angle OLB = \angle CLD = \angle CLH$, because the line segment HD is bisected by the side BC. Therefore the ellipse is tangent to BC at L, and similarly at M and N. It follows that AL, BM, CN are concurrent at a point which will be identified shortly.

This ellipse depends only on O, H and R. It follows that if TUV is any triangle inscribed in Σ with center O, radius R and orthocenter H, then the ellipse touches the sides of TUV. The porism is established.

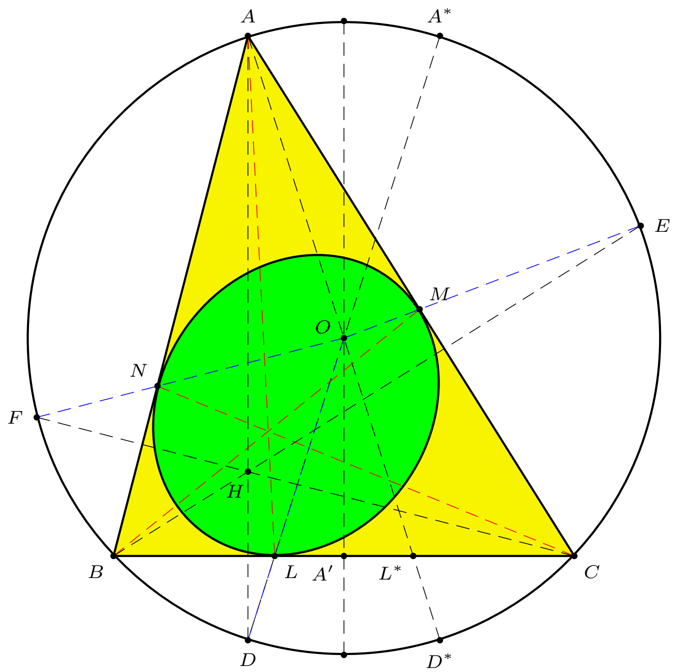

Figure 3. The inconic of the Euler line porism

Identification of the Brianchon point. This is the point of concurrence of AL, BM, CN. Since O and H are isogonal conjugates, it follows that D^* and A^* are reflections of D and A in the line which is the perpendicular bisector of BC. The same applies to B^*, C^*, E^* and F^* with respect to other perpendicular bisectors. Thus A^*D and AD^* are reflections of each other in the perpendicular bisector. Thus L^* is the reflection of L and thus $A'L = A'L^*$. Thus since AL^*, BM^*, CN^* are concurrent at O, the lines AL, BM and CN are concurrent at O_t, the isotomic conjugate of O.

The obtuse triangle case. Refer to Figure 4. Using the same notation as before, now consider the hyperbola defined as the locus of points P such that $|HP - OP| = R$. We now have $HL - OL = LD - OL = R$ so that L lies on the hyperbola.

Also $\angle A^*LB = \angle CLD = \angle HLC$, so the hyperbola touches BC at L, and the argument proceeds as before.

It is a routine matter to obtain the Cartesian equation of this inconic. Scaling so that $R = 1$ we may assume that O is at $(0, 0)$ and H at $(c, 0)$ where $0 \leq c < 3$ but $c \neq 1$.

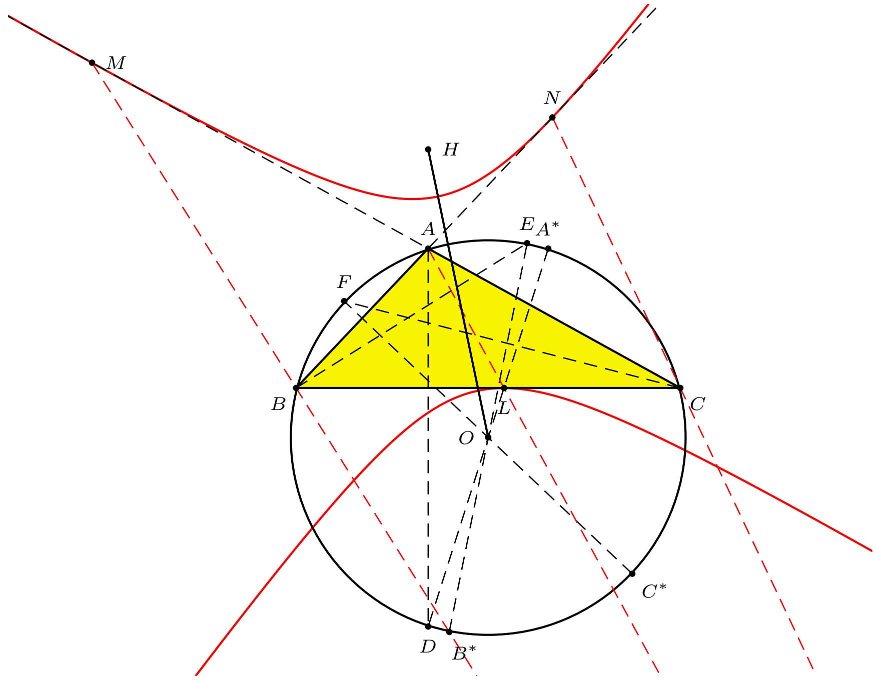

Figure 4. The Euler inconic can be a hyperbola

The inconic then has equation

$$4y^2 + (1-c^2)(2x-c)^2 = (1-c^2). \tag{1}$$

When $c < 1$, so H is internal to Σ, this represents an ellipse, but when $c > 1$ it represents a hyperbola. In all cases the center is at $(\frac{c}{2}, 0)$, which is the nine-point center.

One of the axes of the ellipse is the Euler line itself, whose equation is $y = 0$. We see from Equation (1) that the eccentricity of the inconic is $c = \frac{OH}{R}$ and of course its foci are at O and H. Not every tangent line to the inconic arises as a side of a triangle in the porism if H is outside Σ.

Areal analysis. One can also perform the geometric analysis of the envelope using areal co-ordinates, and we briefly report relevant equations for the reader interested in further areal work. Take ABC as triangle of reference and define $u = \cot B \cot C$, $v = \cot C \cot A$, $w = \cot A \cot B$ so that $H(u, v, w)$ and $O(v+w, w+u, u+v)$. This means that the isotomic conjugate O_t of O has co-ordinates

$$O_t\left(\frac{1}{v+w}, \frac{1}{w+u}, \frac{1}{u+v}\right).$$

The altitudes are AH, BH, CH with equations $wy = vz$, $uz = wx$, $vx = uy$ respectively.

The equation of the inconic is

$$(v+w)^2x^2+(w+u)^2y^2+(u+v)^2z^2-2(w+u)(u+v)yz$$
$$-2(u+v)(v+w)zx-2(v+w)(w+u)xy=0. \quad (2)$$

This curve can be parameterized by the formulas:

$$x=\frac{(1+q)^2}{v+w}, y=\frac{1}{w+u}, z=\frac{q^2}{u+v}, \quad (3)$$

where q has any real value (including infinity). The perpendicular lines l and l through H may be taken to pass through the points at infinity with co-ordinates $((1+t),-t,-1)$ and $((1+s),-s,-1)$ and then the Droz-Farny line has equation

$$-(sw+tw-2v)(2stw-sv-tv)x-(sw+tw+2(u+w))(2stw-sv-tv)y$$
$$+(sw+tw-2v)(2st(u+v)+sv+tv)z=0. \quad (4)$$

In Equation (4) for the midpoints X, Y, Z to be collinear we must take

$$s=-\frac{v(tw+u+w)}{w(t(u+v)+v)}. \quad (5)$$

If we now substitute Equation (3) into Equation (4) and use Equation (5), a discriminant test on the resulting quadratic equation with the help of DERIVE confirms the tangency for all values of t.

Incidentally, nowhere in this areal analysis do we use the precise values of u, v, w in terms of the angles A, B, C. Therefore we have a bonus theorem: if H is replaced by another point K, then given a line through K, there is always a second line through K (but not generally at right angles to it) so that XYZ is a straight line. As the line l rotates, l' also rotates (but not at the same rate). However the rotations of these lines is such that the variable points X, Y, Z remain collinear and the line XYZ also envelops a conic. This affine generalization of the Droz-Farny theorem was discovered independently by Charles Thas [5] in a paper published after the original submission of this article. We happily cede priority.

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Forum Geometricorum
Volume 7 (2007) 19–24.

FORUM GEOM
ISSN 1534-1178

The Edge-Tangent Sphere of a Circumscriptible Tetrahedron

Yu-Dong Wu and Zhi-Hua Zhang

Abstract. A tetrahedron is circumscriptible if there is a sphere tangent to each of its six edges. We prove that the radius ℓ of the edge-tangent sphere is at least $\sqrt{3}$ times the radius of its inscribed sphere. This settles affirmatively a problem posed by Z. C. Lin and H. F. Zhu. We also briefly examine the generalization into higher dimension, and pose an analogous problem for an $n-$dimensional simplex admitting a sphere tangent to each of its edges.

1. Introduction

Every tetrahedron has a circumscribed sphere passing through its four vertices and an inscribed sphere tangent to each of its four faces. A tetrahedron is said to be circumscriptible if there is a sphere tangent to each of its six edges (see [1, §§786–794]). We call this the edge-tangent sphere of the tetrahedron.

Let $\mathscr{P}$ denote a tetrahedron $P_0P_1P_2P_3$ with edge lengths $P_iP_j = a_{ij}$ for $0 \le i < j \le 3$. The following necessary and sufficient condition for a tetrahedron to admit an edge-tangent sphere can be found in [1, §§787, 790, 792]. See also [4, 6].

Theorem 1. *The following statement for a tetrahedron $\mathscr{P}$ are equivalent.*
(1) *$\mathscr{P}$ has an edge-tangent sphere.*
(2) $a_{01} + a_{23} = a_{02} + a_{13} = a_{03} + a_{12}$;
(3) *There exist $x_i > 0$, $i = 0, 1, 2, 3$, such that $a_{ij} = x_i + x_j$ for $0 \le i < j \le 3$.*

For $i = 0, 1, 2, 3$, x_i is the length of a tangent from P_i to the edge-tangent sphere of $\mathscr{P}$. Let ℓ denote the radius of this sphere.

Theorem 2. [1, §793] *The radius of the edge-tangent sphere of a circumscriptible tetrahedron of volume V is given by*

$$\ell = \frac{2x_0x_1x_2x_3}{3V}. \tag{1}$$

Lin and Zhu [4] have given the formula (1) in the form

$$\ell^2 = \frac{(2x_0x_1x_2x_3)^2}{2x_0x_1x_2x_3 \sum\limits_{0\le i<j\le 3} x_ix_j - (x_1^2x_2^2x_3^2 + x_2^2x_3^2x_0^2 + x_3^2x_0^2x_1^2 + x_0^2x_1^2x_2^2)}. \tag{2}$$

Publication Date: January 22, 2007. Communicating Editor: Paul Yiu.
The authors would like to thank Prof. Han-Fang Zhang for his enthusiastic help.

The fact that this latter denominator is $(3V)^2$ follows from the formula for the volume of a tetrahedron in terms of its edges:

$$V^2 = \frac{1}{288}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & (x_0+x_1)^2 & (x_0+x_2)^2 & (x_0+x_3)^2 \\ 1 & (x_0+x_1)^2 & 0 & (x_1+x_2)^2 & (x_1+x_3)^2 \\ 1 & (x_0+x_2)^2 & (x_1+x_2)^2 & 0 & (x_2+x_3)^2 \\ 1 & (x_0+x_3)^2 & (x_1+x_3)^2 & (x_2+x_3)^2 & 0 \end{vmatrix}.$$

Lin and Zhu *op. cit.* obtained several inequalities for the edge-tangent sphere of $\mathscr{P}$. They also posed the problem of proving or disproving $\ell^2 \geq 3r^2$ for a circumscriptible tetrahedron. See also [2]. The main purpose of this paper is to settle this problem affirmatively.

Theorem 3. *For a circumscriptible tetrahedron with inradius r and edge-tangent sphere of radius ℓ, $\ell \geq \sqrt{3}r$.*

2. Two inequalities

Lemma 4. *If $x_i > 0$ for $0 \leq i \leq 3$, then*

$$\begin{aligned}&\left(\frac{x_1+x_2+x_3}{x_1x_2x_3}+\frac{x_2+x_3+x_0}{x_2x_3x_0}+\frac{x_3+x_0+x_1}{x_3x_0x_1}+\frac{x_0+x_1+x_2}{x_0x_1x_2}\right)\\ &\cdot\frac{4(x_0x_1x_2x_3)^2}{2x_0x_1x_2x_3\sum\limits_{0\leq i<j\leq 3}x_ix_j-(x_1^2x_2^2x_3^2+x_2^2x_3^2x_0^2+x_3^2x_0^2x_1^2+x_0^2x_1^2x_2^2)}\geq 6. \end{aligned}\tag{3}$$

Proof. From

$$\begin{aligned}&x_0^2x_1^2(x_2-x_3)^2+x_0^2x_2^2(x_1-x_3)^2+x_0^2x_3^2(x_1-x_2)^2\\ &+x_1^2x_2^2(x_0-x_3)^2+x_1^2x_3^2(x_0-x_2)^2+x_2^2x_3^2(x_0-x_1)^2\geq 0,\end{aligned}$$

we have

$$x_1^2x_2^2x_3^2+x_2^2x_3^2x_0^2+x_3^2x_0^2x_1^2+x_0^2x_1^2x_2^2\geq\frac{2}{3}x_0x_1x_2x_3\sum_{0\leq i<j\leq 3}x_ix_j,$$

and

$$\begin{aligned}&2x_0x_1x_2x_3\sum_{0\leq i<j\leq 3}x_ix_j-(x_1^2x_2^2x_3^2+x_2^2x_3^2x_0^2+x_3^2x_0^2x_1^2+x_0^2x_1^2x_2^2)\\ \leq&\frac{4}{3}x_0x_1x_2x_3\sum_{0\leq i<j\leq 3}x_ix_j,\end{aligned}$$

or

$$\begin{aligned}&\frac{4(x_0x_1x_2x_3)^2}{2x_0x_1x_2x_3\sum\limits_{0\leq i<j\leq 3}x_ix_j-(x_1^2x_2^2x_3^2+x_2^2x_3^2x_0^2+x_3^2x_0^2x_1^2+x_0^2x_1^2x_2^2)}\\ \geq&\frac{4(x_0x_1x_2x_3)^2}{\frac{4}{3}x_0x_1x_2x_3\sum\limits_{0\leq i<j\leq 3}x_ix_j}=\frac{3x_0x_1x_2x_3}{\sum\limits_{0\leq i<j\leq 3}x_ix_j}.\end{aligned}\tag{4}$$

On the other hand, it is easy to see that

$$\frac{x_1+x_2+x_3}{x_1x_2x_3}+\frac{x_2+x_3+x_0}{x_2x_3x_0}+\frac{x_3+x_0+x_1}{x_3x_0x_1}+\frac{x_0+x_1+x_2}{x_0x_1x_2}=\frac{2\sum\limits_{0\le i<j\le 3}x_ix_j}{x_0x_1x_2x_3}. \tag{5}$$

Inequality (3) follows immediately from (4) and (5). □

Corollary 5. *For a circumscriptible tetrahedron $\mathscr{P}$ with an edge-tangent sphere of radius ℓ, and faces with inradii r_0, r_1, r_2, r_3,*

$$\left(\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\right)\ell^2\ge 6.$$

Equality holds if and only if $\mathscr{P}$ is a regular tetrahedron.

Proof. From the famous Heron formula, the inradius of a triangle ABC of side-lengths $a=y+z$, $b=z+x$ and $c=x+y$ is given by

$$r^2=\frac{xyz}{x+y+z}.$$

Applying this to the four faces of $\mathscr{P}$, we see that the first factor on the left hand side of (3) is $\left(\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\right)$. Now the result follows from (2). □

Proposition 6. *Let $\mathscr{P}$ be a circumscriptible tetrahedron of volume V. If, for $i=0,1,2,3$, the opposite face of vertex P_i has area $\triangle_i$ and inradius r_i, then*

$$(\triangle_0+\triangle_1+\triangle_2+\triangle_3)^2\ge\frac{9V^2}{2}\left(\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\right). \tag{6}$$

Equality holds if and only if $\mathscr{P}$ is a regular tetrahedron.

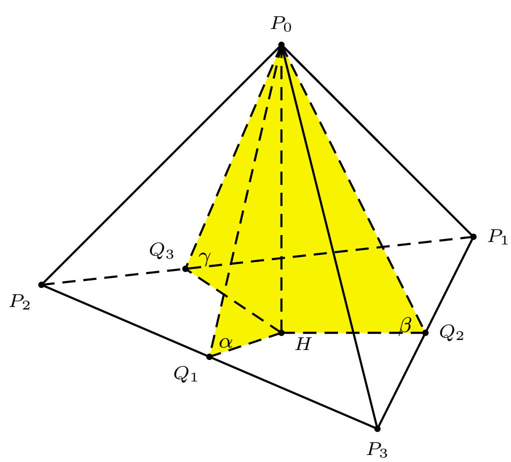

Figure 1.

Proof. Let α be the angle between the planes $P_0P_2P_3$ and $P_1P_2P_3$. If the perpendiculars from P_0 to the line P_2P_3 and to the plane $P_1P_2P_3$ intersect these at Q_1 and H respectively, then $\angle P_0QH = \alpha$. See Figure 1. Similarly, we have the angles β between the planes $P_0P_3P_1$ and $P_1P_2P_3$, and γ between $P_0P_1P_2$ and $P_1P_2P_3$. Note that

$$P_0H = P_0Q_1 \cdot \sin\alpha = P_0Q_2 \cdot \sin\beta = P_0Q_3 \cdot \sin\gamma.$$

Hence,

$$P_0H \cdot P_2P_3 = 2\triangle_1 \sin\alpha = 2\sqrt{(\triangle_1 + \triangle_1\cos\alpha)(\triangle_1 - \triangle_1\cos\alpha)}, \tag{7}$$

$$P_0H \cdot P_3P_1 = 2\triangle_2 \sin\beta = 2\sqrt{(\triangle_2 + \triangle_2\cos\beta)(\triangle_2 - \triangle_2\cos\beta)}, \tag{8}$$

$$P_0H \cdot P_1P_2 = 2\triangle_3 \sin\gamma = 2\sqrt{(\triangle_3 + \triangle_3\cos\gamma)(\triangle_3 - \triangle_3\cos\gamma)}. \tag{9}$$

From (7–9), together with $P_0H = \frac{3V}{\triangle_0}$ and $\frac{\triangle_0}{r_0} = \frac{1}{2}(P_1P_2 + P_2P_3 + P_3P_1)$, we have

$$\begin{aligned}\frac{3V}{r_0} =& \sqrt{(\triangle_1 + \triangle_1\cos\alpha)(\triangle_1 - \triangle_1\cos\alpha)}\\ &+\sqrt{(\triangle_2 + \triangle_2\cos\beta)(\triangle_2 - \triangle_2\cos\beta)}\\ &+\sqrt{(\triangle_3 + \triangle_3\cos\gamma)(\triangle_3 - \triangle_3\cos\gamma)}.\end{aligned} \tag{10}$$

Applying Cauchy's inequality and noting that

$$\triangle_0 = \triangle_1\cos\alpha + \triangle_2\cos\beta + \triangle_3\cos\gamma,$$

we have

$$\begin{aligned}\left(\frac{3V}{r_0}\right)^2 \leq& (\triangle_1 + \triangle_1\cos\alpha + \triangle_2 + \triangle_2\cos\beta + \triangle_3 + \triangle_3\cos\gamma)\\ &\cdot(\triangle_1 - \triangle_1\cos\alpha + \triangle_2 - \triangle_2\cos\beta + \triangle_3 - \triangle_3\cos\gamma)\\ =& (\triangle_1 + \triangle_2 + \triangle_3 + \triangle_0)(\triangle_1 + \triangle_2 + \triangle_3 - \triangle_0)\\ =& (\triangle_1 + \triangle_2 + \triangle_3)^2 - \triangle_0^2,\end{aligned} \tag{11}$$

or

$$(\triangle_1 + \triangle_2 + \triangle_3)^2 - \triangle_0^2 \geq \left(\frac{3V}{r_0}\right)^2. \tag{12}$$

It is easy to see that equality in (12) holds if and only if

$$\frac{\triangle_1 + \triangle_1\cos\alpha}{\triangle_1 - \triangle_1\cos\alpha} = \frac{\triangle_2 + \triangle_2\cos\beta}{\triangle_2 - \triangle_2\cos\beta} = \frac{\triangle_3 + \triangle_3\cos\gamma}{\triangle_3 - \triangle_3\cos\gamma}.$$

Equivalently, $\cos\alpha = \cos\beta = \cos\gamma$, or $\alpha = \beta = \gamma$. Similarly, we have

$$(\triangle_2 + \triangle_3 + \triangle_0)^2 - \triangle_1^2 \geq \left(\frac{3\,V}{r_1}\right)^2, \tag{13}$$

$$(\triangle_3 + \triangle_0 + \triangle_1)^2 - \triangle_2^2 \geq \left(\frac{3\,V}{r_2}\right)^2, \tag{14}$$

$$(\triangle_0 + \triangle_1 + \triangle_2)^2 - \triangle_3^2 \geq \left(\frac{3\,V}{r_3}\right)^2. \tag{15}$$

Summing (12) to (15), we obtain the inequality (6), with equality precisely when all dihedral angles are equal, *i.e.*, when $\mathscr{P}$ is a regular tetrahedron. □

Remark. Inequality (6) is obtained by X. Z. Yang in [5].

3. Proof of Theorem 3

Since $r = \frac{3V}{\triangle_0+\triangle_1+\triangle_2+\triangle_3}$, it follows from Proposition 6 and Corollary 5 that

$$\ell^2 \geq \frac{6}{\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}} \geq \frac{27V^2}{(\triangle_0+\triangle_1+\triangle_2+\triangle_3)^2} = 3r^2.$$

This completes the proof of Theorem 3.

4. A generalization with an open problem

As a generalization of the tetrahedron, we say that an $n-$dimensional simplex is circumscriptible if there is a sphere tangent to each of its edges. The following basic properties of a circumscriptible simplex can be found in [3].

Theorem 7. *Suppose the edge lengths of an n-simplex $\mathscr{P} = P_0P_1\cdots P_n$ are $P_iP_j = a_{ij}$ for $0 \leq i < j \leq n$. The n-simplex has an edge-tangent sphere if and only if there exist x_i, $i = 0, 1, \ldots, n$, satisfying $a_{ij} = x_i + x_j$ for $0 \leq i \neq j \leq n$. In this case, the radius of the edge-tangent sphere is given by*

$$\ell^2 = -\frac{D_1}{2D_2}, \tag{16}$$

where

$$D_1 = \begin{vmatrix} -2x_0^2 & 2x_0x_1 & \cdots & 2x_0x_{n-1} \\ 2x_0x_1 & -2x_1^2 & \cdots & 2x_1x_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 2x_0x_{n-1} & 2x_1x_{n-1} & \cdots & -2x_{n-1}^2 \end{vmatrix},$$

and

$$D_2 = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & \cdot & \cdots & \cdot \\ \vdots & \vdots & D_1 & \vdots \\ 1 & \cdot & \cdots & \cdot \end{vmatrix}.$$

We conclude this paper with an open problem: for a circumscriptible n-simplex with a circumscribed sphere of radius R, an inscribed sphere of radius r and an edge-tangent sphere of radius ℓ, prove or disprove that

$$R \geq \sqrt{\frac{2n}{n-1}}l \geq nr.$$

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Forum Geometricorum
Volume 7 (2007) 25–29.

FORUM GEOM
ISSN 1534-1178

A Stronger Triangle Inequality for Neutral Geometry

Melissa Baker and Robert C. Powers

Abstract. Bailey and Bannister [*College Math. Journal*, 28 (1997) 182–186] proved that a stronger triangle inequality holds in the Euclidean plane for all triangles having largest angle less than $\arctan(\frac{24}{7}) \approx 74°$. We use hyperbolic trigonometry to show that a stronger triangle inequality holds in the hyperbolic plane for all triangles having largest angle less than or equal to $65.87°$.

1. Introduction

One of the most fundamental results of neutral geometry is the triangle inequality. How can this cherished inequality be strengthened? Under certain restrictions, the sum of the lengths of two sides of a triangle is greater than the length of the remaining side plus the length of the altitude to this side.

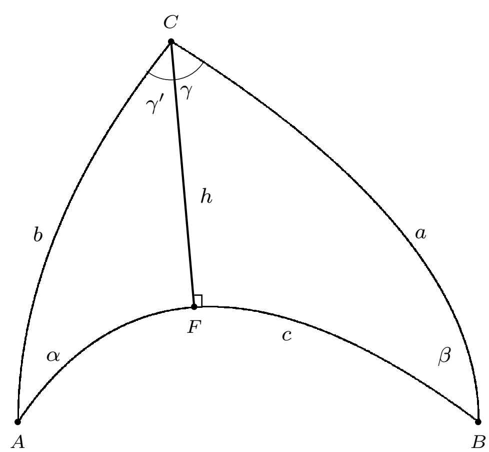

Figure 1. Strong triangle inequality $a + b > c + h$

Let ABC be a triangle belonging to neutral geometry (see Figure 1). Let a, b and c be the lengths of sides BC, AC and AB, respectively. Also, let α, β and γ denote the angles at A, B and C respectively. If we let F be the foot of the perpendicular from C onto side AB and if h is the length of the segment CF, when is it true that $a + b > c + h$? Since $a > h$ and $b > h$, this question is of interest only if c is the length of the longest side of ABC, or, equivalently, if γ is the the largest angle of ABC. With this notation, if the inequality $a + b > c + h$ holds where γ is the largest angle of the triangle ABC, we say that ABC satisfies the *strong triangle inequality*.

Publication Date: January 29, 2007. Communicating Editor: Paul Yiu.

The following result, due to Bailey and Bannister [1], explains what happens if the triangle ABC belongs to Euclidean geometry.

Theorem 1. *If ABC is a Euclidean triangle having largest angle $\gamma < \arctan(\frac{24}{7}) \approx 74°$, then ABC satisfies the strong triangle inequality.*

An elegant trigonometric proof of Theorem 1 can by found in [3]. It should be noted that the bound of $\arctan(\frac{24}{7})$ is the best possible since any isosceles Euclidean triangle with $\gamma = \arctan(\frac{24}{7})$ violates the strong triangle inequality.

The goal of this note is to extend the Bailey and Bannister result to neutral geometry. To get the appropriate bound for the extended result we need the function

$$f(\gamma) := -1 - \cos\gamma + \sin\gamma + \sin\frac{\gamma}{2}\sin\gamma. \tag{1}$$

Observe that $f'(\gamma) = \sin\gamma + \cos\gamma + \sin\frac{\gamma}{2}\cos\gamma + \frac{1}{2}\cos\frac{\gamma}{2}\sin\gamma > 0$ on the interval $\left[0, \frac{\pi}{2}\right]$. Therefore, $f(\gamma)$ is strictly monotone increasing on the interval $(0, \frac{\pi}{2})$. Since $f(0) = -2$, $f(\frac{\pi}{2}) = \frac{\sqrt{2}}{2}$, and f is continuous it follows that f has a unique root r in the interval $\left(0, \frac{\pi}{2}\right)$. In fact, r is approximately 1.15 (radians) or $65.87°$. See Figure 2.

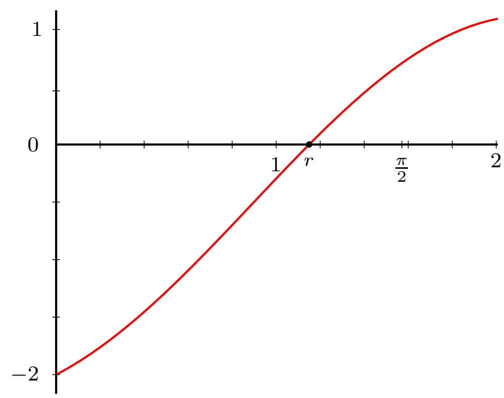

Figure 2. Graph of $f(\gamma)$

Theorem 2. *In neutral geometry a triangle ABC having largest angle γ satisfies the strong triangle inequality if $\gamma \leq r \approx 1.15$ radians or $65.87°$.*

The proof of Theorem 2 is based on the fact that a model of neutral geometry is isomorphic to either the Euclidean plane or a hyperbolic plane. Given Theorem 1, it is enough to establish our result for the case of hyperbolic geometry. Moreover, since the strong triangle inequality holds if and only if $ka + kb > kc + kh$ for any positive constant k, it is enough to assume that the distance scale in hyperbolic geometry is 1. An explanation about the distance scale k and how it is used in hyperbolic geometry can be found in [4].

2. Hyperbolic trigonometry

Recall that the hyperbolic sine and hyperbolic cosine functions are given by

$$\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}.$$

The formulas needed to prove the main result are given below. First, there are the standard identities

$$\cosh^2 x - \sinh^2 x = 1 \tag{2}$$

and

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y. \tag{3}$$

If ABC is a hyperbolic triangle with a right angle at C, *i.e.*, $\gamma = \frac{\pi}{2}$, then

$$\sinh a = \sinh c \sin \alpha \tag{4}$$

and

$$\cosh a \sin \beta = \cos \alpha. \tag{5}$$

For any hyperbolic triangle ABC,

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma, \tag{6}$$

$$\frac{\sin \alpha}{\sinh a} = \frac{\sin \beta}{\sinh b} = \frac{\sin \gamma}{\sinh c}, \tag{7}$$

$$\cosh c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}. \tag{8}$$

See [2, Chapter 10] or [5, Chapter 8] for more details regarding (4 – 8).

3. Proof of Theorem 2

The strong triangle inequality $a + b > c + h$ holds if and only if $\cosh(a+b) > \cosh(c+h)$. Expanding both sides by the identity given in (3) we have

$$\cosh a \cosh b + \sinh a \sinh b > \cosh c \cosh h + \sinh c \sinh h,$$
$$\cosh c + \sinh a \sinh b \cos \gamma + \sinh a \sinh b > \cosh c \cosh h + \sinh c \sinh h, \text{by } (6)$$
$$\cosh c\,(1 - \cosh h) + \sinh a \sinh b\,(\cos \gamma + 1) - \sinh c \sinh h > 0.$$

Since ACF is a right triangle with the length of CF equal to h, it follows from (4) that $\sinh h = \sinh b \sin \alpha$. Applying (7), we have

$$\cosh c\,(1 - \cosh h) + \sinh a \sinh b\,(\cos \gamma + 1) - \frac{\sinh a}{\sin \alpha} \cdot \sin \gamma \sinh b \sin \alpha > 0,$$
$$\cosh c\,(1 - \cosh h) + \sinh a \sinh b\,(\cos \gamma + 1 - \sin \gamma) > 0,$$
$$\cosh c\left(1 - \cosh^2 h\right) + \sinh a \sinh b (1 + \cosh h)\,(\cos \gamma + 1 - \sin \gamma) > 0,$$
$$\cosh c\left(-\sinh^2 h\right) + \sinh a \sinh b (1 + \cosh h)\,(\cos \gamma + 1 - \sin \gamma) > 0, \quad \text{by } (2)$$
$$\cosh c\left(-\sinh^2 b \sin^2 \alpha\right) + \sinh a \sinh b (1 + \cosh h)(\cos \gamma + 1 - \sin \gamma) > 0.$$

Dividing both sides of the inequality by $\sinh b > 0$, we have

$$-\cosh c \sinh b \sin^2 \alpha + \sinh a(1 + \cosh h)(\cos\gamma + 1 - \sin\gamma) > 0.$$

By (7) and (8), we have

$$-\left(\frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}\right)\frac{\sinh a \sin\beta}{\sin\alpha}\sin^2\alpha + \sinh a(1+\cosh h)(\cos\gamma + 1 - \sin\gamma) > 0.$$

Simplifying and dividing by $\sinh a > 0$, we have

$$-(\cos\alpha\cos\beta + \cos\gamma)\sinh a + \sinh a(1 + \cosh h)(\cos\gamma + 1 - \sin\gamma) > 0,$$
$$-(\cos\alpha\cos\beta + \cos\gamma) + (1 + \cosh h)(\cos\gamma + 1 - \sin\gamma) > 0, \tag{9}$$

We have manipulated the original inequality into one involving the original angles, α, β, and γ, and the length of the altitude on AB. In the right triangle ACF, let γ' be the angle at C. We may assume $\gamma' \leq \frac{\gamma}{2}$ (otherwise we can work with the right triangle BCF). Applying (5) to triangle ACF gives $\cosh h = \frac{\cos\alpha}{\sin\gamma'}$. Now continuing with the inequality (9) we get

$$-(\cos\alpha\cos\beta + \cos\gamma) + \left(1 + \frac{\cos\alpha}{\sin\gamma'}\right)(1 + \cos\gamma - \sin\gamma) > 0$$

Multiplying both sides by $-\sin\gamma' < 0$, we have

$$\sin\gamma'(\cos\alpha\cos\beta + \cos\gamma) - \left(\sin\gamma' + \cos\alpha\right)(1 + \cos\gamma - \sin\gamma) < 0,$$

Simplifying this and rearranging terms, we have

$$\cos\alpha\left(\sin\gamma'\cos\beta - 1 - \cos\gamma + \sin\gamma\right) + \sin\gamma'(\sin\gamma - 1) < 0. \tag{10}$$

If $\sin\gamma'\cos\beta - 1 - \cos\alpha + \sin\alpha > 0$, then

$$\begin{aligned}
&\cos\alpha\left(\sin\gamma'\cos\beta - 1 - \cos\gamma + \sin\gamma\right) + \sin\gamma'(\sin\gamma - 1)\\
&< \sin\gamma' - 1 - \cos\gamma + \sin\gamma + \sin\gamma'(\sin\gamma - 1)\\
&= -1 - \cos\gamma + \sin\gamma + \sin\gamma'\sin\gamma\\
&\leq -1 - \cos\gamma + \sin\gamma + \sin\frac{\gamma}{2}\sin\gamma.
\end{aligned}$$

Note that this last expression is $f(\gamma)$ defined in (1). We have shown that

$$\cos\alpha(\sin\gamma'\cos\beta - 1 - \cos\gamma + \sin\gamma) + \sin\gamma'(\sin\gamma - 1) < \max\{0, f(\gamma)\}.$$

For $\gamma \leq r$, we have $f(\gamma) \leq 0$ and the strong triangle inequality holds.

This completes the proof of Theorem 2.

If $r < \gamma < \frac{\pi}{2}$, then $f(\gamma) > 0$. In this case, we can find an angle α such that $0 < \alpha < \frac{\pi}{2} - \frac{\gamma}{2}$ and

$$\cos\alpha\left(\sin\frac{\gamma}{2}\cos\alpha - 1 - \cos\alpha + \sin\alpha\right) + \sin\frac{\gamma}{2}(\sin\gamma - 1) > 0.$$

Since $\gamma + 2\alpha < \pi$ it follows from [5, Theorem 6.7] that there exists a hyperbolic triangle ABC with angles α, α, and γ. Our previous work shows that the triangle ABC satisfies the strong triangle inequality if and only if (10) holds. Consequently, $a + b > c + h$ provided $f(\gamma) \leq 0$. Therefore, the bound r given in Theorem 2 is the best possible.

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Forum Geometricorum
Volume 7 (2007) 31–32.

FORUM GEOM
ISSN 1534-1178

A Simple Construction of the Golden Ratio

Jingcheng Tong and Sidney Kung

Abstract. We construct the golden ratio by using an area bisector of a trapezoid.

Consider a trapezoid $PQRS$ with bases $PQ = b$, $RS = a$, $a < b$. Assume, in Figure 1, that the segment MN of length $\sqrt{\frac{a^2+b^2}{2}}$ is parallel to PQ. Then MN lies between the bases PQ and RS (see [1, p.57]). It is easy to show that MN bisects the area of the trapezoid. It is more interesting to note that M and N divide SP and RQ in the golden ratio if $b = 3a$. To see this, construct a segment SW parallel to RQ and let $V = MN \cap SW$. It is clear that

$$\frac{SM}{SP} = \frac{MV}{PW} = \frac{\sqrt{\frac{a^2+b^2}{2}} - a}{b-a} = \frac{\sqrt{5}-1}{2}$$

if $b = 3a$.

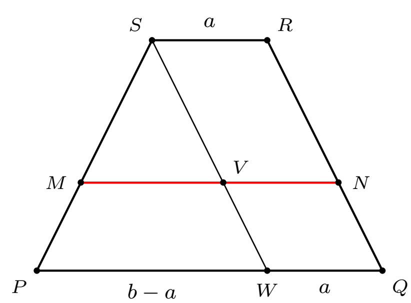

Figure 1

Based upon this result, we present the following simple division of a given segment AB in the golden ratio. Construct
(1) a trapezoid $ABCD$ with $AD//BC$ and $BC = 3 \cdot AD$,
(2) a right triangle BCD with a right angle at C and $CE = AD$,
(3) the midpoint F of BE and a point H on the perpendicular bisector of BE such that $FH = \frac{1}{2}BE$,
(4) a point I on BC such that $BI = BH$.

Complete a parallelogram $BIJG$ with J on DC and G on AB. See Figure 2. Then G divides AB in the golden ratio, *i.e.*, $\frac{AB}{AB} = \frac{\sqrt{5}-1}{2}$.

Publication Date: February 5, 2007. Communicating Editor: Paul Yiu.

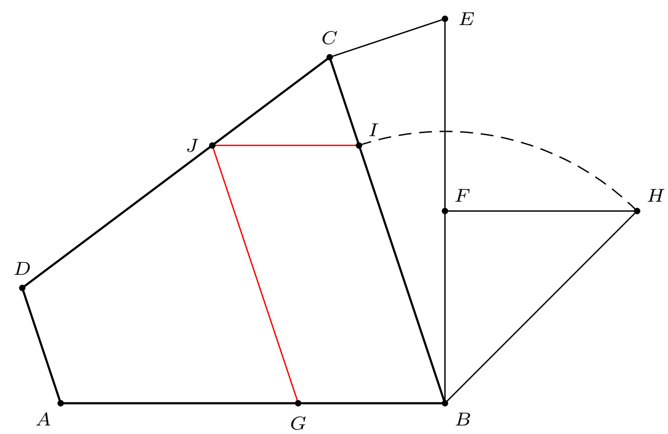

Figure 2

Proof. The trapezoid $ABCD$ has $AD = a$, $BC = b$ with $b = 3a$. The segment JG is parallel to the bases and

$$JG = BI = BH = \sqrt{2} \cdot \frac{\sqrt{a^2 + b^2}}{2} = \sqrt{\frac{a^2 + b^2}{2}}.$$

Therefore, $\frac{AG}{AB} = \frac{\sqrt{5}-1}{2}$. □

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Forum Geometricorum
Volume 7 (2007) 33–52.

FORUM GEOM
ISSN 1534-1178

The Method of Punctured Containers

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Abstract. We introduce the method of punctured containers, which geometrically relates volumes and centroids of complicated solids to those of simpler punctured prismatic solids. This method goes to the heart of some of the basic properties of the sphere, and extends them in natural and significant ways to solids assembled from cylindrical wedges (Archimedean domes) and to more general solids, especially those with nonuniform densities.

1. Introduction

Archimedes (287-212 B.C.) is regarded as the greatest mathematician of ancient times because of his masterful and innovative treatment of a remarkable range of topics in both pure and applied mathematics. One landmark discovery is that the volume of a solid sphere is two-thirds the volume of its circumscribing cylinder, and that the surface area of the sphere is also two-thirds the total surface area of the same cylinder. Archimedes was so proud of this revelation that he wanted the sphere and circumscribing cylinder engraved on his tombstone. He discovered the volume ratio by balancing slices of the sphere against slices of a *larger* cylinder and cone, using centroids and the law of the lever, which he had also discovered.

Today we know that the volume ratio for the sphere and cylinder can be derived more simply by an elementary geometric method that Archimedes overlooked. It is illustrated in Figure 1. By symmetry it suffices to consider a hemisphere, as in Figure 1a, and its circumscribing cylindrical container. Figure 1b shows the cylinder with a solid cone removed. The punctured cylindrical container has exactly the same volume as the hemisphere, because every horizontal plane cuts the hemisphere and the punctured cylinder in cross sections of equal area. The cone's volume is one-third that of the cylinder, hence the hemisphere's volume is two-thirds that of the cylinder, which gives the Archimedes volume ratio for the sphere and its circumscribing cylinder.

This geometric method extends to more general solids we call Archimedean domes. They and their punctured prismatic containers are described below in Section 2. Any plane parallel to the equatorial base cuts such a dome and its punctured container in cross sections of equal area. This implies that two planes parallel to the base cut the dome and the punctured container in slices of equal volumes, equality of volumes being a consequence of the following:

Slicing principle. *Two solids have equal volumes if their horizontal cross sections taken at any height have equal areas.*

Publication Date: February 12, 2007. Communicating Editor: Xiao-Dong Zhang.

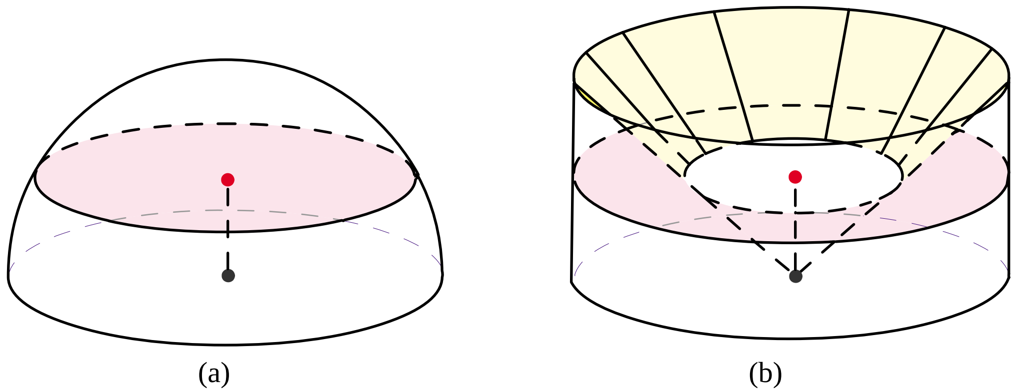


Figure 1. (a) A hemisphere and (b) a punctured cylindrical container of equal volume.

This statement is often called Cavalieri's principle in honor of Bonaventura Cavalieri (1598-1647), who attempted to prove it for general solids. Archimedes used it sixteen centuries earlier for special solids, and he credits Eudoxus and Democritus for using it even earlier in their discovery of the volume of a cone. Cavalieri employed it to find volumes of many solids, and tried to establish the principle for general solids by applying Archimedes' method of exhaustion, but it was not demonstrated rigorously until integral calculus was developed in the 17th century. We prefer using the neutral and more descriptive term *slicing principle*.

To describe the slicing principle in the language of calculus, cut two solids by horizontal planes that produce cross sections of equal area $A(x)$ at an arbitrary height x above a fixed base. The integral $\int_{x_1}^{x_2} A(x)\,dx$ gives the volume of the portion of each solid cut by all horizontal planes as x varies over some interval $[x_1,\, x_2]$. Because the integrand $A(x)$ is the same for both solids, the corresponding volumes are also equal. We could just as well integrate any function $f(x, A(x))$, and the integral over the interval $[x_1,\, x_2]$ would be the same for both solids. For example, $\int_{x_1}^{x_2} xA(x)\,dx$ is the first moment of the area function over the interval $[x_1,\, x_2]$, and this integral divided by the volume gives the altitude of the *centroid* of the slice between the planes $x = x_1$ and $x = x_2$. Thus, not only are the volumes of these slices equal, but also the altitudes of their centroids are equal. Moreover, all moments $\int_{x_1}^{x_2} x^k A(x)\,dx$ with respect to the plane of the base are equal for both slices.

In [1; Theorem 6a] we showed that the lateral surface area of any slice of an Archimedean dome between two parallel planes is equal to the lateral surface area of the corresponding slice of the circumscribing (unpunctured) prism. This was deduced from the fact that Archimedean domes circumscribe hemispheres. It implies that the total surface area of a sphere is equal to the lateral surface area of its circumscribing cylinder which, in turn, is two-thirds the total surface area of the cylinder. The surface area ratio was discovered by Archimedes by a completely different method.

This paper extends our geometric method further, from Archimedean domes to more general solids. First we dilate an Archimedean dome in a vertical direction to produce a dome with elliptic profiles, then we replace its base by an arbitrary polygon, not necessarily convex. This leads naturally to domes with arbitrary curved bases. Such domes and their punctured prismatic containers have equal volumes and equal moments relative to the plane of the base because of the slicing principle, but if these domes do not circumscribe hemispheres the corresponding lateral surface areas will not be equal. This paper relaxes the requirement of equal surface areas and concentrates on solids having the same volume and moments as their punctured prismatic containers. We call such solids *reducible* and describe them in Section 3. Section 4 treats reducible domes and shells with polygonal bases, then Section 5 extends the results to domes with curved bases, and formulates reducibility in terms of mappings that preserve volumes and moments.

The full power of our method, which we call *the method of punctured containers*, is revealed by the treatment of nonuniform mass distributions in Section 6. Problems of calculating masses and centroids of nonuniform wedges, shells, and their slices with elliptic profiles, including those with cavities, are reduced to those of *simpler punctured prismatic containers*. Section 7 gives explicit formulas for volumes and centroids, and Section 8 reveals the surprising fact that uniform domes are reducible to their punctured containers if and only if they have elliptic profiles.

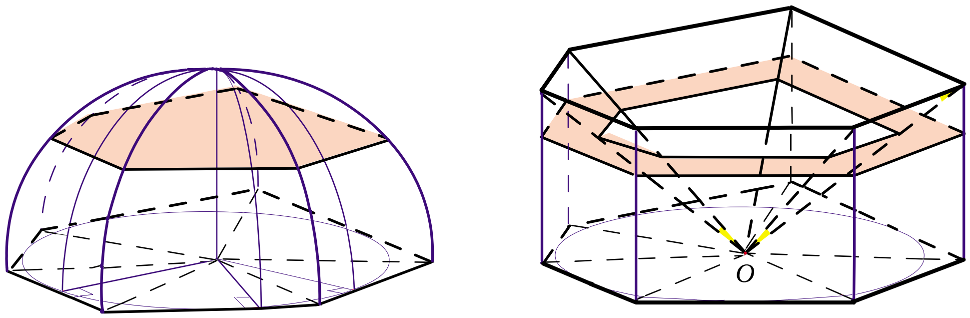
2. Archimedean domes

Archimedean domes are solids of the type shown in Figure 2a, formed by assembling portions of circular cylindrical wedges. Each dome circumscribes a hemisphere, and its horizontal base is a polygon, not necessarily regular, circumscribing the equator of the hemisphere. Cross sections cut by planes parallel to the base are similar polygons circumscribing the cross sections of the hemisphere. Figure 2b shows the dome's punctured prismatic container, a circumscribing prism, from which a pyramid with congruent polygonal base has been removed as indicated. The shaded regions in Figure 2 illustrate the fundamental relation between any Archimedean dome and its punctured prismatic container:

Each horizontal plane cuts both solids in cross sections of equal area.

Hence, by the slicing principle, any two horizontal planes cut both solids in slices of equal volume. Because the removed pyramid has volume one-third that of the unpunctured prism, we see that the volume of any Archimedean dome is two-thirds that of its punctured prismatic container.

We used the name "Archimedean dome" because of a special case considered by Archimedes. In his preface to The Method [3; Supplement, p. 12] Archimedes announced (without proof) that the volume of intersection of two congruent orthogonal circular cylinders is two-thirds the volume of the circumscribing cube. In [3; pp. 48-50], Zeuthen verifies this with the method of centroids and levers employed by Archimedes in treating the sphere. However, if we observe that half the solid of intersection is an Archimedean dome with a square base, and compare its volume with that of its punctured prismatic container, we immediately obtain the required two-thirds volume ratio.



(a) Archimedean dome (b) Punctured prismatic container

Figure 2. Each horizontal plane cuts the dome and its punctured prismatic container in cross sections of equal area.

As a limiting case, when the polygonal cross sections of an Archimedean dome become circles, and the punctured container becomes a circumscribing cylinder punctured by a cone, we obtain a purely geometric derivation of the Archimedes volume ratio for a sphere and cylinder.

When an Archimedean dome and its punctured container are *uniform* solids, made of material of the same constant density (mass per unit volume), the corresponding horizontal slices also have equal masses, and the center of mass of each slice lies at the same height above the base [1; Section 9].

3. Reducible solids

This paper extends the method of punctured containers by applying it first to general dome-like structures far removed from Archimedean domes, and then to domes with *nonuniform* mass distributions. The generality of the structures is demonstrated by the following examples.

Cut any Archimedean dome and its punctured container into horizontal slices and assign to each pair of slices the same constant density, which can differ from pair to pair. Because the masses are equal slice by slice, the total mass of the dome is equal to that of its punctured container, and the centers of mass are at the same height. Or, cut the dome and its punctured container into wedges by vertical half planes through the polar axis, and assign to each pair of wedges the same constant density, which can differ from pair to pair. Again, the masses are equal wedge by wedge, so the total mass of the dome is equal to that of its punctured container, and the centers of mass are at the same height. Or, imagine an Archimedean dome divided into thin concentric shell-like layers, like those of an onion, each assigned its own constant density, which can differ from layer to layer. The punctured container is correspondingly divided into coaxial prismatic layers, each assigned the same constant density as the corresponding shell layer. In this case the masses are equal shell by shell, so the total mass of the dome is equal to that of its punctured container, and again the centers of mass are at the same height. We are interested in a class of solids, with pyramidically punctured prismatic containers, that share the following property with Archimedean domes:

Definition. (Reducible solid) A solid is called reducible if an arbitrary horizontal slice of the solid and its punctured container have equal volumes, equal masses, and hence centers of mass at the same height above the base.

Every uniform Archimedean done is reducible, and in Section 5 we exhibit some nonuniform Archimedean domes that are reducible as well.

The method of punctured containers enables us to reduce both volume and mass calculations of domes to those of simpler prismatic solids, thus generalizing the profound volume relation between the sphere and cylinder discovered by Archimedes. Another famous result of Archimedes [3; Method, Proposition 6] states that the centroid of a uniform solid hemisphere divides its altitude in the ratio 5:3. Using the method of punctured containers we show that the same ratio holds for uniform Archimedean domes and other more general domes (Theorem 7), and we also extend this result to the center of mass of a more general class of nonuniform reducible domes (Theorem 8).

4. Polygonal elliptic domes and shells

To easily construct a more general class of reducible solids, start with any Archimedean dome, and dilate it and its punctured container in a vertical direction by the same scaling factor $\lambda > 0$. The circular cylindrical wedges in Figure 2a become elliptic cylindrical wedges, as typified by the example in Figure 3a. A circular arc of radius a is dilated into an elliptic arc with horizontal semi axis a and vertical semi axis λa. Dilation changes the altitude of the prismatic wedge from a to λa (Figure 3b). The punctured container is again a prism punctured by a pyramid.

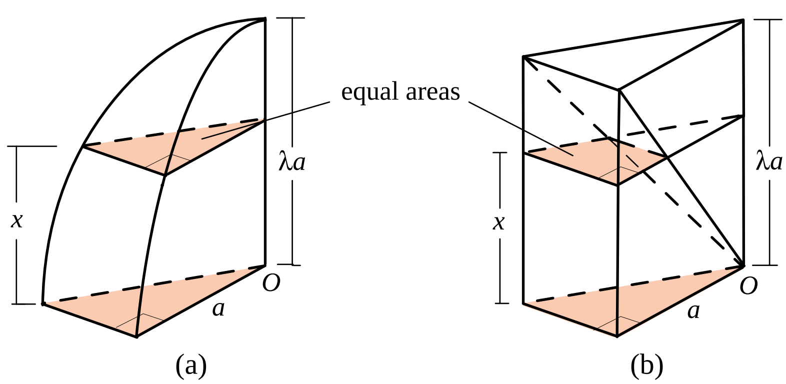


Figure 3. (a) Vertical dilation of a cylindrical wedge by a factor λ. (b) Its punctured prismatic container.

Each horizontal plane at a given height above the base cuts both the elliptic wedge and the corresponding punctured prismatic wedge in cross sections of equal area. Consequently, any two horizontal planes cut both solids in slices of equal volume.

If the elliptic and prismatic wedges have the same constant density, then they also have the same mass, and their centers of mass are at the same height above the base. In other words, we have:

Theorem 1. *Every uniform elliptic cylindrical wedge is reducible.*

Now assemble a finite collection of nonoverlapping elliptic cylindrical wedges with their horizontal semi axes, possibly of different lengths, in the same horizontal plane, but having a *common vertical semi axis*, which meets the base at a point O called the *center*. We assume the density of each component wedge is constant, although this constant may differ from component to component. For each wedge, the punctured circumscribing prismatic container with the same density is called its *prismatic counterpart*. The punctured containers assembled in the same manner produce the counterpart of the wedge assemblage. We call an assemblage *nonuniform* if some of its components can have different constant densities. This includes the special case of a *uniform* assemblage where all components have the same constant density. Because each wedge is reducible we obtain:

Corollary 1. *Any nonuniform assemblage of elliptic cylindrical wedges is reducible.*

Polygonal elliptic domes. Because the base of a finite assemblage is a polygon (a union of triangles with a common vertex O) we call the assemblage a *polygonal elliptic dome*. The polygonal base need not circumscribe a circle and it need not be convex. Corollary 1 gives us:

Corollary 2. *The volume of any polygonal elliptic dome is equal to the volume of its circumscribing punctured prismatic container, that is, two-thirds the volume of the unpunctured prismatic container, which, in turn, is the area of the base times the height.*

In the special limiting case when the equatorial polygonal base of the dome turns into an ellipse with center at O, the dome becomes half an ellipsoid, and the circumscribing prism becomes an elliptic cylinder. In this limiting case, Corollary 2 reduces to:

Corollary 3. *The volume of any ellipsoid is two-thirds that of its circumscribing elliptic cylinder.*

In particular, we have Archimedes' result for "spheroids" [3; Method, Proposition 3]:

Corollary 4. (Archimedes) *The volume of an ellipsoid of revolution is two-thirds that of its circumscribing circular cylinder.*

Polygonal elliptic shells. A *polygonal elliptic shell* is the solid between two concentric similar polygonal elliptic domes. From Theorem 1 we also obtain:

Theorem 2. *The following solids are reducible:*

(a) *Any uniform polygonal elliptic shell.*
(b) *Any wedge of a uniform polygonal elliptic shell.*
(c) *Any horizontal slice of a wedge of type* (b).
(d) *Any nonuniform assemblage of shells of type* (a).
(e) *Any nonuniform assemblage of wedges of type* (b).
(f) *Any nonuniform assemblage of slices of type* (c).

By using as building blocks horizontal slices of wedges cut from a polygonal elliptic shell, we can see intuitively how one might construct, from such building blocks, very general polygonal elliptic domes that are reducible and have more or less arbitrary mass distribution. By considering limiting cases of polygonal bases with many edges, and building blocks with very small side lengths, we can imagine elliptic shells and domes whose bases are more or less arbitrary plane regions, for example, elliptic, parabolic or hyperbolic segments.

The next section describes an explicit construction of general reducible domes with curvilinear bases.

5. General elliptic domes

Replace the polygonal base by any plane region bounded by a curve whose polar coordinates (r, θ) relative to a "center" O satisfy an equation $r = \rho(\theta)$, where ρ is a given piecewise continuous function, and θ varies over an interval of length 2π. Above this base we build an elliptic dome as follows. First, the altitude of the dome is a segment of fixed height $h > 0$ along the polar axis perpendicular to the base at O. We assume that each vertical half plane through the polar axis at angle θ cuts the surface of the dome along a quarter of an ellipse with horizontal semi axis $\rho(\theta)$ and the same vertical semi axis h, as in Figure 4a. The ellipse will be degenerate at points where $\rho(\theta) = 0$. Thus, an elliptic wedge is a special case of an elliptic dome.

When $\rho(\theta) > 0$, the cylindrical coordinates (r, θ, z) of points on the surface of the dome satisfy the equation of an ellipse:

$$\left(\frac{r}{\rho(\theta)}\right)^2 + \left(\frac{z}{h}\right)^2 = 1. \tag{1}$$

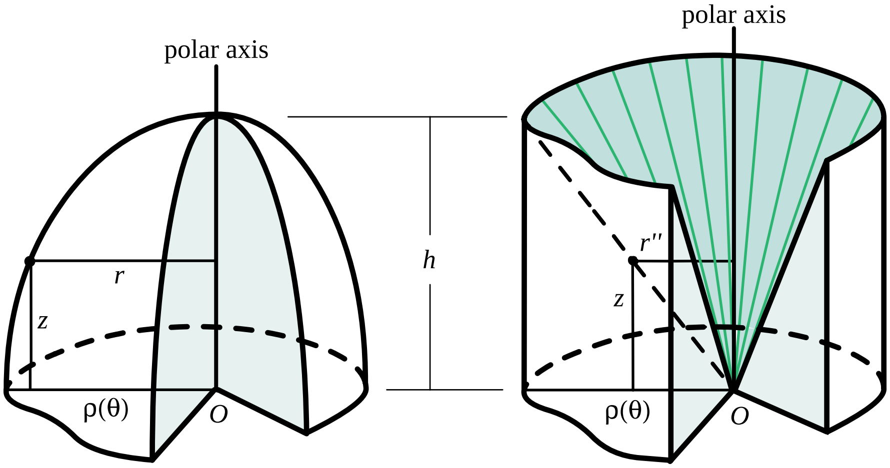
The dome is circumscribed by a cylindrical solid of altitude h whose base is congruent to that of the dome (Figure 4b). Incidentally, we use the term "cylindrical solid" with the understanding that the solid is a prism when the base is polygonal.

Each point (r', θ', z') on the lateral surface of the cylinder in Figure 4b is related to the corresponding point (r, θ, z) on the surface of the dome by the equations

$$\theta' = \theta, \quad z' = z, \quad r' = \rho(\theta).$$

From this cylindrical solid we remove a conical solid whose surface points have cylindrical coordinates (r'', θ, z), where $z/h = r''/\rho(\theta)$, or

$$r'' = z\rho(\theta)/h.$$



(a) (b)

Figure 4. An elliptic dome (a), and its circumscribing punctured prismatic container (b).

When $z = h$, this becomes $r'' = \rho(\theta)$, so the base of the cone is congruent to the base of the elliptic dome. When the base is polygonal, the conical solid is a pyramid.

More reducible domes. The polar axis of an elliptic dome depends on the location of center O. For a given curvilinear base, we can move O to any point inside the base, or even to the boundary. Moving O will change the function $\rho(\theta)$ describing the boundary of the base, with a corresponding change in the shape of the ellipse determined by (1). Thus, this construction generates not one, but *infinitely many elliptic domes* with a given base. For any such dome, we can generate another family as follows: Imagine the dome and its prismatic counterpart made up of very thin horizontal layers, like two stacks of cards. Deform each solid by a horizontal translation and rotation of each horizontal layer. The shapes of the solids will change, but their cross-sectional areas will not change. In general, such a deformation may alter the shape of each ellipse on the surface to some other curve, and the deformed dome will no longer be elliptic. The same deformation applied to the prismatic counterpart will change the punctured container to a nonprismatic punctured counterpart. Nevertheless, all the results of this paper (with the exception of Theorem 11) will hold for such deformed solids and their counterparts.

However, if the deformation is a linear shearing that leaves the base fixed but translates each layer by a distance proportional to its distance from the base, then straight lines are mapped onto straight lines and the punctured prismatic solid is deformed into another prism punctured by a pyramid with the same base. The correspondingly sheared dome will be *elliptic* because each elliptic curve on the surface of the dome is deformed into an elliptic curve. To visualize a physical model of such a shearing, imagine a general elliptic dome and its counterpart sliced horizontally to form stacks of cards. Pierce each stack by a long pin along the polar axis, and let O be the point where the tip of the pin touches the base. Tilting the pin away from the vertical polar axis, keeping O fixed, results in horizontal linear shearing of the stacks and produces infinitely many elliptic domes, all with the same polygonal base. The prismatic containers are correspondingly tilted, and the domes are reducible.

Reducibility mapping. For a given general elliptic dome, we call the corresponding circumscribing punctured cylindrical solid *its punctured container*. Our goal is to show that *every uniform general elliptic dome is reducible*. This will be deduced from a more profound property, stated below in Theorem 3. It concerns a mapping that relates elliptic domes and their punctured containers.

To determine this mapping, regard the dome as a collection of layers of similar elliptic domes, like layers of an onion. Choose O as the center of similarity, and for each scaling factor $\mu \le 1$, imagine a surface $E(\mu)$ such that a vertical half plane through the polar axis at angle θ intersects $E(\mu)$ along a quarter of an ellipse with semiaxes $\mu\rho(\theta)$ and μh. When $\rho(\theta) > 0,$ the coordinates r and z of points on this similar ellipse satisfy

$$\left(\frac{r}{\mu\rho(\theta)}\right)^2 + \left(\frac{z}{\mu h}\right)^2 = 1. \tag{2}$$

Regard the punctured container as a collection of coaxial layers of similar punctured cylindrical surfaces $C(\mu)$.

It is easy to relate the cylindrical coordinates $(r',\, \theta',\, z')$ of each point on $C(\mu)$ to the coordinates $(r,\, \theta,\, z)$ of the corresponding point on $E(\mu)$. First, we have

$$\theta' = \theta, \quad z' = z, \quad r' = \mu\rho(\theta). \tag{3}$$

From (2) we find $r^2 + z^2\rho\,(\theta)^2\,/h^2 = \mu^2\rho(\theta)^2$, hence (3) becomes

$$\theta' = \theta, \quad z' = z, \quad r' = \sqrt{r^2 + z^2\rho\,(\theta)^2\,/h^2}. \tag{4}$$

The three equations in (4), which are independent of μ, describe a *mapping* from each point $(r,\, \theta,\, z)$, not on the polar axis, of the solid elliptic dome to the corresponding point $(r',\, \theta',\, z')$ on its punctured container. On the polar axis, $r = 0$ and θ is undefined.

Using (2) in (4) we obtain (3), hence points on the ellipse described by (2) are mapped onto the vertical segment of length μh through the base point $(\mu\rho(\theta),\, \theta)$. It is helpful to think of the solid elliptic dome as made up of *elliptic fibers* emanating from the points on the base. Mapping (4) converts each elliptic fiber into a vertical fiber through the corresponding point on the base of the punctured container.

Preservation of volumes. Now we show that mapping (4) preserves volumes. The volume element in the $(r,\, \theta,\, z)$ system is given by $r\,dr\,d\theta\,dz$, while that in the $(r',\, \theta', z')$ system is $r'dr'd\theta'dz'$. From (4) we have

$$(r')^2 = r^2 + z^2\rho\,(\theta)^2\,/h^2$$

which, for fixed z and θ, gives $r'dr' = r dr$. From (4) we also have $d\theta' = d\theta$ and $dz' = dz$, so the volume elements are equal: $r\,dr\,d\theta\,dz = r'dr'd\theta'dz'$. This proves:

Theorem 3. *Mapping* (4)*, from a general elliptic dome to its punctured prismatic container, preserves volumes. In particular, every general uniform elliptic dome is reducible.*

As an immediate consequence of Theorem 3 we obtain:

Corollary 5. *The volume of a general elliptic dome is equal to the volume of its circumscribing punctured cylindrical container, that is, two-thirds the volume of the circumscribing unpunctured cylindrical container which, in turn, is simply the area of the base times the height.*

The same formulas show that for a fixed altitude z, we have $r\,dr\,d\theta = r'dr'd\theta'$. In other words, the mapping also preserves areas of horizontal cross sections cut from the elliptic dome and its punctured container. This also implies Corollary 5 because of the slicing principle.

Lambert's classical mapping as a special case. Our mapping (4) generalizes Lambert's classical mapping [2], which is effected by wrapping a tangent cylinder about the equator, and then projecting the surface of the sphere onto this cylinder by rays through the axis which are parallel to the equatorial plane. Lambert's mapping takes points on the spherical surface (not at the north or south pole) and maps them onto points on the lateral cylindrical surface in a way that preserves areas. For a solid sphere, our mapping (4) takes each point not on the polar axis and maps it onto a point of the punctured solid cylinder in a way that preserves volumes. Moreover, analysis of a thin shell (similar to that in [1; Section 6]) shows that (4) also preserves areas when the surface of an Archimedean dome is mapped onto the lateral surface of its prismatic container. Consequently, we have:

Theorem 4. *Mapping* (4)*, from the surface of an Archimedean dome onto the lateral surface of its prismatic container, preserves areas.*

In the limiting case when the Archimedean dome becomes a hemisphere we get:

Corollary 6. (Lambert) *Mapping* (4)*, from the surface of a sphere to the lateral surface of its tangent cylinder, preserves areas.*

If the hemisphere in this limiting case has radius a, it is easily verified that (4) reduces to Lambert's mapping: $\theta' = \theta, \;\; z' = z, \;\; r' = a.$

6. Nonuniform elliptic domes

Mapping (4) takes each point P of an elliptic dome and carries it onto a point P' of its punctured container. Imagine an arbitrary mass density assigned to P, and assign the same mass density to its image P'. If a set of points P fills out a portion of the dome of volume v and total mass m, say, then the image points P' fill out a solid, which we call the *counterpart*, having the same volume v and the same total mass m. This can be stated as an extension of Theorem 3:

Theorem 5. *Any portion of a general nonuniform elliptic dome is reducible.*

By analogy with Theorem 3, we can say that mapping (4) “with weights” also preserves masses.

Fiber-elliptic and shell-elliptic domes. Next we describe a special way of assigning variable mass density to the points of a general elliptic dome and its punctured container so that corresponding portions of the dome and its counterpart have the same mass. The structure of the dome as a collection of similar domes plays an essential role in this description.

First assign mass density $f(r,\theta)$ to each point (r,θ) on the base of the dome and of its cylindrical container. Consider the elliptic fiber that emanates from any point $(\mu\rho(\theta),\,\theta)$ on the base, and assign the same mass density $f(\mu\rho(\theta),\,\theta)$ to each point of this fiber. In other words, the mass density along the elliptic fiber has a constant value inherited from the point at which the fiber meets the base. Of course, the constant may differ from point to point on the base. The elliptic fiber maps into a vertical fiber in the punctured container (of length μh, where h is the altitude of the dome), and we assign the same mass density $f(\mu\rho(\theta),\,\theta)$ to each point on this vertical fiber. In this way we produce a nonuniform elliptic dome and its punctured container, each with variable mass density inherited from the base. We call such a dome *fiber-elliptic*. The punctured container with density assigned in this manner is called the *counterpart* of the dome. The volume element multiplied by mass density is the same for both the dome and its counterpart.

An important special case occurs when the assigned density is also constant along the base curve $r = \rho(\theta)$ and along each curve $r = \mu\rho(\theta)$ similar to the base curve, where the constant density depends only on μ. Then each elliptic surface $E(\mu)$ will have its own constant density. We call domes with this assignment of mass density *shell-elliptic*. For fiber-elliptic and shell-elliptic domes, horizontal slices cut from any portion of the dome and its counterpart have equal masses, and their centers of mass are at the same height above the base. Thus, as a consequence of Theorem 3 we have:

Corollary 7. (a) *Any portion of a fiber-elliptic dome is reducible.*

(b) *In particular, any portion of a shell-elliptic dome is reducible.*

(c) *In particular, a sphere with spherically symmetric mass distribution is reducible.*

The reducibility properties of an elliptic dome also hold for the more general case in which we multiply the mass density $f(\mu\rho(\theta),\,\theta)$ by any function of z. Such change of density could be imposed, for example, by an external field (such as atmospheric density in a gravitational field that depends only on the height z). Consequently, not only are the volume and mass of any portion of this type of nonuniform elliptical dome equal to those of its counterpart, but the same is true for all moments with respect to the horizontal base.

Elliptic shells and cavities. Consider a general elliptic dome of altitude h, and denote its elliptic surface by $E(1)$. Scale $E(1)$ by a factor μ, where $0 < \mu < 1$, to produce a similar elliptic surface $E(\mu)$. The region between the two surfaces $E(\mu)$ and $E(1)$ is called an *elliptic shell*. It can be regarded as an elliptic dome with a cavity, or, equivalently, as a shell-elliptic dome with density 0 assigned to each point between $E(\mu)$ and the center.

Figure 5a shows an elliptic shell element, and Figure 5b shows its counterpart. Each base in the equatorial plane is bounded by portions of two curves with polar equations $r = \rho(\theta)$ and $r = \mu\rho(\theta)$, and two segments with $\theta = \theta_1$ and $\theta = \theta_2$. The shell element has two vertical plane faces, each consisting of a region between two similar ellipses. If μ is close to 1 and if θ_1 and θ_2 are nearly equal, the elliptic shell element can be thought of as a thin elliptic fiber, as was done earlier.

Consider a horizontal slice between two horizontal planes that cut both the inner and outer elliptic boundaries of the shell element. In other words, both planes are pierced by the cavity. The prismatic counterpart of this slice has horizontal cross sections congruent to the base, so its centroid lies *midway* between the two cutting planes. The same is true for the slice of the shell and for the center of mass of a slice cut from an assemblage of uniform elliptic shell elements, each with its own constant density.

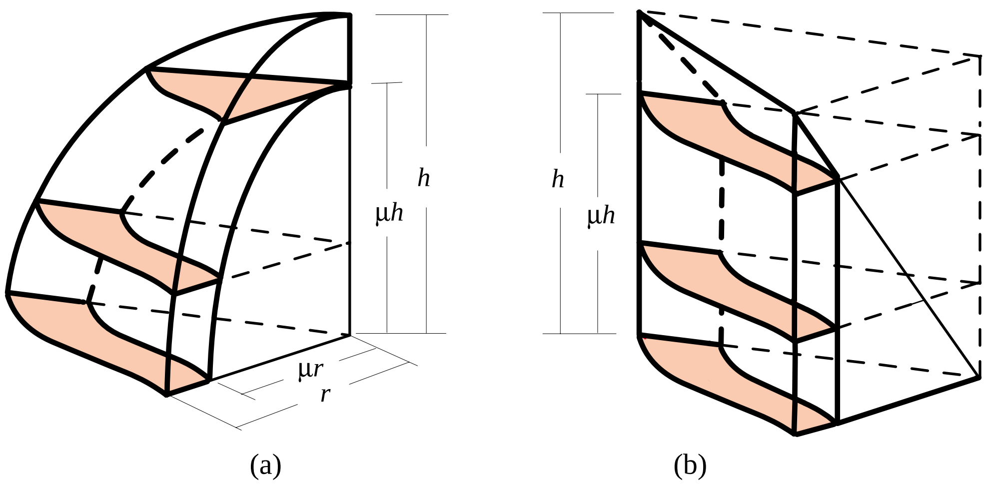


Figure 5. An elliptic shell element (a) and its counterpart (b).

In the same way, if we build a nonuniform shell-elliptic solid with a finite number of similar elliptic shells, each with its density inherited from the base, then any horizontal slice pierced by the cavity has its center of mass midway between the two horizontal cutting planes. Moreover, the following theorem holds for every such shell-elliptic wedge.

Theorem 6. *Any horizontal slice pierced by the cavity of a nonuniform shell-elliptic wedge has volume and mass equal, respectively, to those of its prismatic counterpart. Each volume and mass is independent of the height above the base and each is proportional to the thickness of the slice. Consequently, the center of mass of such a slice lies midway between the two cutting planes.*

Corollary 8. (Sphere with cavity) *Consider a spherically symmetric distribution of mass inside a solid sphere with a concentric cavity. Any slice between parallel* *planes pierced by the cavity has volume and mass proportional to the thickness of the slice, and is independent of the location of the slice.*

Corollary 8 implies that the one-dimensional vertical projection of the density is constant along the cavity. This simple result has profound consequences in tomography, which deals with the inverse problem of reconstructing spatial density distributions from a knowledge of their lower dimensional projections. Details of this application will appear elsewhere.

7. Formulas for volume and centroid

This section uses reducibility to give specific formulas for volumes and centroids of various building blocks of elliptic domes with an arbitrary curvilinear base.

Volume of a shell element. We begin with the simplest case. Cut a wedge from an elliptic dome of altitude h by two vertical half planes $\theta = \theta_1$ and $\theta = \theta_2$ through the polar axis, and then remove a similar wedge scaled by a factor μ, where $0 < \mu < 1$, as shown in Figure 5a. Assume the unpunctured cylindrical container in Figure 5b has volume V. By Corollary 5 the outer wedge has volume $2V/3$, and the similar inner wedge has volume $2\mu^3 V/3$, so the volume v of the shell element and its prismatic counterpart is the difference

$$v = \frac{2}{3}V(1 - \mu^3). \tag{5}$$

Now $V = Ah$, where A is the area of the base of both the elliptic wedge and its container. The base of the elliptic shell element and its unpunctured container have area $B = A - \mu^2 A$, so $A = B/(1 - \mu^2)$, $V = Bh/(1 - \mu^2)$, and (5) can be written as

$$v = \frac{2}{3}Bh\frac{1 - \mu^3}{1 - \mu^2}. \tag{6}$$

Formula (6) also holds for the total volume of any assemblage of elliptic shell elements with a given h and μ, with B representing the total base area. The product Bh is the volume of the corresponding unpunctured cylindrical container of altitude h, so (6) gives us the formula

$$v_\mu(h) = \frac{2}{3}v_{cyl}\frac{1 - \mu^3}{1 - \mu^2}\,, \tag{7}$$

where $v_\mu(h)$ is the volume of the assemblage of elliptic shell elements and of the counterpart, and v_{cyl} is the volume of its *unpunctured* cylindrical container. When $\mu = 0$ in (7), the assemblage of elliptic wedges has volume $v_0(h) = 2v_{cyl}/3$, so we can write (7) in the form

$$v_\mu(h) = v_0(h)\frac{1 - \mu^3}{1 - \mu^2}, \tag{8}$$

where $v_0(h)$ is the volume of the outer dome of the assemblage and its counterpart. If μ approaches 1 the shell becomes very thin, the quotient $(1 - \mu^3)/\left(1 - \mu^2\right)$ approaches 3/2, and (7) shows that $v_\mu(h)$ approaches v_{cyl}. In other words, a very thin elliptic shell element has volume very nearly equal to that of its very thin unpunctured cylindrical container. An Archimedean shell has constant thickness equal to that of the prismatic container, so the lateral surface area of any assemblage of Archimedean wedges is equal to the lateral surface area of its prismatic container, a result derived in [1]. Note that this argument cannot be used to find the surface area of an nonspherical elliptic shell because it does not have constant thickness.

Next we derive a formula for the height of the centroid of any uniform elliptic wedge above the plane of its base.

Theorem 7. *Any uniform elliptic wedge or dome of altitude h has volume two-thirds that of its unpunctured prismatic container. Its centroid is located at height c above the plane of the base, where*

$$c = \frac{3}{8}h. \tag{9}$$

Proof. It suffices to prove (9) for the prismatic counterpart. For any prism of altitude h, the centroid is at a distance $h/2$ above the plane of the base. For a cone or pyramid with the same base and altitude it is known that the centroid is at a distance $3h/4$ from the vertex. To determine the height c of the centroid of a punctured prismatic container above the plane of the base, assume the unpunctured prismatic container has volume V and equate moments to get

$$c\left(\frac{2}{3}V\right) + \frac{3h}{4}\left(\frac{1}{3}V\right) = \frac{h}{2}V,$$

from which we find (9). By Theorem 5, the centroid of the inscribed elliptic wedge is also at height $3h/8$ above the base. The result is also true for any uniform elliptic dome formed as an assemblage of wedges. □

Equation (9) is equivalent to saying, in the style of Archimedes, that the centroid divides the altitude in the ratio 3:5.

Corollary 9. (a) *The centroid of a uniform Archimedean dome divides its altitude in the ratio* 3:5*.*

(b) (Archimedes) *The centroid of a uniform hemisphere divides its altitude in the ratio* 3:5*.*

Formula (9) is obviously true for the center of mass of any nonuniform assemblage of elliptic wedges of altitude h, each with its own constant density.

Centroid of a shell element. Now we can find, for any elliptic shell element, the height $c_\mu(h)$ of its centroid above the plane of its base. The volume and centroid results are summarized as follows:

Theorem 8. *Any nonuniform assemblage of elliptic shell elements with common altitude h and scaling factor μ has volume $v_\mu(h)$ given by* (8)*. The height $c_\mu(h)$ of the centroid above the plane of its base is given by*

$$c_\mu(h) = \frac{3}{8}h\frac{1-\mu^4}{1-\mu^3}. \tag{10}$$

Proof. Consider first a single uniform elliptic shell element. Again it suffices to do the calculation for the prismatic counterpart. The inner wedge has altitude $\mu h,$ so by (9) its centroid is at height $3\mu h/8$. The centroid of the outer wedge is at height $3h/8$. If the outer wedge has volume V_{outer}, the inner wedge has volume $\mu^3 V_{outer}$, and the shell element between them has volume $(1-\mu^3)V_{outer}$. Equating moments and canceling the common factor V_{outer} we find

$$\left(\frac{3}{8}\mu h\right)\mu^3 \;+\; c_\mu(h)(1-\mu^3) \;=\frac{3}{8}h,$$

from which we obtain (10). Formula (10) also holds for any nonuniform assemblage of elliptic shell elements with the same h and μ, each of constant density, although the density can differ from element to element. □

When $\mu = 0$, (10) gives $c_0(h) = 3h/8$.

When $\mu \;\to\; 1$, the shell becomes very thin and the limiting value of $c_\mu(h)$ in (10) is $h/2$. This also follows from Theorem 6 when the shell is very thin and the slice includes the entire dome. It is also consistent with Corollary 15 of [1], which states that the centroid of the surface area of an Archimedean dome is at the midpoint of its altitude.

Centroid of a slice of a wedge. More generally, we can determine the centroid of any slice of altitude z of a uniform elliptic wedge. By reducing this calculation to that of the prismatic counterpart, shown in Figure 6, the analysis becomes very simple. For clarity, the base in Figure 6 is shown as a triangle, but the same argument applies to a more general base like that in Figure 5. The slice in question is obtained from a prism of altitude z and volume $V(z) \,=\, \lambda V$, where V is the volume of the unpunctured prismatic container of altitude h, and $\lambda \,=\, z/h$. The centroid of the slice is at an altitude $z/2$ above the base. We remove from this slice a pyramidal portion of altitude z and volume $v(z) \,=\, \lambda^3 V/3$, whose centroid is at an altitude $3z/4$ above the base. The portion that remains has volume

$$V(z) - v(z) \;= \left(\lambda - \frac{1}{3}\lambda^3\right) V \tag{11}$$

and centroid at altitude $c(z)$ above the base. To determine $c(z)$, equate moments to obtain

$$\frac{3z}{4}v(z) + c(z)(V(z) - v(z)) = \frac{z}{2}V(z),$$

which gives

$$c(z) = \frac{\dfrac{z}{2}\,V(z) - \dfrac{3z}{4}v(z)}{V(z) - v(z)}.$$

Because $V(z) = \lambda V$, and $v(z) = \lambda^3 V/3,$ we obtain the following theorem.

Theorem 9. *Any slice of altitude z cut from a uniform elliptic wedge of altitude h has volume given by* (11)*, where $\lambda = z/h$ and V is the volume of the unpunctured prismatic container. The height $c(z)$ of the centroid is given by*

$$c(z) = \frac{3}{4}z\frac{2-\lambda^2}{3-\;\lambda^2}. \tag{12}$$

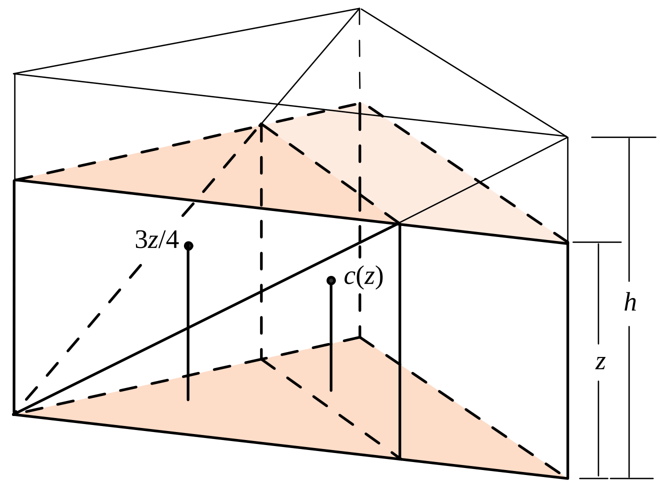


Figure 6. Calculating the centroid of a slice of altitude z cut from a wedge of altitude h.

When $z = h$ then $\lambda = 1$ and this reduces to (9). For small z the right member of (12) is asymptotic to $z/2$. This is reasonable because for small z the walls of the dome are nearly perpendicular to the plane of the equatorial base, so the dome is almost cylindrical near the base.

Centroid of a slice of a wedge shell element. There is a common generalization of (10) and (12). Cut a slice of altitude z from a shell element having altitude h and scaling factor μ, and let $c_\mu(z)$ denote the height of its centroid above the base. Again, we simplify the calculation of $c_\mu(z)$ by reducing it to that of its prismatic counterpart. The slice in question is obtained from an unpunctured prism of altitude z, whose centroid has altitude $z/2$ above the base. As in Theorem 9, let $\lambda = z/h$. If $\lambda \leq \mu$, the slice lies within the cavity, and the prismatic counterpart is the same unpunctured prism of altitude z, in which case we know from Theorem 6 that

$$c_\mu(z) = \frac{z}{2}, \quad (\lambda \leq \mu). \tag{13}$$

But if $\lambda \geq \mu$, the slice cuts the outer elliptic dome as shown in Figure 7a. In this case the counterpart slice has a slant face due to a piece removed by the puncturing pyramid, as indicated in Figure 7b.

Let V denote the volume of the unpunctured prismatic container of the outer dome. Then λV is the volume of the unpunctured prism of altitude z. Remove from this prism the puncturing pyramid of volume $\lambda^3 V/3$, leaving a solid whose volume is

$$V(z) = \lambda V - \frac{1}{3}\lambda^3 V, \quad (\lambda \geq \mu) \tag{14}$$

and whose centroid is at altitude $c(z)$ given by (12). This solid, in turn, is the union of the counterpart slice in question, and an adjacent pyramid with vertex O,

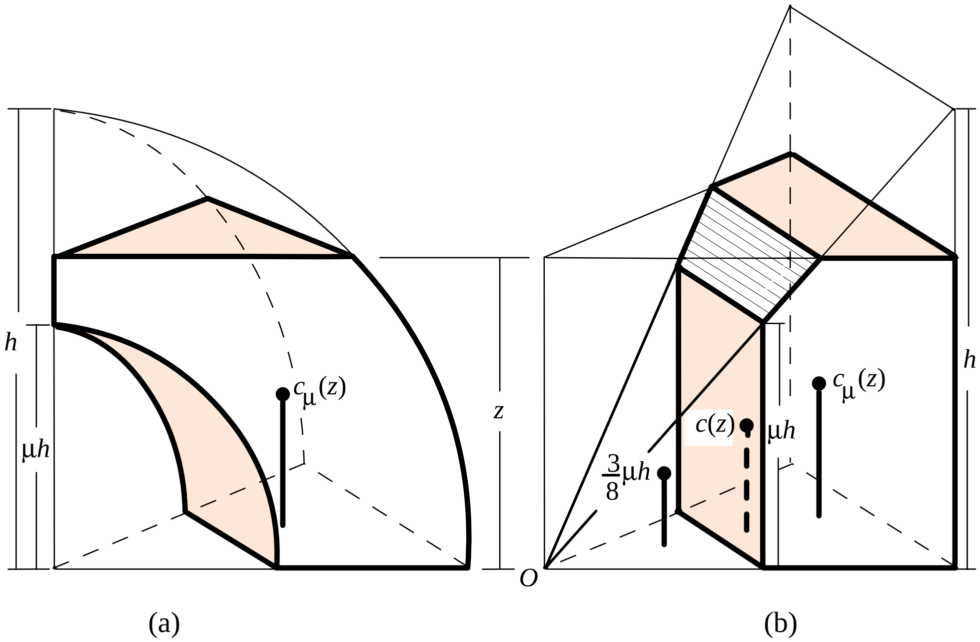


Figure 7. Determining the centroid of a slice of altitude $z \geq \mu h$ cut from an elliptic shell element.

altitude μh, volume

$$v_\mu = \frac{2}{3}\mu^3 V, \tag{15}$$

and centroid at altitude $3\mu h/8$. The counterpart slice in question has volume

$$V(z) - v_\mu = \left(\lambda - \frac{1}{3}\lambda^3 - \frac{2}{3}\mu^3\right) V. \tag{16}$$

To find the altitude $c_\mu(z)$ of its centroid we equate moments and obtain

$$\left(\frac{3}{8}\mu h\right) v_\mu + c_\mu(z)(V(z) - v_\mu) = c(z)V(z),$$

from which we find

$$c_\mu(z) = \frac{c(z)V(z) - \left(\frac{3}{8}\mu h\right) v_\mu}{V(z) - v_\mu}.$$

Now we use (12), (14), (15)and (16). After some simplification we find the result

$$c_\mu(z) = \frac{3}{4}h\,\frac{\lambda^2(2-\lambda^2) - \mu^4}{\lambda(3-\lambda^2) - 2\mu^3} \quad (\lambda \geq \mu). \tag{17}$$

When $\lambda = \mu$, (17) reduces to (13); when $\lambda = 1$ then $z = h$ and (17) reduces to (10); and when $\mu = 0$, (17) reduces to (12). The results are summarized by the following theorem.

Theorem 10. *Any horizontal slice of altitude $z \geq \mu h$ cut from a wedge shell element of altitude h and scaling factor μ has volume given by* (16)*, where $\lambda = z/h$. The altitude of its centroid above the base is given by* (17)*. In particular these formulas hold for any slice of a shell of an Archimedean, elliptic, or spherical dome.*

Note: Theorem 6 covers the case $z \leq \mu h$.

In deriving the formulas in this section we made no essential use of the fact that the shell elements are elliptic. The important fact is that each shell element is the region between two similar objects.

8. The necessity of elliptic profiles

We know that every horizontal plane cuts an elliptic dome and its punctured cylindrical container in cross sections of equal area. This section reveals the surprising fact that the elliptical shape of the dome is actually a consequence of this property.

Consider a dome of altitude h, and its punctured prismatic counterpart having a congruent base bounded by a curve satisfying a polar equation $r = \rho(\theta)$. Each vertical half plane through the polar axis at angle θ cuts the dome along a curve we call a *profile*, illustrated by the example in Figure 8a. This is like the elliptic dome in Figure 5a, except that we do not assume that the profiles are elliptic. Each profile passes through a point $(\rho(\theta),\, \theta)$ on the outer edge of the base. At altitude z above the base a point on the profile is at distance r from the polar axis, where r is a function of z that determines the shape of the profiles. We define a general profile dome to be one in which each horizontal cross section is similar to the base. Figure 8a shows a portion of a dome in which $\rho(\theta) > 0$. This portion is a wedge with two vertical plane faces that can be thought of as "walls" forming part of the boundary of the wedge.

Suppose that a horizontal plane at distance z above the base cuts a region of area $A(z)$ from the wedge and a region of area $B(z)$ from the punctured prism. We know that $A(0) = B(0)$. Now we assume that $A(z) = B(z)$ for some $z > 0$ and deduce that the point on the profile with polar coordinates (r, θ, z) satisfies the equation

$$\left(\frac{r}{\rho(\theta)}\right)^2 + \left(\frac{z}{h}\right)^2 = 1 \tag{18}$$

if $\rho(\theta) > 0$. In other words, the point on the profile at a height where the areas are equal lies on an ellipse with vertical semi axis of length h, and horizontal semi axis of length $\rho(\theta)$. Consequently, if $A(z) = B(z)$ for every z from 0 to h, the profile will fill out a quarter of an ellipse and the dome will necessarily be elliptic. Note that (18) implies that $r \to 0$ as $z \to h$.

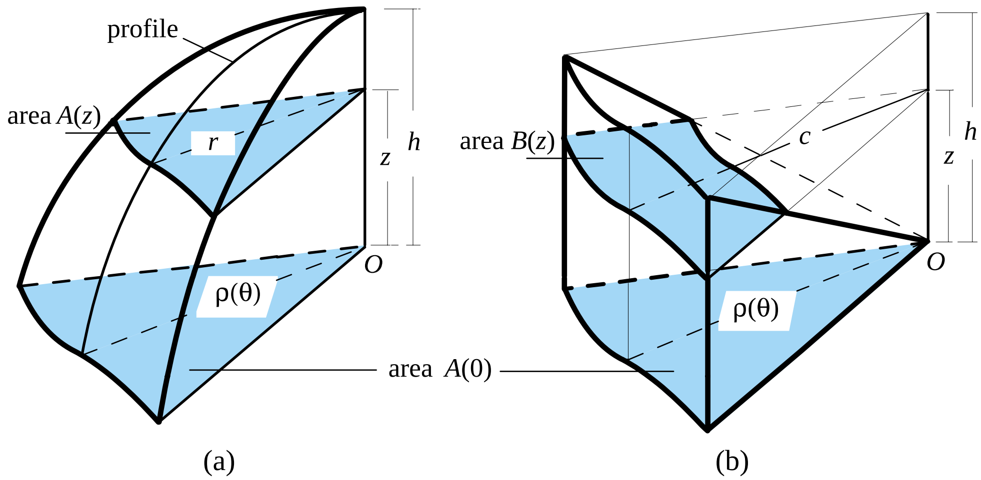


Figure 8. Determining the elliptic shape of the profiles as a consequence of the relation $A(z) = B(z)$.

To deduce (18), note that the horizontal cross section of area $A(z)$ in Figure 8a is similar to the base with similarity ratio $r/\rho(\theta)$, where $\rho(\theta)$ denotes the radial distance to the point where the profile intersects the base, and r is the length of the radial segment at height z. By similarity, $A(z) = (r/\rho(\theta))^2 A(0)$. In Figure 8b, area $B(z)$ is equal to $A(0)$ minus the area of a smaller similar region with similarity ratio $c/\rho(\theta)$, where c is the length of the parallel radial segment of the smaller similar region at height z. By similarity, $c/\rho(\theta) = z/h$, hence $B(z) = (1 - (z/h)^2)A(0)$. Equating this to $A(z)$ we find $(1 - (z/h)^2)A(0) = (r/\rho(\theta))^2 A(0)$, which gives (18). And, of course, we already know that (18) implies $A(z) = B(z)$ for every z. Thus we have proved:

Theorem 11. *Corresponding horizontal cross sections of a general profile uniform dome and its punctured prismatic counterpart have equal areas if, and only if, each profile is elliptic.*

As already remarked in Section 5, an elliptic dome can be deformed in such a way that areas of horizontal cross sections are preserved but the deformed dome no longer has elliptic profiles. At first glance, this may seem to contradict Theorem 11. However, such a deformation will distort the vertical walls; the dome will not satisfy the requirements of Theorem 11, and also the punctured counterpart will no longer be prismatic.

An immediate consequence of Theorem 11 is that any reducible general profile dome necessarily has elliptic profiles, because if all horizontal slices of such a dome and its counterpart have equal volumes then the cross sections must have equal areas. We have also verified that Theorem 11 can be extended to nonuniform general profile domes built from a finite number of general profile similar shells, each with its own constant density, under the condition that corresponding horizontal slices of the dome and its counterpart have equal masses, with no requirements on volumes or reducibility.

Concluding remarks. The original motivation for this research was to extend to more general solids classical properties which seemed to be unique to spheres and hemispheres. Initially an extension was given for Archimedean domes and a further extension was made by simply dilating these domes in a vertical direction. These extensions could also have been analyzed by using properties of inscribed spheroids.

A significant extension was made when we introduced polygonal elliptic domes whose bases could be arbitrary polygons, not necessarily circumscribing the circle. In this case there are no inscribed spheroids to aid in the analysis, but the method of punctured containers was applicable. This led naturally to general elliptic domes with arbitrary base, and the method of punctured containers was formulated in terms of mappings that preserve volumes.

But the real power of the method is revealed by the treatment of nonuniform mass distributions. Problems of determining volumes and centroids of elliptic wedges, shells, and their slices, including those with cavities, were reduced to those of simpler prismatic containers. Finally, we showed that domes with elliptic profiles are essentially the only ones that are reducible.

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Forum Geometricorum
Volume 7 (2007) 53–65.

FORUM GEOM
ISSN 1534-1178

Midcircles and the Arbelos

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Abstract. We begin with a study of inversions mapping one given circle into another. The results are applied to the famous configuration of an arbelos. In particular, we show how to construct three infinite Pappus chains associated with the arbelos.

1. Inversions swapping two circles

Given two circles $O_i(r_i)$, $i = 1, 2$, in the plane, we seek the inversions which transform one of them into the other. Set up a cartesian coordinate system such that for $i = 1,\ 2$, O_i is the point $(a_i,\ 0)$. The endpoints of the diameters of the circles on the x-axis are $(a_i \pm r_i,\ 0)$. Let $(a,\ 0)$ and Φ be the center and the power of inversion. This means, for an appropriate choice of $\varepsilon = \pm 1$,

$$(a_1 + \varepsilon \cdot r_1 - a)(a_2 + r_2 - a) = (a_1 - \varepsilon \cdot r_1 - a)(a_2 - r_2 - a) = \Phi.$$

Solving these equations we obtain

$$a = \frac{r_2 a_1 + \varepsilon \cdot r_1 a_2}{r_2 + \varepsilon \cdot r_1}, \tag{1}$$

$$\Phi = \frac{\varepsilon \cdot r_1 r_2((r_2 + \varepsilon \cdot r_1)^2 - (a_1 - a_2)^2)}{(r_2 + \varepsilon \cdot r_1)^2}. \tag{2}$$

From (1) it is clear that the center of inversion is a center of similitude of the two circles, internal or external according as $\varepsilon = +1$ or -1. The two circles of inversion, real or imaginary, are given by $(x - a)^2 + y^2 = \Phi$, or more explicitly,

$$r_2((x - a_1)^2 + y^2 - r_1^2) + \varepsilon \cdot r_1((x - a_2)^2 + y^2 - r_2^2) = 0. \tag{3}$$

They are members of the pencil of circles generated by the two given circles. Following Dixon [1, pp.86–88], we call these the *midcircles* $\mathcal{M}_\varepsilon$, $\varepsilon = \pm 1$, of the two given circles $O_i(r_i),\ i = 1, 2$. From (2) we conclude that
(i) the *internal* midcircle $\mathcal{M}_+$ is real if and only if $r_1 + r_2 > d$, the distance between the two centers, and
(ii) the *external* midcircle $\mathcal{M}_-$ is real if and only if $|r_1 - r_2| < d$.
In particular, if the two given circles intersect, then there are two real circles of inversion through their common points, with centers at the centers of similitudes. See Figure 1.

Lemma 1. *The image of the circle with center B, radius r, under inversion at a point A with power Φ is the circle of radius $\left|\frac{\Phi}{d^2 - r^2}\right| r$, and center dividing AB at the ratio $AP : PB = \Phi : d^2 - r^2 - \Phi$, where d is the distance between A and B.*

Publication Date: February 19, 2007. Communicating Editor: Paul Yiu.

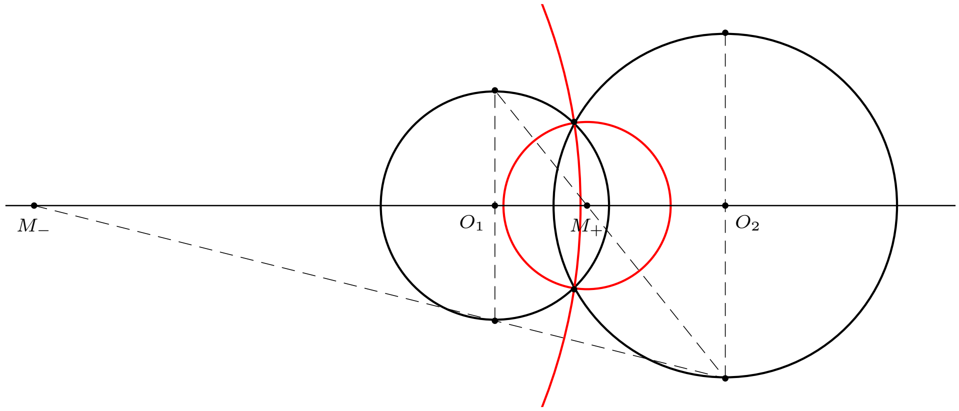

Figure 1.

2. A locus property of the midcircles

Proposition 2. *The locus of the center of inversion mapping two given circles $O_i(a_i)$, $i = 1, 2$, into two congruent circles is the union of their midcircles $\mathcal{M}_+$ and $\mathcal{M}_-$.*

Proof. Let $d(P, Q)$ denote the distance between two points P and Q. Suppose inversion in P with power Φ transforms the given circles into congruent circles. By Lemma 1,

$$\frac{d(P, O_1)^2 - r_1^2}{d(P, O_2)^2 - r_2^2} = \varepsilon \cdot \frac{r_1}{r_2} \tag{4}$$

for $\varepsilon = \pm 1$. If we set up a coordinate system so that $O_i = (a_i, 0)$ for $i = 1, 2$, $P = (x, y)$, then (4) reduces to (3), showing that the locus of P is the union of the midcircles $\mathcal{M}_+$ and $\mathcal{M}_-$. $\square$

Corollary 3. *Given three circles, the common points of their midcircles taken by pairs are the centers of inversion that map the three given circles into three congruent circles.*

For $i, j = 1, 2, 3$, let $\mathcal{M}_{ij}$ be a midcircle of the circles $\mathcal{C}_i = O_i(R_i)$ and $\mathcal{C}_j = O_j(R_j)$. By Proposition 2 we have $\mathcal{M}_{ij} = R_j \cdot \mathcal{C}_i + \varepsilon_{ij} \cdot R_i \cdot \mathcal{C}_j$ with $\varepsilon_{ij} = \pm 1$. If we choose ε_{ij} to satisfy $\varepsilon_{12} \cdot \varepsilon_{23} \cdot \varepsilon_{31} = -1$, then the centers of $\mathcal{M}_{12}$, $\mathcal{M}_{23}$ and $\mathcal{M}_{31}$ are collinear. Since the radical center P of the triad $\mathcal{C}_i$, $i = 1, 2, 3$, has the same power with respect to these circles, they form a pencil and their common points X and Y are the poles of inversion mapping the circles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ into congruent circles.

The number of common points that are the poles of inversion mapping the circles $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ into a triple of congruent circles depends on the configuration of these circles.

(1) The maximal number is 8 and occurs when each pair of circles $\mathcal{C}_i$ and $\mathcal{C}_j$ have two distinct intersections. Of these 8 points, two correspond to

the three external midcircles while each pair of the remaining six points correspond to a combination of one external and two internal midcircles.

(2) The minimal number is 0. This occurs for instance when the circles belong to a pencil of circles without common points.

Corollary 4. *The locus of the centers of the circles that intersect three given circles at equal angles are* $0, 1, 2, 3$ *or* 4 *lines through their radical center* P *perpendicular to a line joining three of their centers of similitude.*

Proof. Let $\mathcal{C}_1 = A(R_1)$, $\mathcal{C}_2 = B(R_2)$, and $\mathcal{C}_3 = C(R_3)$ be the given circles. Consider three midcircles with collinear centers. If X is an intersection of these midcircles, reflection in the center line gives another common point Y. Consider an inversion τ with pole X that maps circle $\mathcal{C}_3$ to itself. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ become $\mathcal{C}_1' = A'(R_3)$ and $\mathcal{C}_2 = B'(R_3)$. If P' is the radical center of the circles $\mathcal{C}_1$, $\mathcal{C}_2'$ and $\mathcal{C}_3'$, then every circle $\mathcal{C} = P'(R)$ will intersect these 3 circles at equal angles. When we apply the inversion τ once again to the circles $\mathcal{C}_1'$, $\mathcal{C}_2'$, $\mathcal{C}_3$ and $\mathcal{C}$ we get the 3 original circles $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and a circle $\mathcal{C}'$ and since an inversion preserves angles circle $\mathcal{C}'$ will also intersect these original circles at equal angles.

The circles orthogonal to all circles $\mathcal{C}'$ are mapped by τ to lines through P'. This means that the circles orthogonal to $\mathcal{C}'$ all pass through the inversion pole X. By symmetry they also pass through Y, and thus form the pencil generated by the triple of midcircles we started with. The circles $\mathcal{C}'$ form therefore a pencil as well, and their centers lie on XY as X and Y are the limit-points of this pencil. □

Remark. Not every point on the line leads to a real circle, and not every real circle leads to real intersections and real angles.

As an example we consider the A-, B- and C-Soddy circles of a triangle ABC. Recall that the A-Soddy circle of a triangle is the circle with center A and radius $s - a$, where s is the semiperimeter of triangle ABC. The area enclosed in the interior of ABC by the A-, B- and C-Soddy circles form a skewed arbelos, as defined in [5]. The circles $\mathcal{F}_\phi$ making equal angles to the A-, B- and C-Soddy circles form a pencil, their centers lie on the Soddy line of ABC, while the only real line of three centers of midcircles is the tripolar of the Gergonne point X_7. [1]

The points X and Y in the proof of Corollary 4 are the limit points of the pencil generated by $\mathcal{F}_\phi$. In barycentric coordinates, these points are, for $\varepsilon = \pm 1$,

$$(4R + r) \cdot X_7 + \varepsilon \cdot \sqrt{3}s \cdot I = \left(2r_a + \varepsilon \cdot \sqrt{3}a : 2r_b + \varepsilon \cdot \sqrt{3}b : 2r_c + \varepsilon \cdot \sqrt{3}c\right),$$

where R, r, r_a, r_b, r_c are the circumradius, inradius, and inradii. The midpoint of XY is the Fletcher-point X_{1323}. See Figure 2.

3. The Arbelos

Now consider an arbelos, consisting of two interior semicircles $O_1(r_1)$ [2] and $O_2(r_2)$ and an exterior semicircle $O(r) = O_0(r)$, $r = r_1 + r_2$. Their points of

[1] The numbering of triangle centers following numbering in [2, 3].

[2] We adopt notations as used in [4]: By (PQ) we denote the circle with diameter PQ, by $P(r)$ the circle with center P and radius r, while $P(Q)$ is the circle with center P through Q and (PQR)

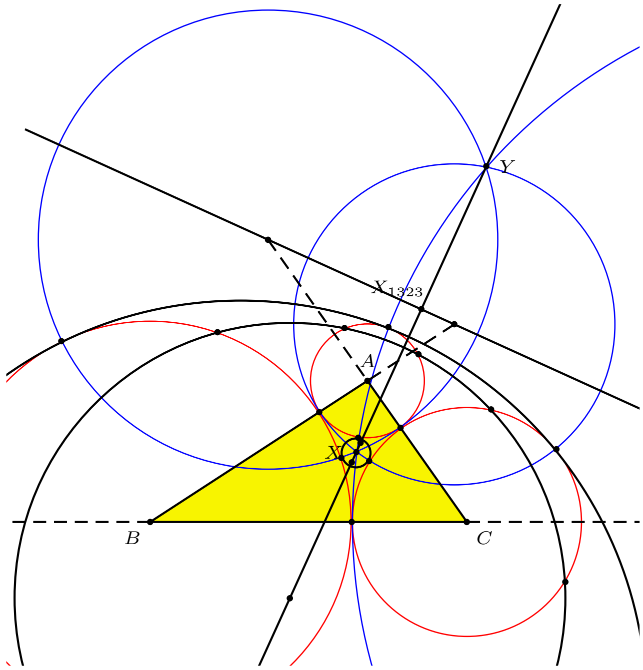

Figure 2.

tangency are A, B and C as indicated in Figure 3. The arbelos has an incircle (O'). For simple constructions of (O'), see [7, 8].

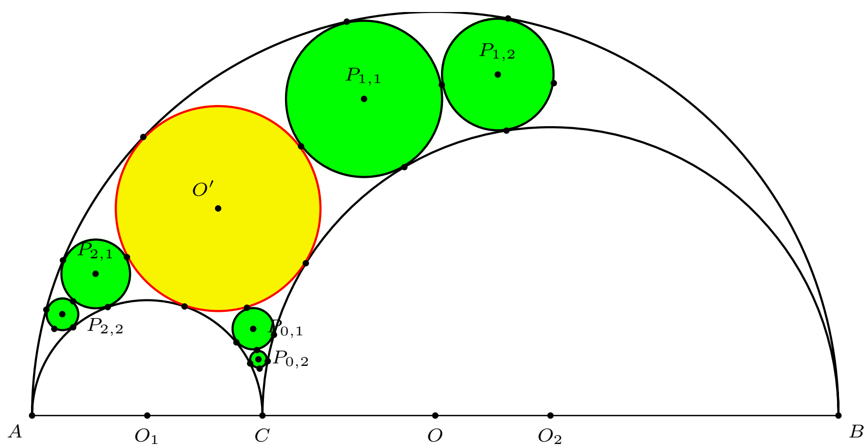

Figure 3.

We consider three Pappus chains $(\mathcal{P}_{i,n})$, $i = 0, 1, 2$. If (i, j, k) is a permutation of $(0, 1, 2)$, the Pappus chain $(\mathcal{P}_{i,n})$ is the sequence of circles tangent to both (O_j)

is the circle through P, Q and R. The circle (P) is the circle with center P, and radius clear from context.

and (O_k) defined recursively by
(i) $\mathcal{P}_{i,0} = (O')$, the incircle of the arbelos,
(ii) for $n \geq 1$, $\mathcal{P}_{i,n}$ is tangent to $\mathcal{P}_{i,n-1}$, (O_j) and (O_k),
(iii) for $n \geq 2$, $\mathcal{P}_{i,n}$ and $\mathcal{P}_{i,n-2}$ are distinct circles.

These Pappus chains are related to the centers of similitude of the circles of the arbelos. We denote by M_0 the external center of similitude of (O_1) and (O_2), and, for $i, j = 1, 2$, by M_i the internal center of similitude of (O) and (O_j). The midcircles are $M_0(C)$, $M_1(B)$ and $M_2(A)$. Each of the three midcircles leaves (O') and its reflection in AB invariant, so does each of the circles centered at A, B and C respectively and orthogonal to (O'). These six circles are thus members of a pencil, and O' lies on the radical axis of this pencil. Each of the latter three circles inverts two of the circles forming the arbelos to the tangents to (O') perpendicular to AB, and the third circle into one tangent to (O'). See Figure 4.

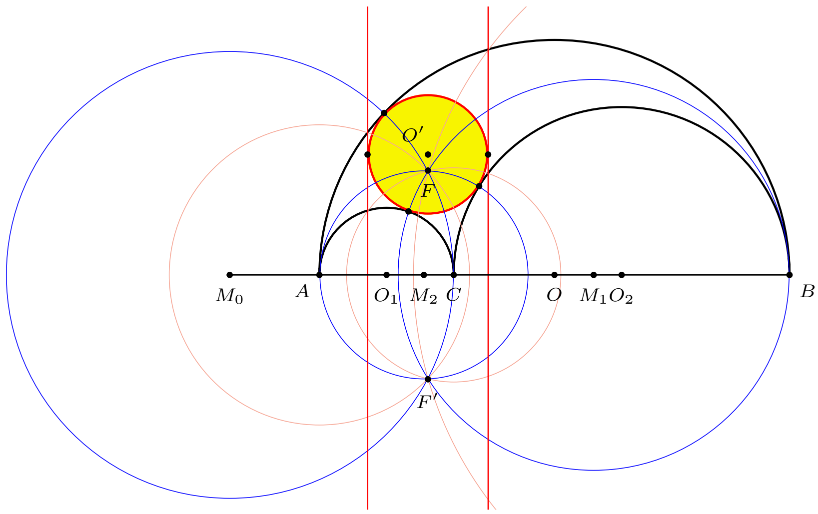

Figure 4.

We make a number of interesting observations pertaining to the construction of the Pappus chains. Denote by $P_{i,n}$ the center of the circle $\mathcal{P}_{i,n}$.

3.1. For $i = 0, 1, 2$, inversion in the midcircle (M_i) leaves $(P_{i,n})$ invariant. Consequently,

(1) the point of tangency of $(P_{i,n})$ and $(P_{i,n+1})$ lies on (M_i) and their common tangent passes through M_i;
(2) for every permuation (i, j, k) of $(0, 1, 2)$, the points of tangency of $(P_{i,n})$ with (O_j) and (O_k) are collinear with M_i. See Figure 5.

3.2. For every permutation (i, j, k) of $(0, 1, 2)$, inversion in (M_i) swaps $(P_{j,n})$ and $(P_{k,n})$. Hence,

(1) M_i, $P_{j,n}$ and $P_{k,n}$ are collinear;

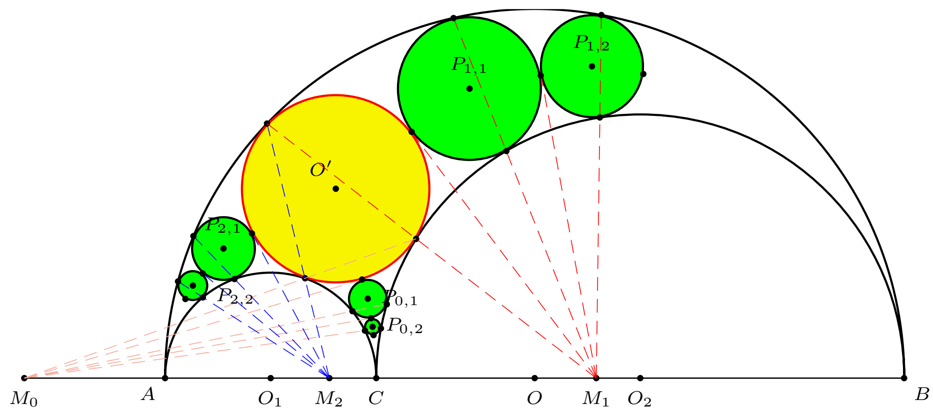

Figure 5.

(2) the points of tangency of $(P_{j,n})$ and $(P_{k,n})$ with (O_i) are collinear with M_i;
(3) the points of tangency of $(P_{j,n})$ with $(P_{j,n+1})$, and of $(P_{k,n})$ with $(P_{k,n+1})$ are collinear with M_i;
(4) the points of tangency of $(P_{j,n})$ with (O_k), and of $(P_{k,n})$ with (O_j) are collinear with M_i. See Figure 6.

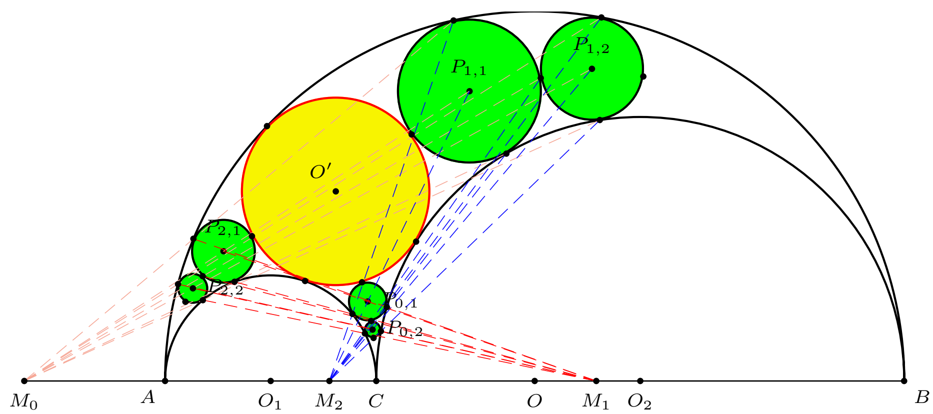

Figure 6.

3.3. Let (i, j, k) be a permutation of $(0, 1, 2)$. There is a circle $\mathcal{I}_i$ which inverts (O_j) and (O_k) respectively into the two tangents ℓ_1 and ℓ_2 of (O') perpendicular to AB. The Pappus chain $(P_{i,n})$ is inverted to a chain of congruent circles (Q_n) tangent to ℓ_1 and ℓ_2 as well, with $(Q_0) = (O')$. See Figure 7. The lines joining A to
(i) the point of tangency of (Q_n) with ℓ_1 (respectively ℓ_2) intersect $\mathcal{C}_0$ (respectively $\mathcal{C}_1$) at the points of tangency with $\mathcal{P}_{2,n}$,

(ii) the point of tangency of (Q_n) and (Q_{n-1}) intersect $\mathcal{M}_2$ at the point of tangency of $\mathcal{P}_{2,n}$ and $\mathcal{P}_{2,n-1}$.

From these points of tangency the circle $(P_{2,n})$ can be constructed.

Similarly, the lines joining B to

(iii) the point of tangency of (Q_n) with ℓ_1 (respectively ℓ_2) intersect $\mathcal{C}_2$ (respectively $\mathcal{C}_0$) at the points of tangency with $\mathcal{P}_{1,n}$,

(iv) the point of tangency of (Q_n) and (Q_{n-1}) intersect $\mathcal{M}_1$ at the point of tangency of $(P_{1,n})$ and $(P_{1,n-1})$.

From these points of tangency the circle $(P_{1,n})$ can be constructed.

Finally, the lines joining C to

(v) the point of tangency of (Q_n) with ℓ_i, $i = 1, 2$, intersect $\mathcal{C}_i$ at the points of tangency with $\mathcal{P}_{0,n}$,

(vi) the point of tangency of (Q_n) and (Q_{n-1}) intersect $\mathcal{M}_0$ at the point of tangency of $(P_{0,n})$ and $(P_{0,n-1})$.

From these points of tangency the circle $(P_{0,n})$ can be constructed.

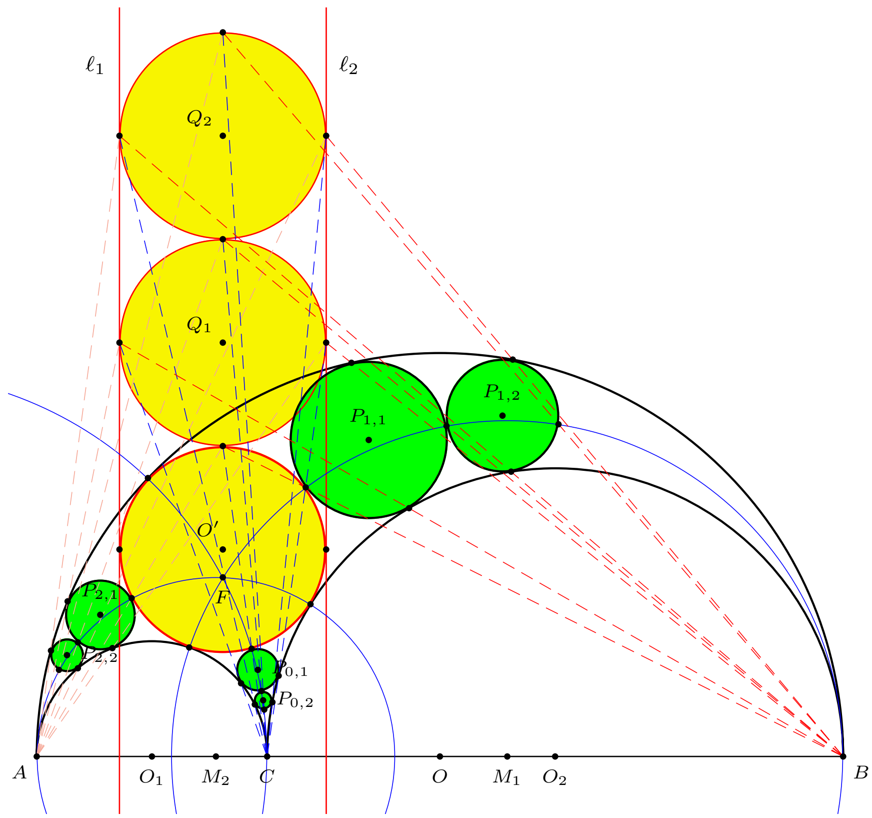

Figure 7.

3.4. Now consider the circle $\mathcal{K}_n$ through the points of tangency of $(P_{i,n})$ with (O_j) and (O_k) and orthogonal to $\mathcal{I}_i$. Then by inversion in $\mathcal{I}_i$ we see that $\mathcal{K}_n$ also passes through the points of tangency of (Q_n) with ℓ_1 and ℓ_2. Consequently the center K_n of $\mathcal{K}_n$ lies on the line through O' parallel to ℓ_1 and ℓ_2, which is the radical axis of the pencil of $\mathcal{I}_i$ and (M_i). By symmetry $\mathcal{K}_n$ passes through the points of tangency $(P_{i',n})$ with $(O_{j'})$ and $(O_{k'})$ for other permutations (i', j', k') of $(0, 1, 2)$ as well. The circle $\mathcal{K}_n$ thus passes through eight points of tangency, and all $\mathcal{K}_n$ are members of the same pencil.

With a similar reasoning the circle $\mathcal{L}_n = (L_n)$ tangent to $P_{i,n}$ and $P_{i,n+1}$ at their point of tangency as well as to (Q_n) and (Q_{n+1}) at their point of tangency, belongs to the same pencil as $\mathcal{K}_n$. See Figure 8.

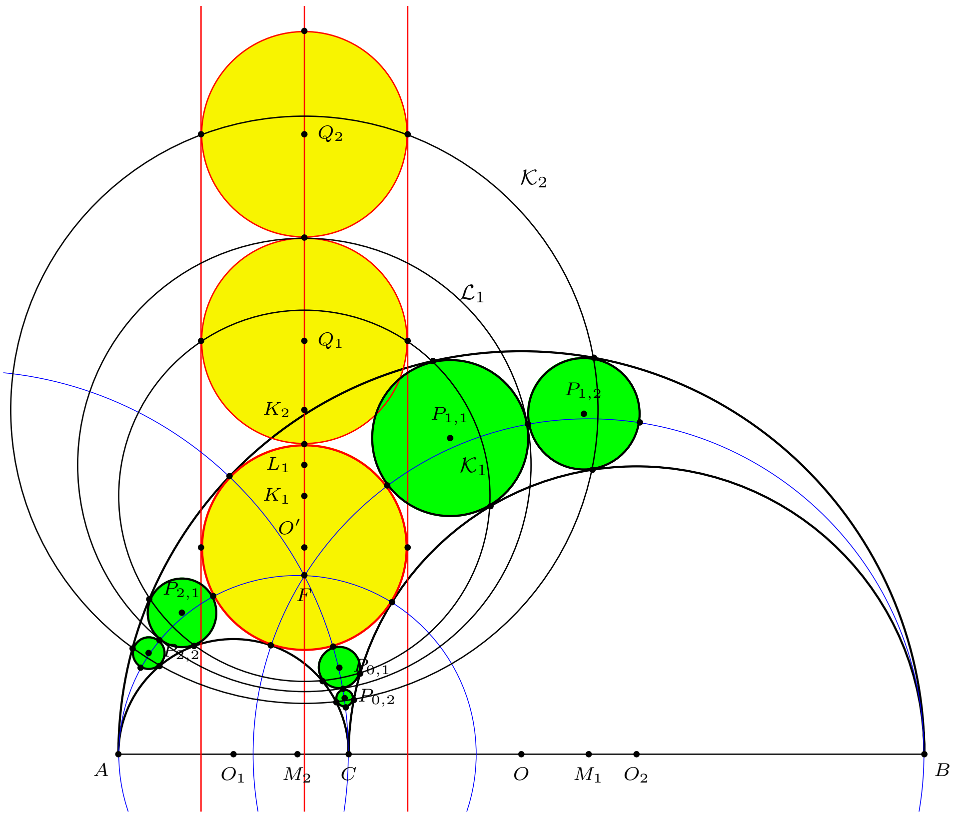

Figure 8.

The circles $\mathcal{K}_n$ and $\mathcal{L}_n$ make equal angles to the three arbelos semicircles (O), (O_1) and (O_2). In §5 we dive more deeply into circles making equal angles to three given circles.

4. λ-Archimedean circles

Recall that in the arbelos the twin circles of Archimedes have radius $r_A = \frac{r_1 r_2}{r}$. Circles congruent to these twin circles with relevant additional properties in the arbelos are called Archimedean.

Now let the homothety $\mathsf{h}(A, \mu)$ map O and O_1 to O' and O'_1. In [4] we have seen that the circle tangent to O' and O'_1 and to the line through C perpendicular to AB is Archimedean for any μ within obvious limitations. On the other hand from this we can conclude that when we apply the homothety $\mathsf{h}(A, \lambda)$ to the line through C perpendicular to AB, to find the line ℓ, then the circle tangent to ℓ, O and O' has radius λr_A. These circles are described in a different way in [6]. We call circles with radius λr_A and with additional relevant properties λ-Archimedean.

We can find a family of λ-Archimedean circles in a way similar to Bankoff's triplet circle. A proof showing that Bankoff's triplet circle is Archimedean uses the inversion in $A(B)$, that maps O and O_1 to two parallel lines perpendicular to AB, and (O_2) and the Pappus chain $(P_{2,n})$ to a chain of tangent circles enclosed by these two lines. The use of a homothety through A mapping Bankoff's triplet circle (W_3) to its inversive image shows that it is Archimedean. We can use this homothety as (W_3) circle is tangent to AB. This we know because (W_3) is invariant under inversion in (M_0), and thus intersects (M_0) orthogonally at C. In the same way we find λ-Archimedean circles.

Proposition 5. *For $i, j = 1, 2$, let $V_{i,n}$ be the point of tangency of (O_i) and $(P_{j,n})$. The circle $(CV_{1,n}V_{2,n})$ is $(n+1)$-Archimedean.*

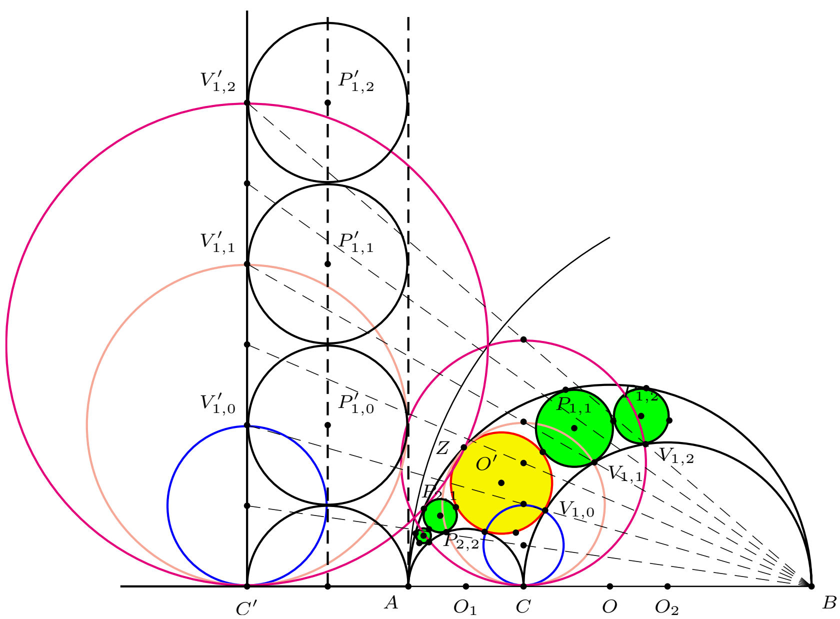

Figure 9.

A special circle of this family is $(L) = (CV_{1,1}V_{2,1})$, which tangent to (O) and (O') at their point of tangency Z, as can be easily seen from the figure after inversion. See Figure 9. We will meet again this circle in the final section.

Let $W_{1,n}$ be the point of tangency of $(P_{0,n})$ and (O_1). Similarly let $W_{2,n}$ be the point of tangency of $(P_{0,n})$ and (O_2). The circles $(CW_{1,n}W_{2,n})$ are invariant under inversion through (M_0), hence are tangent to AB. We may consider AB itself as preceding element of these circles, as we may consider (O) as $(P_{0,-1})$. Inversion through C maps $(P_{0,n})$ to a chain of tangent congruent circles tangent to two lines perpendicular to AB, and maps the circles $(CW_{1,n}W_{2,n})$ to equidistant lines parallel to AB and including AB. The diameters through C of $(CW_{1,n}W_{2,n})$ are thus, by inversion back of these equidistant lines, proportional to the harmonic sequence. See Figure 10.

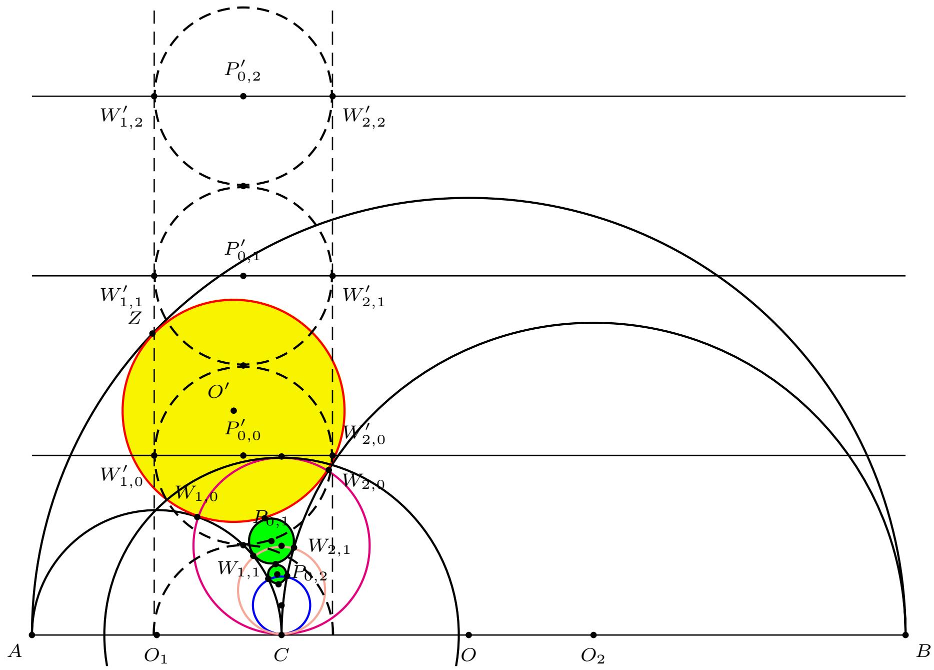

Figure 10.

Proposition 6. *The circle* $(CW_{1,n}W_{2,n})$ *is* $\frac{1}{n+1}$*-Archimedean.*

5. Inverting the arbelos to congruent circles

Let F_1 and F_2 be the intersection points of the midcircles (M_0), (M_1) and (M_2) of the arbelos. Inversion through F_i maps the circles (O), (O_1) and (O_2) to three congruent and pairwise tangent circles $(E_{i,0})$, $(E_{i,1})$ and $(E_{i,2})$. Triangle $E_{i,0}E_{i,1}E_{i,2}$ of course is equilateral, and stays homothetic independent of the power of inversion.

The inversion through F_i maps (M_0) to a straight line which we may consider as the midcircle of the two congruent circles $(E_{i,1})$ and $(E_{i,2})$. The center M'_0 of this degenerate midcircle we may consider at infinity. It follows that the line $F_iM_0 = F_iM'_0$ is parallel to the central $E_{i,1}E_{i,2}$ of these circles. Hence the lines through F_i parallel to the sides of $E_{i,1}E_{i,2}E_{i,3}$ pass through the points M_0, M_1 and M_2.

Now note that A, B, and C are mapped to the midpoints of triangle $E_{i,0}E_{i,1}E_{i,2}$, and the line AB thus to the incircle of $E_{i,0}E_{i,1}E_{i,2}$. The point F_i is thus on this circle, and from inscribed angles in this incircle we see that the directed angles (F_iA, F_iB), (F_iB, F_iC), (F_iC, F_iA) are congruent modulo π.

Proposition 7. *The points F_1 and F_2 are the Fermat-Torricelli points of degenerate triangles ABC and $M_0M_1M_2$.*

Let the diameter of (O') parallel AB meet (O') in G_1 and G_2 and Let G_1' and G_2' be their feet of the perpendicular altitudes on AB. From Pappus' theorem we know that $G_1G_2G_1'G_2'$ is a square. Construction 4 in [7] tells us that O' and its reflection through AB can be found as the Kiepert centers of base angles $\pm\arctan 2$. Multiplying all distances to AB by $\frac{\sqrt{3}}{2}$ implies that the points F_i form equilateral triangles with G_1' and G_2'. See Figure 11.

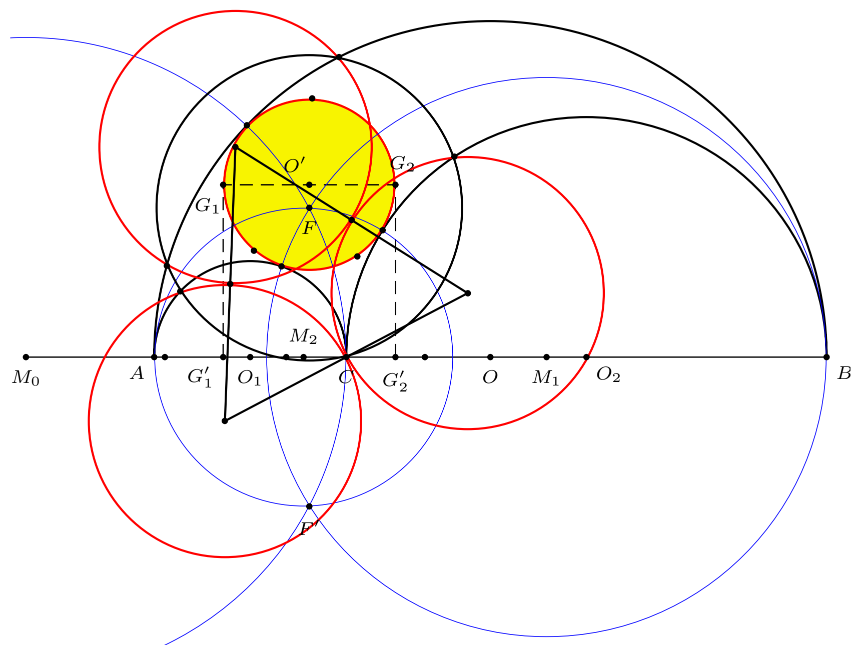

Figure 11.

A remarkable corollary of this and Proposition 7 is that the arbelos erected on $M_0M_1M_2$ shares its incircle with the original arbelos. See Figure 12.

Let F_1 be at the same side of ABC as the Arbelos semicircles. The inversion in $F_1(C)$ maps (O), (O_1) and (O_2) to three 2-Archimedean circles (E_0), (E_1) and (E_2), which can be shown with calculations, that we omit here. The 2-Archimedean circle (L) we met earlier meets (E_1) and (E_2) in their "highest" points H_1 and H_2 respectively. This leads to new Archimedean circles (E_1H_1) and (E_2H_2), which are tangent to Bankoff's triplet circle. Note that the points E_1, E_2, L, the point of tangency of (E_0) and (E_1) and the point of tangency of (E_0)

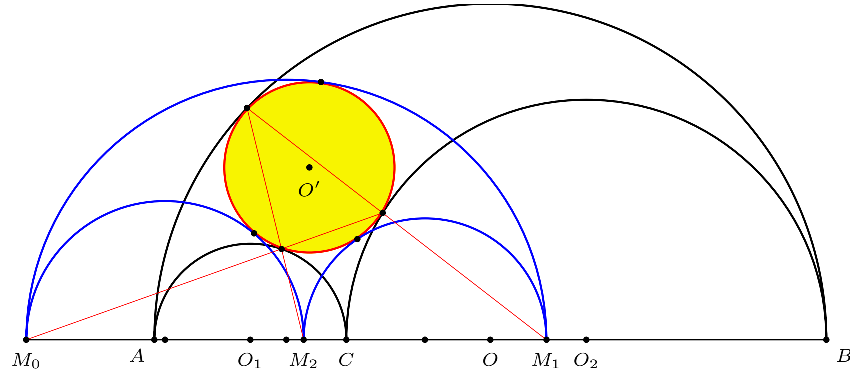

Figure 12.

and (E_2) lie on the 2-Archimedean circle with center C tangent to the common tangent of (O_1) and (O_2). See Figure 13.

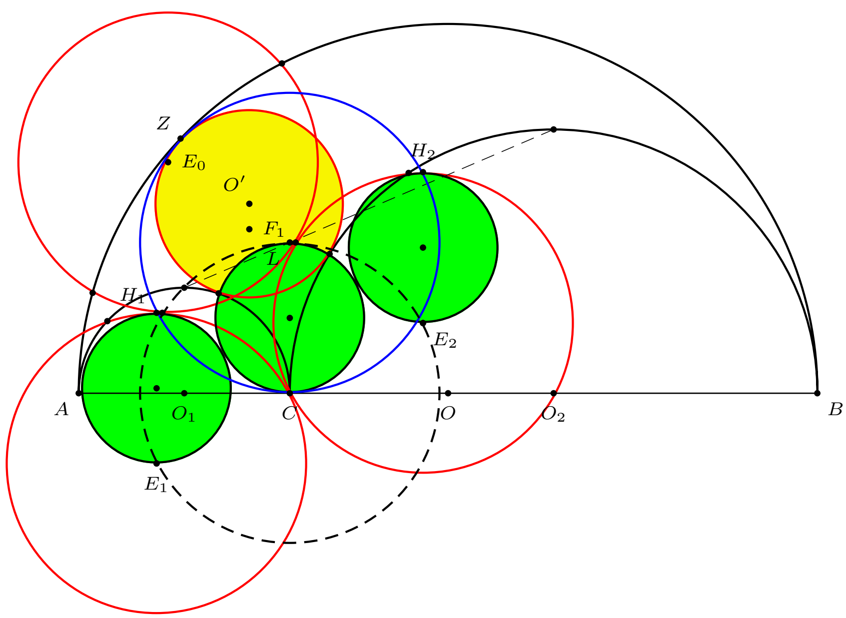

Figure 13.

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Forum Geometricorum
Volume 7 (2007) 67–72.

FORUM GEOM
ISSN 1534-1178

Ceva Collineations

Clark Kimberling

Abstract. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are lines. There exists a unique point U such that if $X \in \mathcal{L}_1$, then $X^{-1}©U \in \mathcal{L}_2$, where X^{-1} denotes the isogonal conjugate of X and $X^{-1}©U$ is the X^{-1}-Ceva conjugate of U. The mapping $X \mapsto X^{-1}©U$ is the U-Ceva collineation. It maps every line onto a line and in particular maps $\mathcal{L}_1$ onto $\mathcal{L}_2$. Examples are given involving the line at infinity, the Euler line, and the Brocard axis. Collineations map cubics to cubics, and images of selected cubics under certain U-Ceva collineations are briefly considered.

1. Introduction

One of the great geometry books of the twentieth century states [1, p.221] that "Möbius's invention of homogeneous coordinates was one of the most far-reaching ideas in the history of mathematics". In triangle geometry, two systems of homogeneous coordinates are in common use: barycentric and trilinear. Trilinears are especially useful when the angle bisectors of a reference triangle ABC play a central role, as in this note.

Suppose that $X = x : y : z$ is a point. If at most one of x, y, z is 0, then the point

$$X^{-1} = yz : zx : xy$$

is the isogonal conjugate of X, and if none of x, y, z is 0, we can write

$$X^{-1} = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

A traditional construction for X^{-1} depends on interior angle bisectors: reflect line AX in the A-bisector, BX in the B-bisector, CX in the C-bisector; then the reflected lines concur in X^{-1}.

The triangle $A_X B_X C_X$ with vertices

$$A_X = AX \cap BC, \quad B_X = BX \cap CA, \quad C_X = CX \cap AB$$

is the *cevian triangle of* X, and

$$A_X = 0 : y : z, \quad B_X = x : 0 : z, \quad C_X = x : y : 0.$$

If $U = u : v : w$ is a point, then the triangle $A^U B^U C^U$ with vertices

$$A^U = -u : v : w, \quad B^U = u : -v : w, \quad C^U = u : v : -w$$

Publication Date: February 26, 2007. Communicating Editor: Paul Yiu.

is the *anticevian triangle* of U. The lines $A_X A^U$, $B_X B^U$, $C_X C^U$ concur in the point

$$u(-uyz + vzx + wxy) : \ v(uyz - vzx + wxy) : \ w(uyz + vzx - wyz),$$

called the X*-Ceva conjugate* of U and denoted by $X©U$ (see [2, p. 57]). It is easy to verify algebraically that $X©(X©U) = U$ and that if $P = p : q : r$ is a point, then the equation $P = X©U$ is equivalent to

$$\begin{aligned} X =&(ru + pw)(pv + qu) : (pv + qu)(qw + rv) : (qw + rv)(ru + pw) \qquad (1) \\ =&\text{cevapoint}(P, U). \end{aligned}$$

A construction of cevapoint (P, U) is given in the Glossary of [3].

One more preliminary will be needed. A *circumconic* is a conic that passes through the vertices, A, B, C. Every point $P = p : q : r$, where $pqr \neq 0$, has its own circumconic, given by the equation $p\beta\gamma + q\gamma\alpha + r\alpha\beta = 0$; indeed, this curve is, loosely speaking, the isogonal conjugate of the line $p\alpha + q\beta + r\gamma = 0$, and the curve is an ellipse, parabola, or hyperbola according as the line meets the circumcircle in 0, 1, or 2 points. The circumcircle is the circumconic having equation $a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$.

2. The Mapping $X \mapsto X^{-1}©U$

In this section, we present first a lemma: that for given circumconic $\mathcal{P}$ and line $\mathcal{L}$, there is a point U such that the mapping $X \mapsto X©U$ takes each point X on $\mathcal{P}$ to a point on $\mathcal{L}$. The lemma easily implies the main theorem of the paper: that the mapping $X \mapsto X^{-1}©U$ takes each point of a certain line to $\mathcal{L}$.

Lemma 1. *Suppose $L = l : m : n$ and $P = p : q : r$ are points. Let $\mathcal{P}$ denote the circumconic $p\beta\gamma + q\gamma\alpha + r\alpha\beta = 0$ and $\mathcal{L}$ the line $l\alpha + m\beta + n\gamma = 0$. There exists a unique point U such that if $X \in \mathcal{P}$, then $X©U \in \mathcal{L}$. In fact,*

$$U = L^{-1}©P = p(-lp + mq + nr) : q(lp - mq + nr) : r(lp + mq - nr).$$

Proof. We wish to solve the containment $X©U \in \mathcal{L}$ for U, given that $X \in \mathcal{P}$. That is, we seek $u : v : w$ such that

$$u(-uyz+vzx+wxy)l+v(uyz-vzx+wxy)m+w(uyz+vzx-wxy)n = 0, \quad (2)$$

given that $X = x : y : z$ is a point satisfying

$$pyz + qzx + qxy = 0. \quad (3)$$

Equation (2) is equivalent to

$$u(-ul + vm + wn)yz + v(ul - vm + wn)zx + w(ul + vm - wn)xy = 0, \quad (4)$$

so that, treating $x : y : z$ as a variable point, equations (3) and (4) represent the same circumconic. Consequently,

$$u(-lu + mv + nw)qr = v(lu - mv + nw)rp = w(lu + mv - nw)pq.$$

In order to solve for $u : v : w$, we assume, as a first of two cases, that p and q are not both 0. Then the equation

$$u(-lu + mv + nw)qr = v(lu - mv + nw)rp$$

gives

$$w = \frac{(mv - lu)\,(pv + qu)}{n(pv - qu)}. \tag{5}$$

Substituting for w in

$$u(-lu + mv + nw)qr - w(lu + mv - nw)pq = 0$$

gives

$$\frac{\left(mpqv - lpqu + nprv - nqru - lp^2v + mq^2u\right)(mv - lu)\,uv}{2nr(pv - qu)^2} = 0,$$

so that

$$u = \frac{(mq - lp + nr)\,pv}{q(lp - mq + nr)}. \tag{6}$$

Consequently, for given v, we have

$$u : v : w = \frac{(mq - lp + nr)\,pv}{q(lp - mq + nr)} : v : \frac{(mv - lu)\,(pv + qu)}{n(pv - qu)}.$$

Substituting for u from (6), canceling v, and simplifying lead to

$$u : v : w = p\,(-lp + mq + nr) : q\,(lp - mq + nr) : r\,(lp + mq - nr)\,,$$

so that $U = L^{-1}\copyright P$.

If, as the second case, we have $p = q = 0$, then $r \neq 0$ because $p : q : r$ is assumed to be a point. In this case, one can start with

$$u(-lu + mv + nw)qr = w(lu + mv - nw)pq$$

and solve for v (instead of w as in (5)) and continue as above to obtain $U = L^{-1}\copyright P$.

The method of proof shows that the point U is unique. $\square$

Theorem 2. *Suppose $\mathcal{L}_1$ is the line $l_1\alpha + m_1\beta + n_1\gamma = 0$ and $\mathcal{L}_2$ is the line $l_2\alpha + m_2\beta + n_2\gamma = 0$. There exists a unique point U such that if $X \in \mathcal{L}_1$, then $X^{-1}\copyright U \in \mathcal{L}_2$.*

Proof. The hypothesis that $X \in \mathcal{L}_1$ is equivalent to $X^{-1} \in \mathcal{P}$, the circumconic having equation $l_1\beta\gamma + m_1\gamma\alpha + n_1\alpha\beta = 0$. Therefore, the lemma applies to the circumconic $\mathcal{P}$ and the line $\mathcal{L}_2$. $\square$

We write the mapping $X \mapsto X^{-1}\copyright U$ as $\mathcal{C}_U(X) = X^{-1}\copyright U$ and call $\mathcal{C}_U$ the U-Ceva collineation. That $\mathcal{C}_U$ is indeed a collineation follows as in [4] from the linearity of x, y, z in the trilinears

$$\mathcal{C}_U(X) = u(-ux + vy + wz) :\ v(ux - vy + wz) :\ w(ux + vy - wz).$$

This collineation is determined by its action on the four points A, B, C, U^{-1}, with respective images A^U, B^U, C^U, U.

Regarding the surjectivity, or onto-ness, of $\mathcal{C}^U$, suppose F is a point on $\mathcal{L}_2$; then the equation $X^{-1}©U = F$ has as solution

$$X = \text{cevapoint}(F, U))^{-1}.$$

3. Corollaries

Lemma 1 tells how to find U for given $\mathcal{L}$ and $\mathcal{P}$. Here, we tell how to find $\mathcal{L}$ from given $\mathcal{P}$ and U and how to find $\mathcal{P}$ from given U and $\mathcal{L}$.

Corollary 3. *Given a circumconic $\mathcal{P}$ and a point U, there exists a line $\mathcal{L}$ such that if $X \in \mathcal{P}$, then $X©U \in \mathcal{L}$.*

Proof. Assuming there is such a $\mathcal{L}$, we have the point $U = L^{-1}©P$ as Theorem 2, so that $L^{-1} = \text{cevapoint}(U, P)$, and

$$L = (\text{cevapoint}(U, P))^{-1},$$

so that $\mathcal{L}$ is the line $(wq + vr)\alpha + (ur + wp)\beta + (vp + uq)\gamma = 0$. It is easy to check that if $X \in \mathcal{P}$, then $X©U \in \mathcal{L}$. □

Corollary 4. *Given a line $\mathcal{L}$ and a point U, there exists a circumconic $\mathcal{P}$ such that if $X \in \mathcal{P}$, then $X©U \in \mathcal{L}$.*

Proof. Assuming there is such a $\mathcal{L}$, we have the point $U = L^{-1}©P$, and $P = L^{-1}©U$, so that $\mathcal{P}$ is the circumconic

$$u(-ul + vm + wn)\beta\gamma + v(ul - vm + wn)\gamma\alpha + w(ul + vm - wn)\alpha\beta = 0.$$

It is easy to check that if $X \in \mathcal{P}$, then $X©U \in \mathcal{L}$. □

4. Examples

4.1. Let $L = P = 1 : 1 : 1$, so that $\mathcal{L}_1 = \mathcal{L}_2$ is the line $\alpha + \beta + \gamma = 1$. We find $U = 1 : 1 : 1$, so that

$$\mathcal{C}_U(X) = -x + y + z : \ x - y + z : \ x + y - z.$$

It is easy to check that $\mathcal{C}_U(X) = X$ for every X on the line $\alpha + \beta + \gamma = 1$, such as X_{44} and X_{513}. On the line X_1X_2 we have

$$\mathcal{C}_U(X) = X \text{ for } X \in \{X_1, X_{899}\},$$

so that $\mathcal{C}_U$ maps X_1X_2 onto itself; e.g., $\mathcal{C}_U(X_2) = X_{43}$, and $\mathcal{C}_U(X_{1201}) = X_8$, and $\mathcal{C}_U(X_8) = X_{972}$. On X_1X_6 we have fixed points X_1 and X_{44}, so that $\mathcal{C}_U$ maps the line X_1X_{44} to itself. Abbreviating $\mathcal{C}_U(X_i) = X_j$ as $X_i \mapsto X_j$, we have, among points on X_1X_{44},

$$X_{1100} \mapsto X_{37} \mapsto X_6 \mapsto X_9 \mapsto X_{1743}.$$

The Euler line, X_2X_3, is a link in a chain as indicated by

$$\cdots \mapsto X_{42}X_{65} \mapsto X_2X_3 \mapsto X_{43}X_{46} \mapsto \cdots$$

4.2. Let $L = L_1 = X_6 = a : b : c$, so that $\mathcal{L}_1$ is the line at infinity and $\mathcal{P}$ is the circumcircle. Let $\mathcal{L}_2$ be the Euler line, given by taking L_2 in the statement of the theorem to be

$$X_{647} = a(b^2-c^2)(b^2+c^2-a^2) : b(c^2-a^2)(c^2+a^2-b^2) : c(a^2-b^2)(a^2+b^2-c^2).$$

The Ceva collineation $\mathcal{C}_U$ that maps $\mathcal{L}_1$ onto $\mathcal{L}_2$ is given by

$$\begin{aligned} U = X_{523} =& a(b^2 - c^2) : b(c^2 - a^2) : c(a^2 - b^2) \\ =& \sin(B - C) : \sin(C - A) : \sin(A - B), \end{aligned}$$

and we find

$$\begin{array}{lll} X_{512} \mapsto X_2, & X_{520} \mapsto X_4, & X_{523} \mapsto X_5, \\ X_{526} \mapsto X_{30}, & X_{2574} \mapsto X_{1312}, & X_{2575} \mapsto X_{1313}. \end{array}$$

The penultimate of these, namely $X_{2574} \mapsto X_{1312}$, is of particular interest, as $X_{2574} = X_{1113}^{-1}$, where X_{1113} is a point of intersection of the Euler line and the circumcircle and X_{1312} is a point of intersection of the Euler line and the nine-point circle; and similarly for $X_{2575} \mapsto X_{1313}$. The mapping $\mathcal{C}_U$ carries the Brocard axis, X_3X_6 onto the line $X_{115}X_{125}$, where X_{115} and X_{125} are the centers of the Kiepert and Jerabek hyperbolas, respectively.

4.3. Let $L_1 = X_{523}$, so that $\mathcal{L}_1$ is the Brocard axis, X_3X_6, and let $\mathcal{L}_2$ be the Euler line, X_2X_3. Then $U = X_6 = a : b : c$. The mapping of $\mathcal{L}_1$ to $\mathcal{L}_2$ is a link in a chain:

$$\cdots \mapsto X_2X_{39} \mapsto X_2X_6 \mapsto X_3X_6 \mapsto X_2X_3 \mapsto X_6X_{25} \mapsto X_3X_{66} \mapsto \cdots$$

4.4. Here, we reverse the roles played by the Brocard axis and Euler line in Example 3: let $\mathcal{L}_1$ be the Euler line and $\mathcal{L}_2$ be the Brocard axis. Then $U = X_{184} = a^2 \cos A : b^2 \cos B : c^2 \cos C$. A few images of the X_{184}-Ceva collineation are given here:

$$\begin{array}{lll} X_2 \mapsto X_{32}, & X_3 \mapsto X_{571}, & X_4 \mapsto X_{577}, \\ X_5 \mapsto X_6, & X_{30} \mapsto X_{50}, & X_{427} \mapsto X_3. \end{array}$$

4.5. Let $\mathcal{L}_1 = \mathcal{L}_2 =$ Brocard axis. Here,

$$U = X_5 = \cos(B - C) : \cos(C - A) : \cos(A - B),$$

the center of the nine-point circle, and

$$X_{389} \mapsto X_3 \mapsto X_{52} \quad \text{and} \quad X_{570} \mapsto X_6 \mapsto X_{216}.$$

4.6. Let $\mathcal{L}_1 = \mathcal{L}_2 =$ the line at infinity, $X_{30}X_{511}$. Here,

$$U = X_3 = \cos A : \cos B : \cos C,$$

the circumcenter. Among line-to-line images under X_3-collineation are these:

$$\begin{aligned} &X_4X_{51} \mapsto \text{Euler line} \mapsto X_3X_{49}, \\ &X_6X_{64} \mapsto X_4X_6 \mapsto \text{Brocard axis} \mapsto X_6X_{155}. \end{aligned}$$

5. Cubics

Collineations map cubics to cubics (e.g. [4, p. 23]). In particular, a U-Ceva collineation maps a cubic Λ that passes through the vertices A, B, C to a cubic $\mathcal{C}_U(\Lambda)$ that passes through the vertices A^U, B^U, C^U of the anticevian triangle of U.

5.1. Let $U = X_1$, as in §4.1, and let Λ be the Thompson cubic, $Z(X_2, X_1)$, with equation

$$bc\alpha(\beta^2 - \gamma^2) + ca\beta(\gamma^2 - \alpha^2) + ab\gamma(\alpha^2 - \beta^2) = 0.$$

Then $\mathcal{C}_U(\Lambda)$ circumscribes the excentral triangle, and for selected X_i on Λ, the image $\mathcal{C}_U(X_i)$ is as shown here:

X_i	1	2	3	4	6	9	57	223
$\mathcal{C}_U(X_i)$	1	43	46	1745	9	1743	165	1750

5.2. Let $U = X_1$, and let Λ be the cubic $Z(X_1, X_{75})$, with equation

$$\alpha(c^2\beta^2 - b^2\gamma^2) + \beta(a^2\gamma^2 - c^2\alpha^2) + \gamma(b^2\alpha^2 - a^2\beta^2) = 0.$$

For selected X_i on Λ, the image $\mathcal{C}_U(X_i)$ is as shown here:

X_i	1	6	19	31	48	55	56	204	221
$\mathcal{C}_U(X_i)$	1	9	610	63	19	57	40	2184	84

5.3. Let $U = X_6$, as in §4.3, and let Λ be the Thompson cubic. Then $\mathcal{C}_U(\Lambda)$ circumscribes the tangential triangle, and for selected X_i on Λ, the image $\mathcal{C}_U(X_i)$ is as shown here:

X_i	1	2	3	4	6	9	57	223	282	1073	1249
$\mathcal{C}_U(X_i)$	55	6	25	154	3	56	198	1436	1035	1033	64

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Forum Geometricorum
Volume 7 (2007) 73–86.

FORUM GEOM
ISSN 1534-1178

Orthocycles, Bicentrics, and Orthodiagonals

Paris Pamfilos

Abstract. We study configurations involving a circle (orthocycle) intimately related to a cyclic quadrilateral. As an illustration of the usefulness of this circle we explore its connexions with bicentric (bicentrics) and orthodiagonal quadrilaterals (orthodiagonals) reviewing the more or less known facts and revealing some other properties of these classes of quadrilaterals.

1. Introduction

Consider a generic convex cyclic quadrilateral $q = ABCD$ inscribed in the circle $k(K, r)$ and having finite intersection points F, G of opposite sides. Line $e = FG$ is the polar of the intersection point E of the diagonals AC, BD. The circle c with diameter FG is orthogonal to k. Also, the midpoints X, Y of the diagonals and the center H of c are collinear. We call c the **orthocycle** of the cyclic quadrilateral q. Consider also the circle f with diameter EK. This is the

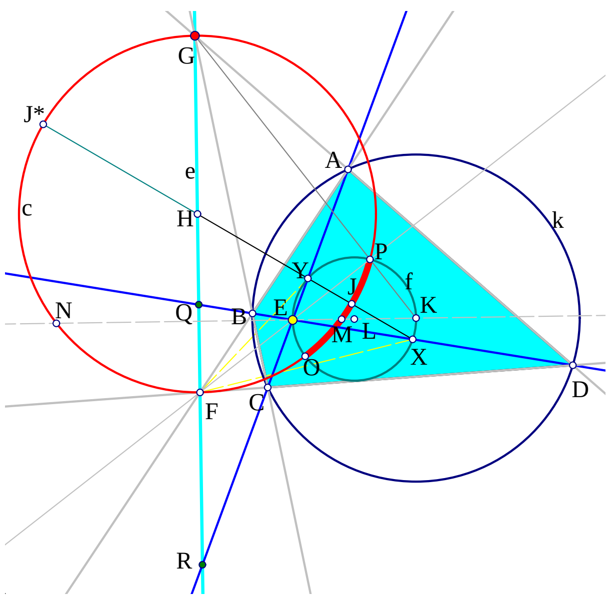

Figure 1. The orthocycle c of the cyclic quadrilateral $ABCD$

locus of the midpoints of chords of k passing through E. It is also the inverse of e with respect to k and is orthogonal to c. Thus, c belongs to the circle-bundle $\mathscr{C}'$, which is orthogonal to the bundle $\mathscr{C}(k, f)$ generated by k and f. The bundle $\mathscr{C}$ is of non intersecting type with limit points M, N, symmetric with respect to

Publication Date: March 12, 2007. Communicating Editor: Paul Yiu.

e, and $\mathscr{C}'$ is a bundle of intersecting type, all of whose members pass through M and N. If we fix the data (k, E, c), then all cyclic quadrilaterals q having these as *circumcircle, diagonals-intersection-point, orthocycle* respectively form a one-parameter family. A member q of this family is uniquely determined by a point J on the circular arc (OMP) of the orthocycle c. Thus the set of all q inscribed in the circle k and having diagonals through E is parameterized through pairs (c, J), c (the orthocycle) being a circle of bundle $\mathscr{C}'$ and J a point on the corresponding arc (OMP) intercepted on the orthocycle by f. In the following sections we consider these facts more closely and investigate (i) the bicentrics inscribed in k, and (ii) a certain 1-1 correspondence of cyclics to orthodiagonals in which the orthocycle plays an essential role.

Regarding the proofs of the statements made, everything (is or) follows immediately from standard, well known material. In fact, the statement on the polar relies on its usual construction from two intersecting chords ([3, p.103]). The statement on the collinearity follows from Newton's theorem on a complete quadrilateral ([3, p.62]). From the harmonic ratios appearing in complete quadrilaterals follows also that the intersection points Q, R of the diagonals with line e divide F, G harmonically. Consequently the circle with diameter QR is also orthogonal to c ([2, §1237]). The orthogonality of c, k follows from the fact that PF is the polar of G, which implies that P, G are inverse with respect to k. Besides, by measuring angles at P, circles f, c are shown to be orthogonal. The statement on the parametrization is analyzed in the following section.

The orthocycle gives a means to establish unity in apparently unrelated properties. For example the well known formula

$$\frac{1}{r^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}$$

is proved to be, essentially, a case of *Stewart's formula* (see next paragraph).

Furthermore, the orthogonality of c to f can be used to characterize the cyclics. To formulate the characterization we consider more general the *orthocycle* of a generic convex quadrilateral to be the circle on the diameter defined by the two intersection points of its pairs of opposite sides.

Proposition 1. *The quadrilateral $q = ABCD$ is cyclic if and only if its orthocycle c is orthogonal to the circle f passing through the midpoints X, Y of its diagonals and their intersection point E.*

Proof. If q is cyclic, then we have already seen that its orthocycle c belongs to the bundle $\mathscr{C}'$ which is orthogonal to the one generated by its circumcircle k and the circle f passing through the diagonal midpoints and their intersection point.

Conversely, if the orthocycle c and f intersect orthogonally, then Y and X are inverse with respect to c. Since the same is true with the intersection points R, Q of the diagonals of q with line FG (see Figure 2), there is a circle a passing through the four points X, Y, R and Q. Then we have

$$|ER| \cdot |EX| = |EY| \cdot |EQ|. \tag{1}$$

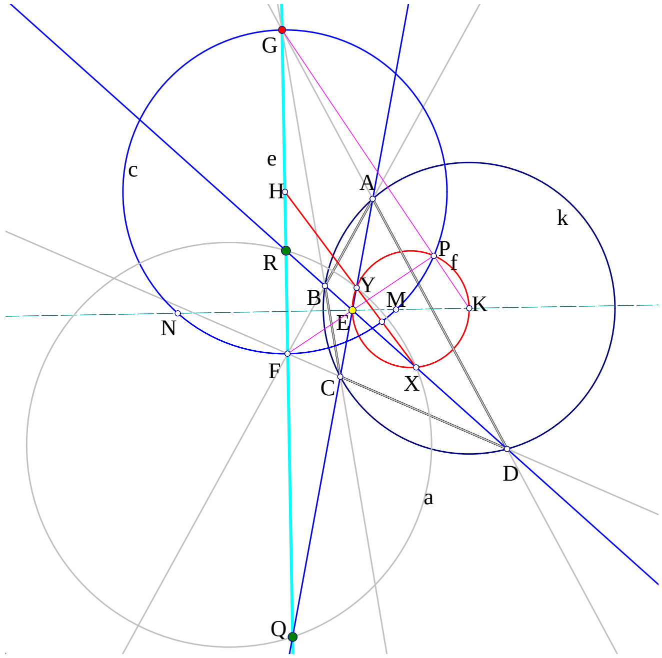

Figure 2. Cyclic characterization

But from the general properties of the complete quadrilaterals we have also that $(Q, E, C, A) = -1$ is a harmonic division, hence

$$|EC| \cdot |EA| = |EQ| \cdot |EY|. \tag{2}$$

Analogously, $(R, E, B, D) = -1$ implies

$$|EB| \cdot |ED| = |ER| \cdot |EX|. \tag{3}$$

Relations (1) to (3) imply that $|EB| \cdot |ED| = |EC| \cdot |EA|$, proving the proposition. □

For a classical treatment of the properties discussed below see Chapter 10 of Paul Yiu's Geometry Notes [5]. Zaslavsky (see [6], [1]) uses the term *orthodiagonal* for a map between quadrangles and gives characterizations of cyclics in another context than the one discussed below.

2. Bicentrics

Denote by (k, E, c) the family of quadrilaterals characterized by these elements (*circumcircle*, *diagonal-intersection-point*, *orthocycle*) correspondingly. Referring to Figure 1 we have the following properties ([2, §§674, 675, 1276]).

Proposition 2. (1) *There is a 1-1 correspondance between the members of the family* (k, E, c) *and the points* J *of the open arc* (OMP) *of circle* c.

(2) *Let* X, Y *be the intersection points of* f *with line* HJ. X, Y *are the midpoints of the diagonals of* q *and are inverse with respect to* c.

(3) *Then FJ bisects angles AFD and XFY. Analogously, GJ bisects angles BGD and XGY.*

Proof. In fact, from the Introduction, it is plain that each member q of the family (k, E, c) defines a J as required. Conversely, a point J on arc (OMP) of c defines two intersection points X, Y of HJ with f, which are inverse with respect to c, since f and c are orthogonal. The chords EX and EY define the cyclic $q = ABCD$, having these as diagonals and X, Y as the midpoints of these diagonals. Consider the orthocycle c' of this q. By the analysis made in the Introduction, c' belongs to the bundle $\mathscr{C}'$ and is also orthogonal to the circle with diameter QR. Thus c' is uniquely defined by the chords XE, YE and must coincide with c. This proves (1).

(2) is already discussed in the Introduction.

(3) follows from the orthogonality of circles k, c. In fact, this implies that J, J^* divide X, Y harmonically. Then (FJ^*, FJ, FY, FX) is a harmonic bundle of lines and FJ^*, FJ are orthogonal. Hence, they bisect $\angle XFY$. They also bisect $\angle BFC$. This follows immediately from the similarity of triangles AFC and DFB. Analogous is the situation with the angles at G. □

Referring to figure 1, denote by $q(c)$ the particular quadrilateral of the family (k, E, c), constructed with the recipe of the previous proposition, for $J \equiv M$. The following two lemmas imply that $q(c)$ is bicentric.

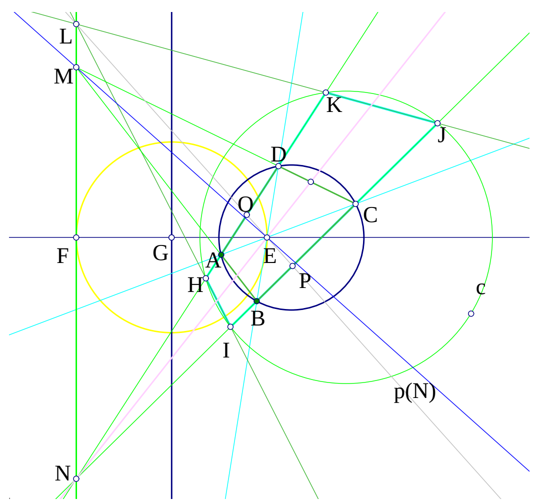

Figure 3. Bundle quadrilaterals

Lemma 3. *Consider a circle bundle of non intersecting type and two chords of a member circle passing through the limit point E of the bundle* (see Figure 3)*. The chords define a quadrilateral $q = ABCD$ having these as diagonals. Extend two opposite sides AD, BC until they intersect a second circle member c of the bundle. The intersection points form a quadrilateral $r = HIJK$. Then the intersection* *point L of the sides HI, JK lies on the polar MN of E with respect to a circle of the bundle (all circles c of the bundle have the same polar with respect to E).*

Indeed, N, M can be taken as the intersection points of opposite sides of q. Then N is on the polar of E, hence the polar $p(N)$ of N contains E. Consider the intersection points O, P of this polar with sides HK, IJ respectively. Then,
(a) these sides intersect at a point L lying on $p(N)$,
(b) L is also on line MN.

(a) follows from the standard theorem on cyclic quadrilaterals.

(b) follows from the fact that the quadruple of lines (NL, NH, NE, NI) at N is harmonic. But (NM, NH, NE, NI) is also harmonic, hence L is contained in line MN.

Lemma 4. *q(c) is bicentric.*

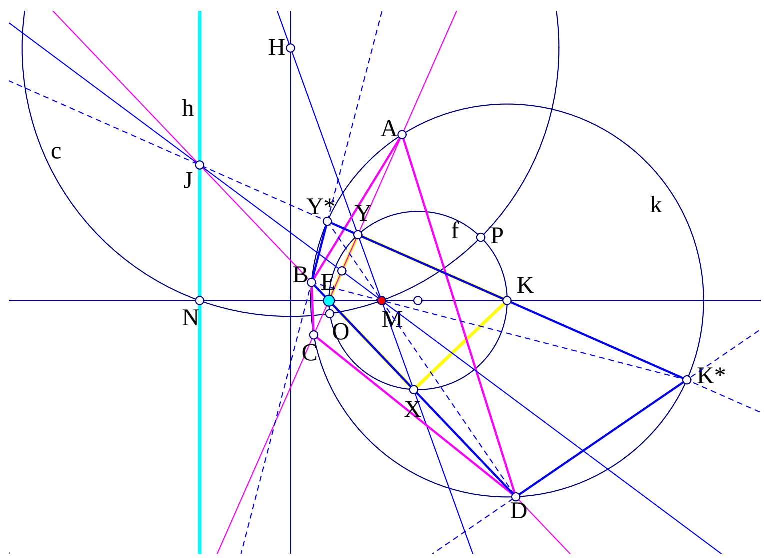

Figure 4. Bisectors of $q(c)$

Indeed, by Proposition 2 the bisectors of angles $\angle AGB, \angle BFC$ will intersect at M. It suffices to show that the bisectors of two opposite angles of $q(c)$ intersect also at M. Let us show that the bisector of angle $\angle ABC$ passes through M (Figure 4). We start with the quadrilateral $q_1 = EXKY$. Its diagonals intersect at M. According to the previous lemma the extensions of its sides will define a quadrilateral $q_2 = BDK^*Y^*$ inscribed in k and having its opposite sides intersecting on line h the common polar of E with respect to every member circle of the bundle I. This implies that the diagonals of q_2 intersect at the pole E of h. But BK^* joins B to the middle K^* of the arc (CK^*B), hence is the bisector of angle $\angle ABC$ and passes through M.

Proposition 5. (1) *There is a unique member $q = q(c) = ABCD$, of the family (k, E, c) which is bicentric. The corresponding J is the limit point M of bundle I contained in the circle k.*

(2) *There is a unique member $o = o(c) = A^*B^*C^*D^*$, of the family (k, E, c) which is orthodiagonal. The corresponding HJ passes through the center L of the circle f.*

(3) *For every bicentric the incenter M is on the line joining the intersection point of the diagonals with the circumcenter.*

(4) *For every bicentric the incenter M is on the line joining the midpoints of the diagonals.*

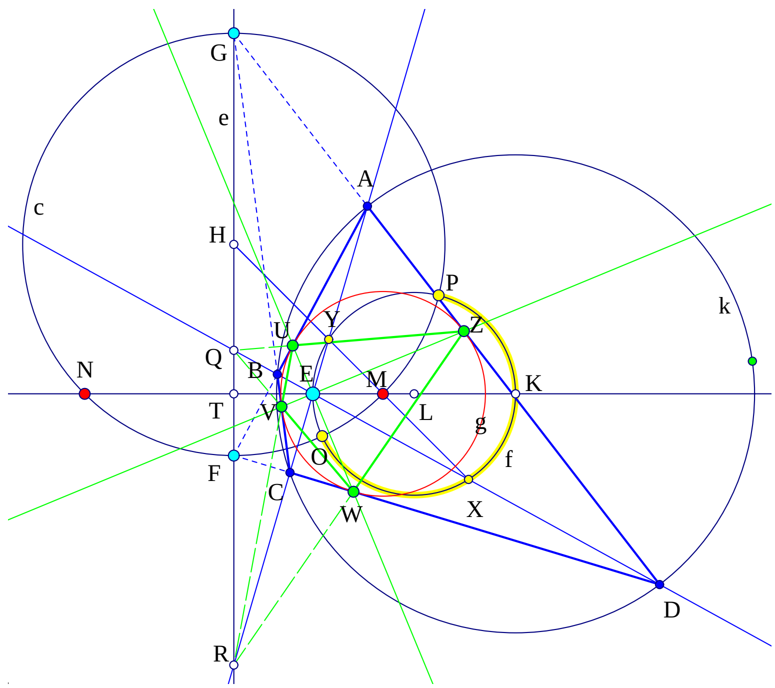

Figure 5. The bicentric member in (k, E, c)

Proof. In fact, by the previous lemmas we know that $q(c)$ is bicentric. To prove the uniqueness we assume that $q = ABCD$ is bicircular and consider the incircle g and the tangential quadrilateral $q' = UVWZ$. From Brianchon's theorem the diagonals of q' intersect also at E. Thus, the poles of the diagonals UW, VZ being correspondingly F, G, line e will be also polar of E with respect to g. In particular the center of g will be on line MN and the pairs of opposite sides of the tangential q' will intersect on e at points Q, R say. The diagonals of q pass through Q, R respectively. In fact, D being the pole of line WZ and B the pole of UV, BD is the polar of R with respect to g. By the standard construction of the polar it follows that Q is on BD. Analogously R is on AC. The center of g will be the intersection M' of the bisectors of angles BGA and BFC. By measuring the angles at M' we find easily that the bisectors form there a right angle. Thus, M' will be on the orthocycle c, hence, being also on line MN, it will coincide with M. In that case line HYX passes through M. This follows from proposition 1 which identifies the bisector of angle YFX with FM. This proves (1).

To prove (2) consider the quadrangle $s = EXKY$. If the diagonals intersect orthogonally then s is a rectangle. Consequently XY is a diameter of f and passes through L. The converse is also valid. If XY passes through L then s is a rectangle and o is orthodiagonal.

The other statements are immediate consequences. Notice that property 2 holds more generally for every circumscriptible quadrilateral ([2, §1614]). □

Proposition 6. *Consider all tripples (k, E, c) with fixed k, E and c running through the members of the circle bundle $\mathscr{C}'$. Denote by $q(c)$ the bicentric member of the corresponding family (k, E, c) and by $q' = UVWZ$ the tangential quadrangle of $q(c)$* (see Figure 5). *The following statements are consequences of the previous considerations:*

(1) *All tangential quadrilaterals $q' = UVWZ$ are orthodiagonal, the diagonals being each time parallel to the bisectors of angles $\angle BGA, \angle BFC$.*

(2) *The pairs of opposite sides or q' intersect at the points Q, R, which are the intersection points of the diagonals of $q(c)$ with e.*

(3) *The orthocycle c' of the tangential q' is the circle on the diameter QR and intersects the incircle g of $q(c)$ orthogonally. The radius r_g of the incircle satisfies $r_g^2 = |ME||MT|$.*

(4) *The bicentrics $\{q(c) : c \in \mathscr{C}'\}$ are precisely the inscribed in circle k and having their diagonals pass through E. They, all, have the same incircle g, depending only on k and E.*

(5) *The radii r_g of the inscribed circle g, r of circumscribed k, and the distance $d = |MK|$ of their centers satisfy the relation $\dfrac{1}{r_g^2} = \dfrac{1}{(r+d)^2} + \dfrac{1}{(r-d)^2}$.*

Proof. (1) follows from the fact that UV is orthogonal to the bisector FM of angle BFC. Analogously VZ is orthogonal to GM and FM, GM are orthogonal ([2, §674]).

(2) follows from the standard construction of the polar of E with respect to g. Thus e is also the polar of E with respect to the incircle g ([2, §1274]).

(3) follows also from (2) and the definition of the orthocycle. The relation for r_g is a consequence of the orthogonality of circles c', g.

(4) is a consequence of (3) and (5) is proved below by specializing to a particular bicentric $q(c)$ which is simultaneously orthodiagonal ([5, p.159]). Since the radius and the center of the incircle g is the same for all $q(c)$ this is legitimate. □

Proposition 7. (1) *For fixed (k, E), the set of all bicentrics $\{q(c) : c \in \mathscr{C}'\}$ contains exactly one member which is simultaneously bicentric and orthodiagonal. It corresponds to the minimum circle of bundle $\mathscr{C}'$, is kite-shaped and symmetric with respect to MN* (see Figure 6).

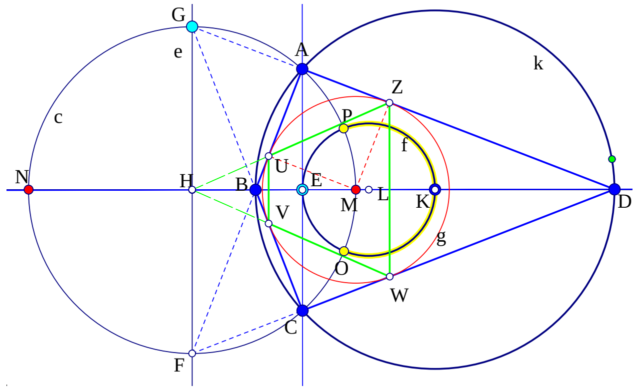

Figure 6. The orthodiagonal bicentric

(2) *All the bicentric orthodiagonals are constructed by reflecting an arbitrary right-angled triangle ABD on its hypotenuse* (see Figure 7). *The center of the incircle coincides with the trace E of the bisector with the hypotenuse, the length of this bisector w satisying $\frac{2}{w^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}$. Here R is the circumradius of ABD and $d = |EK|$ is the distance of the bisector's trace from the middle of BD.*

(3) *There is a particular bicentric orthodiagonal constructed directly from a regular octagon, with inradius $r = \frac{R}{\sqrt{2+\sqrt{2}}}$ and sides equal to w and $w + 2r$ respectively* (see Figure 8).

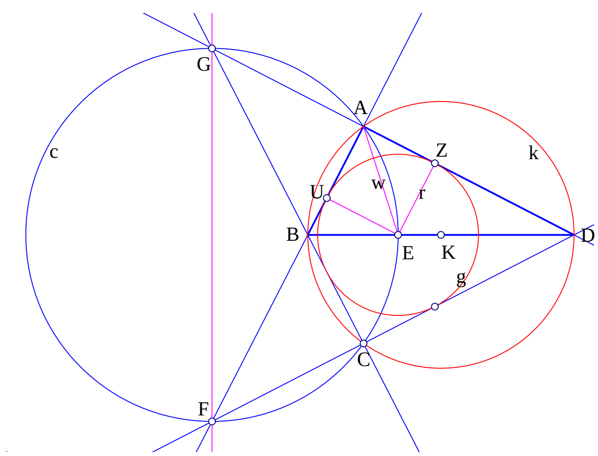

Figure 7. The general orthodiagonal bicentric

Proof. In fact, by applying the previous results to the tangential quadrilateral q' of $q(c)$, we know that the orthocycle of q' is orthogonal to the fixed incircle g and passes through two fixed points, depending only on (k, E) (the limit points of the corresponding circle bundle $\mathscr{C}'$ for the pair (g, E)). If the diagonals EQ and ER become orthogonal then E must be on the orthocycle of q' and this is possible only in the limiting position in which it coincides with line MN. Then the orthocycle of $q(c)$ has MN as diameter and this implies (1).

(2) follows immediately from (1). The formula is an application of Stewart's general formula (see [5, p.14]) on this particular configuration plus a simple calculation. The formula implies trivially the formula of the previous proposition, since all the bicirculars characterized by the fixed pair (k, E) have the same incircle g and from the square $EZAU$ (Figure 6) we have $w^2 = 2r^2$.

(3) is obvious and underlines the existence of a particular nice kite. □

Notice the necessary inequality between the distance $d = MK$ and the distance $d_1 = |EK|$ of circumcenter from the intersection point of diagonals: $2|MK| > |EK|$, holding for every bicentric ([6, p.44]) and being a consequence of a general propety of circle bundles of non intersecting type.

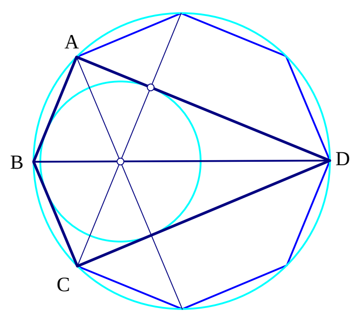

Figure 8. A distinguished kite

3. Circumscribed Quadrilateral

The following proposition give some well known properties of quadrilaterals circumscribed on circles by adding the ingredient of the orthocycle. For convenience we review here these properties and specialize in a subsequent proposition to the case of a bicentric circumscribed.

Proposition 8. *Consider the tangential quadrilateral $q' = QRST$ circumscribed on the circumcircle k of the cyclic quadrilateral $q = ABCD$* (Figure 9). *The following facts are true:*

(1) *The diagonals of q' and q intersect at the same point E.*

(2) *The pairs of opposite sides of q' and pairs of opposite sides of q intersect on the same line e, which is the polar of E with respect to the circumcircle k of q.*

(3) *The diameter UV of the orthocycle of q' is divided harmonically by the diameter FG of the orthocycle of q.*

(4) *The orthocycle of q' is orthogonal to the orthocycle of q.*

(5) *The diagonals of q' (respectively q) pass through the intersection points of opposite sides of q (respectively q').*

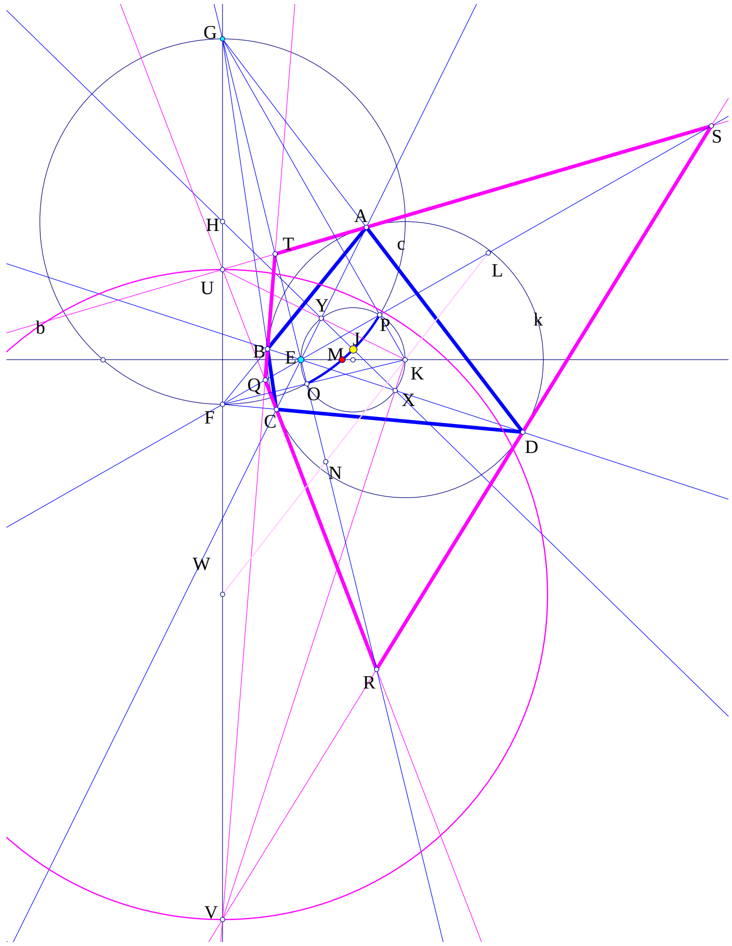

Figure 9. Circumscribed on cyclic

Proof. (1) is a consequence of Brianchon's theorem (see a simpler proof in [5, p.157]). Identify the polar of E with the diameter $e = FG$ of the orthocycle of q. The polar of F is PG and the polar of G is OF. T is the pole of AB which contains F. Hence the polar of F will pass through T, analogously it will pass through R. This proves (2) (see [2, §1275]) and (5).

To show (3) it suffices to see that lines (KP, KY, KO, KX) form a harmonic bundle. But the cross ratio of these four lines through K is independent of the location of K on the circle with diameter EK. Hence is the same with the cross ratio of lines (EP, EY, EO, EX) which is -1 by the general properties of complete quadrilaterals.

(4) is a consequence of (3). Note that the line LN joining the midpoints of the diagonals of q' passes through the center W of the orthocycle and the center of k ([2, §1614]). □

Proposition 9. *For each quadrangle of the family $q \in (k, E, c)$ construct the tangential quadrangle $q' = QRST$ of $q = ABCD$ (Figure 10). The following facts are true.*

(1) *There is exactly one $q_0 \in (k, E, c)$ whose corresponding tangential q' is cyclic. The corresponding line of diagonal midpoints of q' passes through the center K of circle k.*

(2) *The line of diagonal midpoints of q_0 is orthogonal to the corresponding line of diagonal midpoints of q'.*

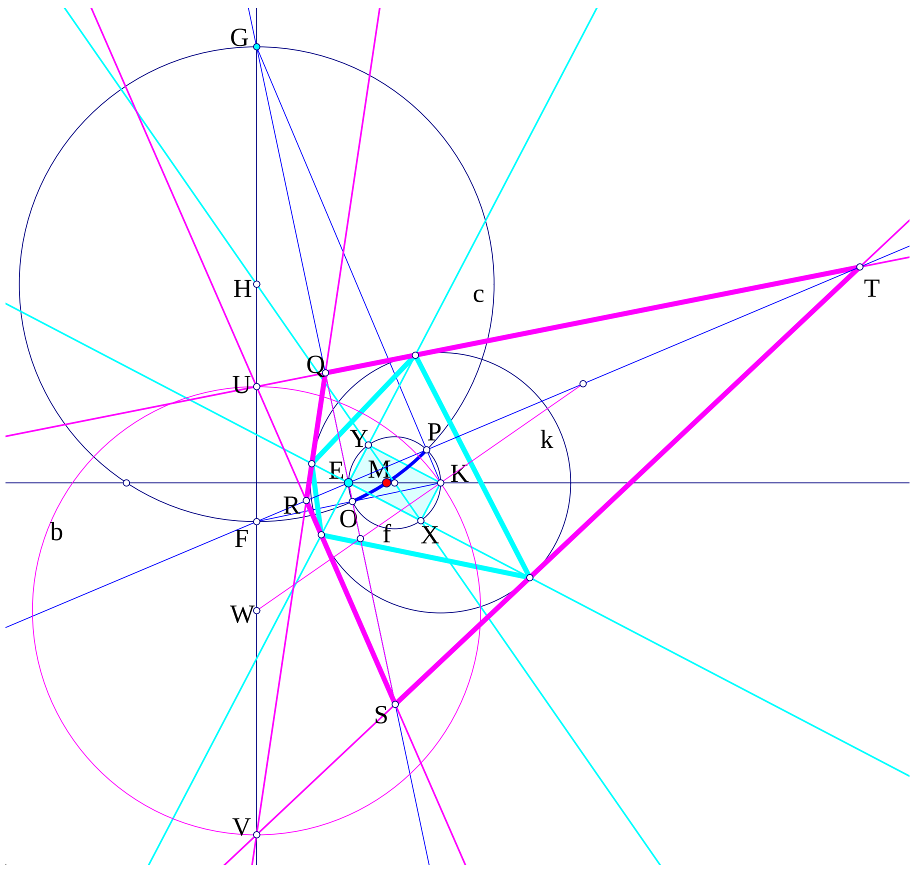

Figure 10. Bicentric circumscribed

Proof. q' being cyclic and circumscriptible it is bicentric. Hence the lines joining opposite contact points must be orthogonal and the orthocycle of q' passes through K. This follows from Proposition 2. Thus $p = EXKY$ is a rectangle, XY being a diameter of the circle f. Inversely, by Proposition 3, if q' is bicentric, then p is a rectangle and K is the limit point of the corresponding bundle I, and K is the center of the incircle. For the other statement notice that circle c being orthogonal simultaneously to circle k and b has its center on the radical axis of b and k. In the particular case of bicentric q', the angles WKK, XYK and EKY are equal and this implies that WL is then orthogonal to HXY which becomes the radical axis of k and b.

Note that the diagonals of all q' are the same and identical with the lines EF, EG which remain fixed for all members q of the family (k, E, c). Also combining this proposition and Proposition 3 we have (see [5, p.162]) that q' is cyclic, if and only if q is orthodiagonal. □

4. Orthodiagonals

The first part of the following proposition constructs an orthodiagonal from a cyclic. This is the inverse procedure of the well known one, which produces a cyclic by projecting the diagonals intersection point of an orthodiagonal to its sides ([4, vol. II p. 358], [6], [1]).

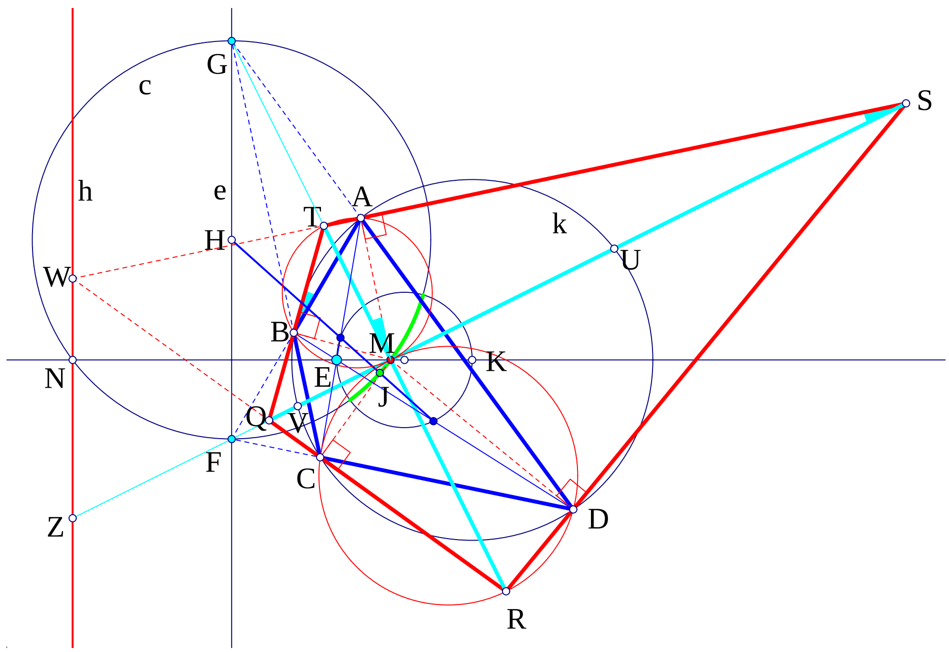

Figure 11. Orthodiagonal from cyclic

Proposition 10. (1) *For each cyclic quadrilateral $q = ABCD$ of the family (k, E, c) there is an orthodiagonal $p = QRST$ whose diagonals coincide with the sides of the right angled triangle $t = FGM$, defined by the limit point M of bundle $\mathscr{C}$ and the intersection points F, G of the pairs of opposite sides of q. The vertices of q are the projections of the intersection point M of the diagonals of p on its sides.*

(2) *The pairs of opposite sides of p intersect at points W, W^* on line h which is the common polar of all circles of bundle I with respect to its limit point M.*

(3) *The orthodiagonal* p *is cyclic if and only if the corresponding* q *is bicentric,* i.e., *point* J *is identical with* M.

(4) *The circumcircle of the orthodiagonal and cyclic* p *belongs to bundle* I.

Proof. Consider the lines orthogonal to MA, MB, MC, MD at the vertices of q (Figure 11). They build a quadrilateral. To show the statement on the diagonals consider the two resulting cyclic quadrilaterals $q_1 = MATB$ and $q_2 = MCRD$. Point F lies on the radical axis of their circumcircles since lines FBA, FCD are chords through F of circle k. Besides, for the same reason $|FB| \cdot |FA| = |FV| \cdot |FU| = |CF| \cdot |FD| = |FM|^2$. The last because circles c, k are orthogonal and M is the limit point of bundle I. From $|FB| \cdot |FA| = |FM|^2$ follows that line FM is tangent to the circumcircle of q_1. Analogously it is tangent to q_2 at M. Thus points G, T, M, R are collinear. Analogously points F, Q, M, S are collinear. This proves (1).

For (2), note that quadrangle $ABQS$ is cyclic, since $\angle TBA = \angle TMA = \angle MSA$. Thus $|FM|^2 = |FQ| \cdot |FS|$ and this implies that points M, Z divide harmonically Q, S, Z being the intersection point of h with the diagonal QS. Analogously the intersection point Z^* of h with the diagonal TR and M will divide T, R harmonically. Thus, by the characteristic property of the diagonals of a complete quadrilateral ZZ^* will be identical with line WW^*.

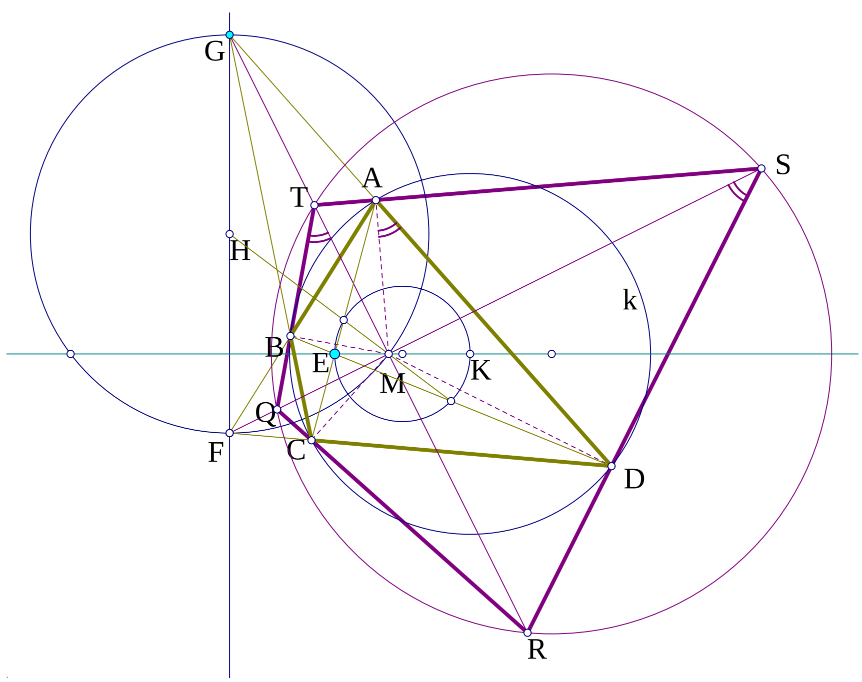

Figure 12. Orthodiagonal and cyclic

For (3), note that p is cyclic if and only if angles $\angle QTR = \angle QSR$ (Figure 12). By the definition of p this is equivalent to $\angle BAM = \angle MAD$, *i.e.*, AM being the bisector of angle A of q. Analogously MB, MC, MD must be bisectors of the corresponding angles of q.

For (4), note that the circumcenter of p must be on the line EM. This follows from the discussion in the first paragraph and the second statement. Indeed, the circumcenter must be on the line which is orthogonal from M to the diameter of the orthocycle of p. Besides the circle with center F and radius FM is a circle of bundle $\mathscr{C}'$ and, according to the proof of first statement, is orthogonal to this circumcircle. Thus the circumcircle of p, being orthogonal to two circles of bundle $\mathscr{C}'$, belongs to bundle $\mathscr{C}$. $\square$

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Forum Geometricorum
Volume 7 (2007) 87–97.

FORUM GEOM
ISSN 1534-1178

Bicevian Tucker Circles

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Abstract. We prove that there are exactly ten bicevian Tucker circles and show several curves containing the Tucker bicevian perspectors.

1. Bicevian Tucker circles

The literature is abundant concerning Tucker circles. We simply recall that a Tucker circle is centered at T on the Brocard axis OK and meets the sidelines of ABC at six points A_b, A_c, B_c, B_a, C_a, C_b such that
(i) the lines X_yY_x are parallel to the sidelines of ABC,
(ii) the lines Y_xZ_x are antiparallel to the sidelines of ABC, *i.e.*, parallel to the sidelines of the orthic triangle $H_aH_bH_c$.

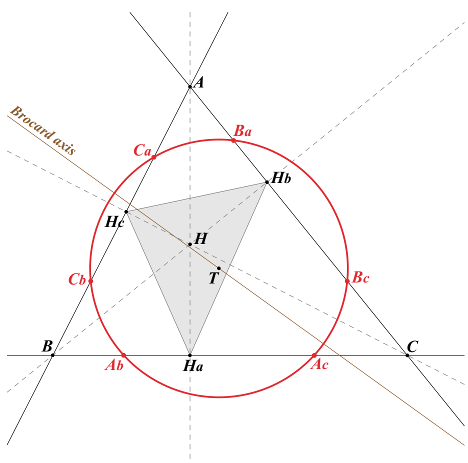

Figure 1. A Tucker circle

If T is defined by $\overrightarrow{OT} = t \cdot \overrightarrow{OK}$, we have

$$B_aC_a = C_bA_b = A_cB_c = \frac{2abc}{a^2+b^2+c^2}\,|t| = R|t|\tan\omega,$$

and the radius of the Tucker circle is

$$R_T = R\sqrt{(1-t)^2 + t^2\,\tan^2\omega}$$

where R is the circumradius and ω is the Brocard angle. See Figure 1.

Publication Date: March 19, 2007. Communicating Editor: Paul Yiu.

One obvious and trivial example consists of the circumcircle of ABC which we do not consider in the sequel. From now on, we assume that the six points are not all the vertices of ABC.

In this paper we characterize the *bicevian Tucker circles*, namely those for which a *Tucker triangle* formed by three of the six points (one on each sideline) is perspective to ABC. It is known that if a Tucker triangle is perspective to ABC, its companion triangle formed by the remaining three points is also perspective to ABC. The two perspectors are then said to be cyclocevian conjugates.

There are basically two kinds of Tucker triangles:
(i) those having one sideline parallel to a sideline of ABC: there are three pairs of such triangles e.g. $A_bB_cC_b$ and its companion $A_cB_aC_a$,
(ii) those not having one sideline parallel to a sideline of ABC: there is only one such pair namely $A_bB_cC_a$ and its companion $A_cB_aC_b$. These are the proper Tucker triangles of the literature.

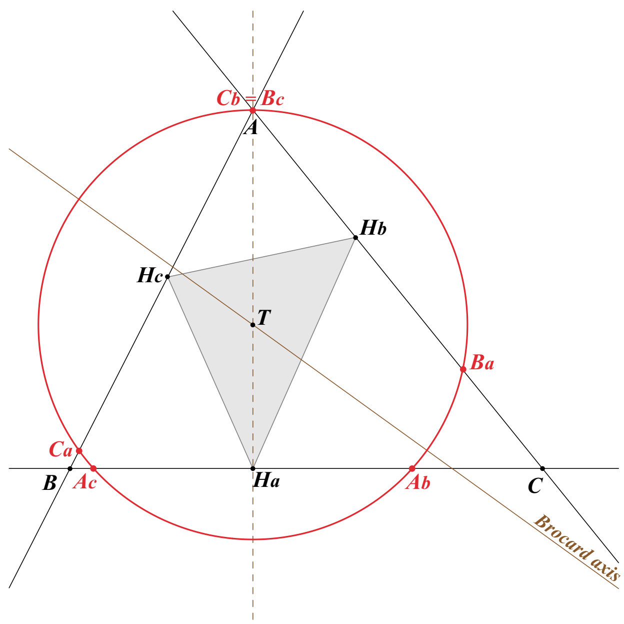

Figure 2. A Tucker circle through a vertex of ABC

In the former case, there are six bicevian Tucker circles which are obtained when T is the intersection of the Brocard axis with an altitude of ABC (which gives a Tucker circle passing through one vertex of ABC, see Figure 2) or with a perpendicular bisector of the medial triangle (which gives a Tucker circle passing through two midpoints of ABC, see Figure 3).

The latter case is more interesting but more difficult. Let us consider the Tucker triangle $A_bB_cC_a$ and denote by X_a the intersection of the lines BB_c and CC_a; define X_b and X_c similarly. Thus, ABC and $A_bB_cC_a$ are perspective (at X) if and only if the three lines AA_b, BB_c and CC_a are concurrent or equivalently the three

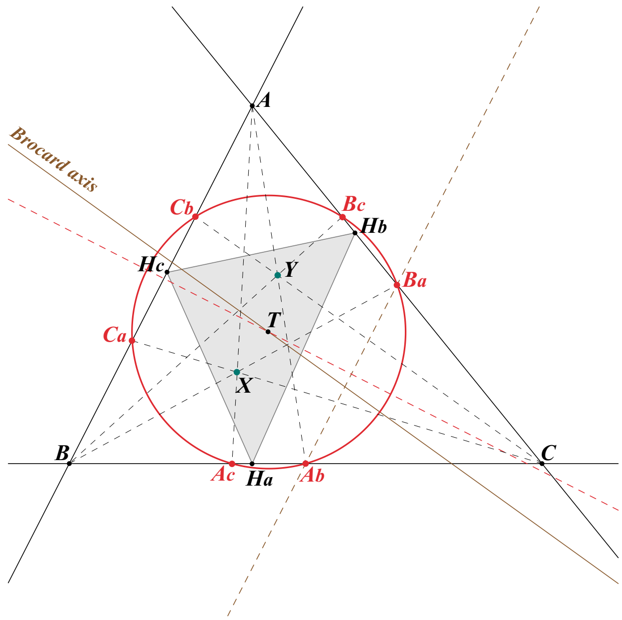

Figure 3. A Tucker circle through two midpoints of ABC

points X_a, X_b and X_c coincide. Consequently, the triangles ABC and $A_cB_aC_b$ are also perspective at Y, the cyclocevian conjugate of X.

Lemma 1. *When T traverses the Brocard axis, the locus of X_a is a conic γ_a.*

Proof. This can be obtained through easy calculation. Here is a synthetic proof. Consider the projections π_1 from the line AC onto the line BC in the direction of H_aH_b, and π_2 from the line BC onto the line AB in the direction of AC. Clearly, $\pi_2(\pi_1(B_c)) = \pi_2(A_c) = C_a$. Hence, the tranformation which associates the line BB_c to the line CC_a is a homography between the pencils of lines passing through B and C. It follows from the theorem of Chasles-Steiner that their intersection X_a must lie on a conic. □

This conic γ_a is easy to draw since it contains B, C, the anticomplement G_a of A, the intersection of the median AG and the symmedian CK and since the tangent at C is the line CA. Hence the perspector X we are seeking must lie on the three conics γ_a, γ_b, γ_c and Y must lie on three other similar conics γ'_a, γ'_b, γ'_c. See Figure 4.

Lemma 2. *γ_a, γ_b and γ_c have three common points X_i, $i = 1, 2, 3$, and one of them is always real.*

Proof. Indeed, γ_b and γ_c for example meet at A and three other points, one of them being necessarily real. On the other hand, it is clear that any point X lying on two conics must lie on the third one. □

This yields the following

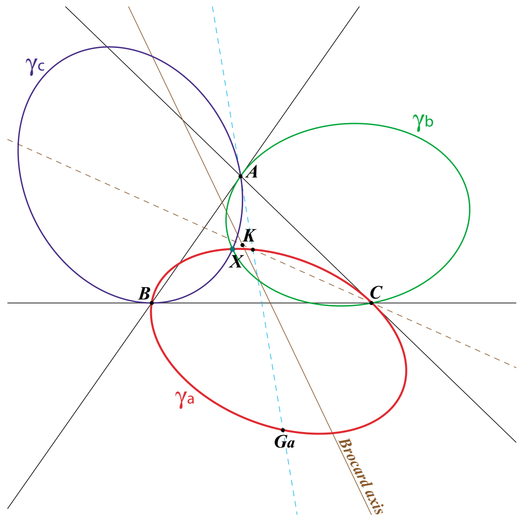

Figure 4. γ_a, γ_b and γ_c with only one real common point X

Theorem 3. *There are three* (*proper*) *bicevian Tucker circles and one of them is always real.*

2. Bicevian Tucker perspectors

The points X_i are not ruler and compass constructible since we need intersect two conics having only one known common point. For each X_i there is a corresponding Y_i which is its cyclocevian conjugate and the Tucker circle passes through the vertices of the cevian triangles of these two points. We call these six points X_i, Y_i the *Tucker bicevian perspectors*.

When X_i is known, it is easy to find the corresponding center T_i of the Tucker circle on the line OK : the perpendicular at T_i to the line H_bH_c meets AK at a point and the parallel through this point to H_bH_c meets the lines AB, AC at two points on the required circle. See Figure 5 where only one X is real and Figure 6 where all three points X_i are real.

We recall that the bicevian conic $\mathcal{C}(P,Q)$ is the conic passing through the vertices of the cevian triangles of P and Q. See [3] for general bicevian conics and their properties.

Theorem 4. *The three lines $\mathcal{L}_i$ passing through X_i, Y_i are parallel and perpendicular to the Brocard axis OK.*

Proof. We know (see [3]) that, for any bicevian conic $\mathcal{C}(P,Q)$, there is an inscribed conic bitangent to $\mathcal{C}(P,Q)$ at two points lying on the line PQ. On the other hand, any Tucker circle is bitangent to the Brocard ellipse and the line through the contacts is perpendicular to the Brocard axis. Hence, any bicevian Tucker circle must be tangent to the Brocard ellipse at two points lying on the line X_iY_i and this completes the proof. □

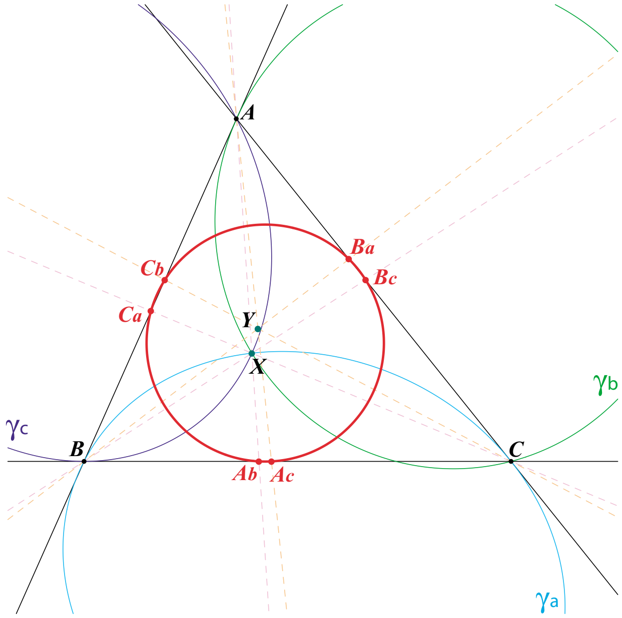

Figure 5. One real bicevian Tucker circle

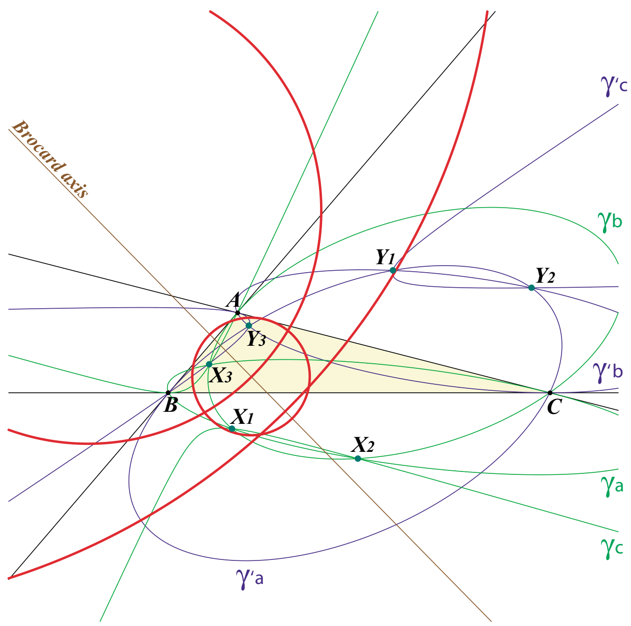

Figure 6. Three real bicevian Tucker circles

Corollary 5. *The two triangles $X_1X_2X_3$ and $Y_1Y_2Y_3$ are perspective at X_{512} and the axis of perspective is the line GK.*

Conversely, any bicevian conic $\mathcal{C}(P,Q)$ bitangent to the Brocard ellipse must verify $Q = K/P$. Such conic has its center on the Brocard line if and only if P lies either
(i) on $\mathrm{p}\mathcal{K}(X_{3051}, K)$ in which case the conic has always its center at the Brocard midpoint X_{39}, but the Tucker circle with center X_{39} is not a bicevian conic, or
(ii) on $\mathrm{p}\mathcal{K}(X_{669}, K)$ = K367 in [4].

This gives the following

Theorem 6. *The six Tucker bicevian perspectors X_i, Y_i lie on $\mathrm{p}\mathcal{K}(X_{669}, X_6)$, the pivotal cubic with pivot the Lemoine point K which is invariant in the isoconjugation swapping K and the infinite point X_{512} of the Lemoine axis.*

See Figure 7. We give another proof and more details on this cubic below.

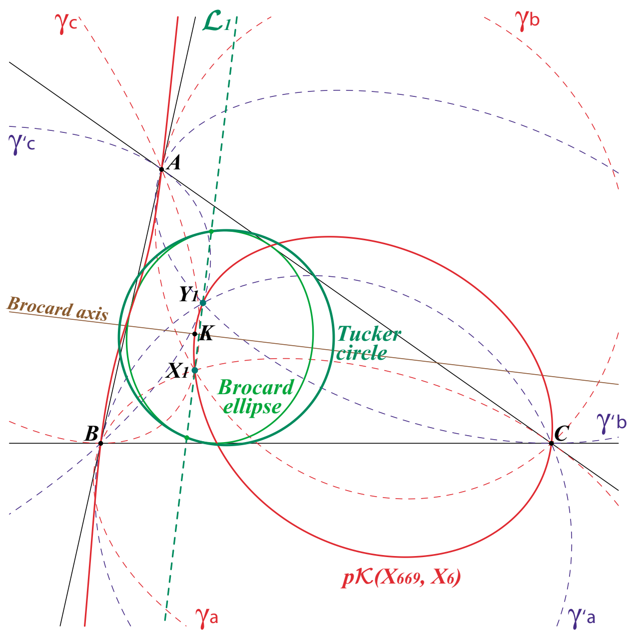

Figure 7. Bicevian Tucker circle and Brocard ellipse

3. Nets of conics associated with the Tucker bicevian perspectors

We now consider curves passing through the six Tucker bicevian perspectors X_i, Y_i. Recall that two of these points are always real and that all six points are two by two cyclocevian conjugates on three lines $\mathcal{L}_i$ perpendicular to the Brocard axis. We already know two nets of conics containing these points:
(i) the net $\mathcal{N}$ generated by γ_a, γ_b, γ_c which contain the points X_i, $i = 1, 2, 3$;
(ii) the net $\mathcal{N}'$ generated by γ'_a, γ'_b, γ'_c which contain the points Y_i, $i = 1, 2, 3$.

The equations of the conics are

$$\gamma_a : \qquad a^2y(x+z) - b^2x(x+y) = 0,$$

$$\gamma'_a: \qquad a^2z(x+y) - c^2x(x+z) = 0;$$

the other equations are obtained through cyclic permutations.

Thus, for any point $P = u : v : w$ in the plane, a conic in $\mathcal{N}$ is

$$\mathcal{N}(P) = u\,\gamma_a + v\,\gamma_b + w\,\gamma_c;$$

similarly for $\mathcal{N}'(P)$. Clearly, $\mathcal{N}(A) = \gamma_a$, etc.

Proposition 7. *Each net of conics ($\mathcal{N}$ and $\mathcal{N}'$) contains one and only one circle. These circles Γ and Γ' contain X_{110}, the focus of the Kiepert parabola.*

These circles are

$$\Gamma: \quad \sum_{\text{cyclic}} b^2c^2(b^2-c^2)(a^2-b^2)x^2 + a^2(b^2-c^2)(c^4+a^2b^2-2a^2c^2)yz = 0$$

and

$$\Gamma': \quad \sum_{\text{cyclic}} b^2c^2(b^2-c^2)(c^2-a^2)x^2 - a^2(b^2-c^2)(b^4+a^2c^2-2a^2b^2)yz = 0.$$

In fact, $\Gamma = \mathcal{N}(P')$ and $\Gamma' = \mathcal{N}'(P'')$ where

$$P' = \frac{c^2}{c^2-a^2} : \frac{a^2}{a^2-b^2} : \frac{b^2}{b^2-c^2},$$
$$P'' = \frac{b^2}{b^2-a^2} : \frac{c^2}{c^2-b^2} : \frac{a^2}{a^2-c^2}.$$

These points lie on the trilinear polar of X_{523}, the line through the centers of the Kiepert and Jerabek hyperbolas and on the circum-conic with perspector X_{76}, which is the isotomic transform of the Lemoine axis. See Figure 8.

Proposition 8. *Each net of conics contains a pencil of rectangular hyperbolas. Each pencil contains one rectangular hyperbola passing through X_{110}.*

Note that these two rectangular hyperbolas have the same asymptotic directions which are those of the rectangular circum-hyperbola passing through X_{110}. See Figure 9.

4. Cubics associated with the Tucker bicevian perspectors

When P has coordinates which are linear in x, y, z, the curves $\mathcal{N}(P)$ and $\mathcal{N}'(P)$ are in general cubics but $\mathcal{N}(z : x : y)$ and $\mathcal{N}'(y : z : x)$ are degenerate. In other words, for any point $x : y : z$ of the plane, we (loosely) may write

$$z\,\gamma_a + x\,\gamma_b + y\,\gamma_c = 0$$

and

$$y\,\gamma'_a + z\,\gamma'_b + x\,\gamma'_c = 0.$$

We obtain two circum-cubics $\mathcal{K}(P)$ and $\mathcal{K}'(P)$ when P takes the form

$$P = q\,z - r\,y : r\,x - p\,z : p\,y - q\,x$$

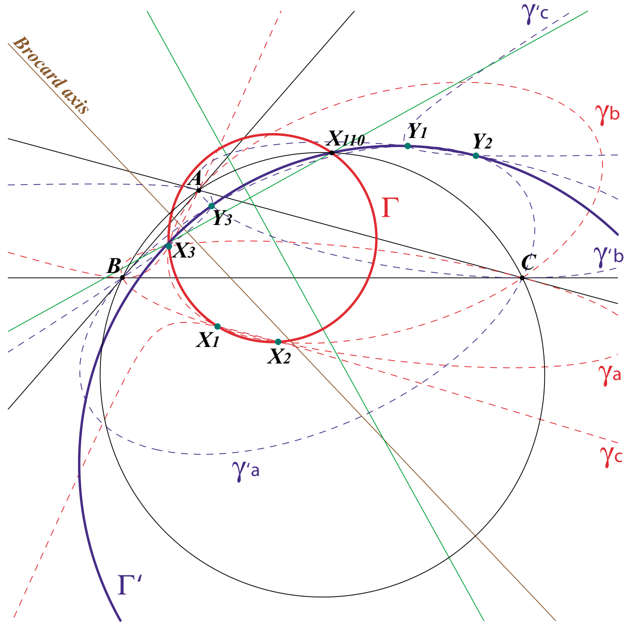

Figure 8. Circles through the Tucker bicevian perspectors

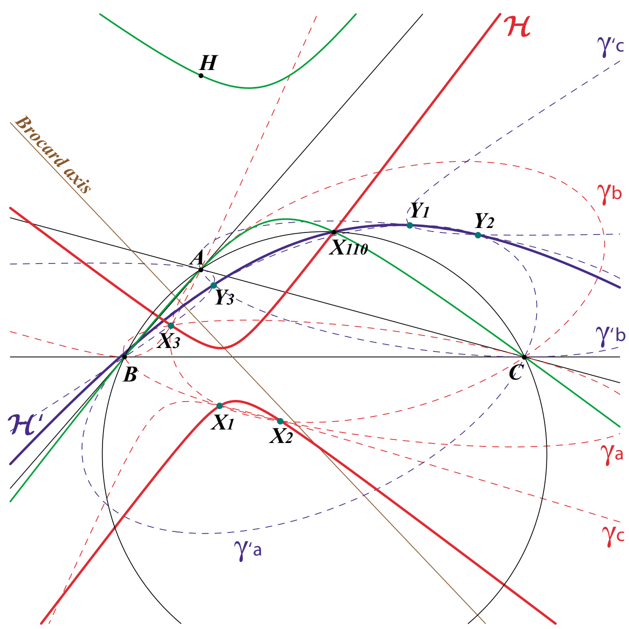

Figure 9. Rectangular hyperbolas through the Tucker bicevian perspectors

associated to the cevian lines of the point $Q = p : q : r$ and both cubics contain Q. Obviously, $\mathcal{K}(P)$ contains the points X_i and $\mathcal{K}'(P)$ contains the points Y_i.

For example, with $Q = G$, we obtain the two cubics $\mathcal{K}(G)$ and $\mathcal{K}'(G)$ passing through G and the vertices of the antimedial triangle $G_aG_bG_c$. See Figure 10.

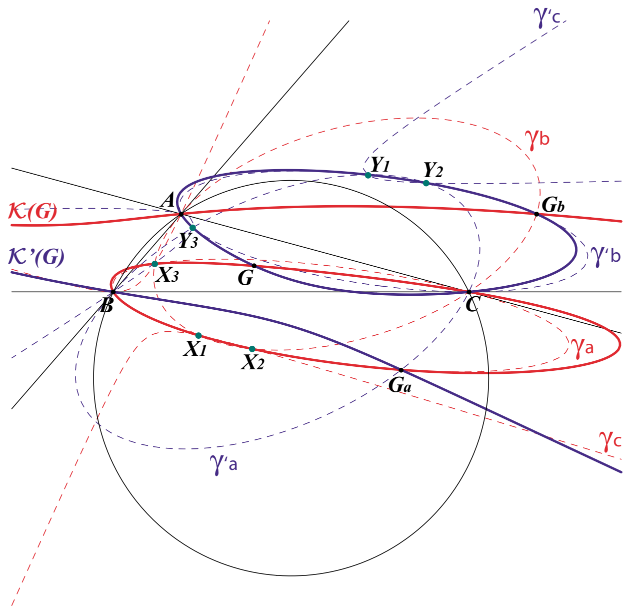

Figure 10. The two cubics $\mathcal{K}(G)$ and $\mathcal{K}'(G)$

These two cubics $\mathcal{K}(P)$ and $\mathcal{K}'(P)$ are isotomic pivotal cubics with pivots the bicentric companions (see [5, p.47] and [2]) of X_{523} respectively

$$X'_{523} = a^2 - b^2 : b^2 - c^2 : c^2 - a^2$$

and

$$X''_{523} = c^2 - a^2 : a^2 - b^2 : b^2 - c^2$$

both on the line at infinity. The two other points at infinity of the cubics are those of the Steiner ellipse.

4.1. *An alternative proof of Theorem 6.* We already know (Theorem 6) that the six Tucker bicevian perspectors X_i, Y_i lie on the cubic $\mathrm{p}\mathcal{K}(X_{669}, X_6)$. Here is an alternative proof. See Figure 11.

Proof. Let U, V, W be the traces of the perpendicular at G to the Brocard axis. We denote by Γ_a the decomposed cubic which is the union of the line AU and the conic γ_a. Γ_a contains the vertices of ABC and the points X_i. Γ_b and Γ_c are defined similarly and contain the same points.

The cubic $c^2\,\Gamma_a + a^2\,\Gamma_b + b^2\,\Gamma_c$ is another cubic through the same points since it belongs to the net of cubics. It is easy to verify that this latter cubic is $\mathrm{p}\mathcal{K}(X_{669}, X_6)$.

Now, if Γ'_a, Γ'_b, Γ'_c are defined likewise, the cubic $b^2\,\Gamma'_a + c^2\,\Gamma'_b + a^2\,\Gamma'_c$ is $\mathrm{p}\mathcal{K}(X_{669}, X_6)$ again and this shows that the six Tucker bicevian perspectors lie on the curve. $\square$

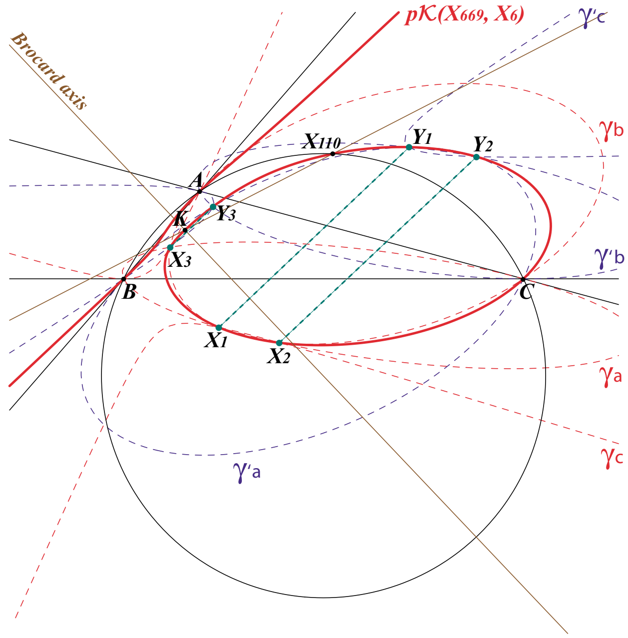

Figure 11. $\mathrm{p}\mathcal{K}(X_{669}, X_6)$ and the three lines $\mathcal{L}_i$

4.2. *More on the cubic* $\mathrm{p}\mathcal{K}(X_{669}, X_6)$. The cubic $\mathrm{p}\mathcal{K}(X_{669}, X_6)$ also contains K, X_{110}, X_{512}, X_{3124} and meets the sidelines of ABC at the feet of the symmedians. Note that the pole X_{669} is the barycentric product of K and X_{512}, the isopivot or secondary pivot (see [1], §1.4). This shows that, for any point M on the cubic, the point K/M (cevian quotient or Ceva conjugate) lies on the cubic and the line $M\ K/M$ contains X_{512} i.e. is perpendicular to the Brocard axis.

We can apply to the Tucker bicevian perspectors the usual group law on the cubic. For any two points P, Q on $\mathrm{p}\mathcal{K}(X_{669}, X_6)$ we define $P \oplus Q$ as the third intersection of the line through K and the third point on the line PQ.

For a permuation $i,\ j,\ k$ of $1,\ 2,\ 3$, we have

$$X_i \oplus X_j = Y_k, \qquad Y_i \oplus Y_j = X_k.$$

Furthermore, $X_i \oplus Y_i = K$. These properties are obvious since the pivot of the cubic is K and the secondary pivot is X_{512}.

The third point of $\mathrm{p}\mathcal{K}(X_{669}, X_6)$ on the line KX_{110} is $X_{3124} = a^2(b^2-c^2)^2 : b^2(c^2-a^2)^2 : c^2(a^2-b^2)^2$, the cevian quotient of K and X_{512} and the third point on the line $X_{110}X_{512}$ is the cevian quotient of K and X_{110}.

The infinite points of $\mathrm{p}\mathcal{K}(X_{669}, X_6)$ are X_{512} and two imaginary points, those of the bicevian ellipse $\mathcal{C}(G, K)$ or, equivalently, those of the circum-ellipse $\mathcal{C}(X_{39})$ with perspector X_{39} and center X_{141}.

The real asymptote is perpendicular to the Brocard axis and meets the curve at $X = K/X_{512}$, the third point on the line KX_{110} seen above. X also lies on the Brocard ellipse, on $\mathcal{C}(G, K)$. See Figure 12.

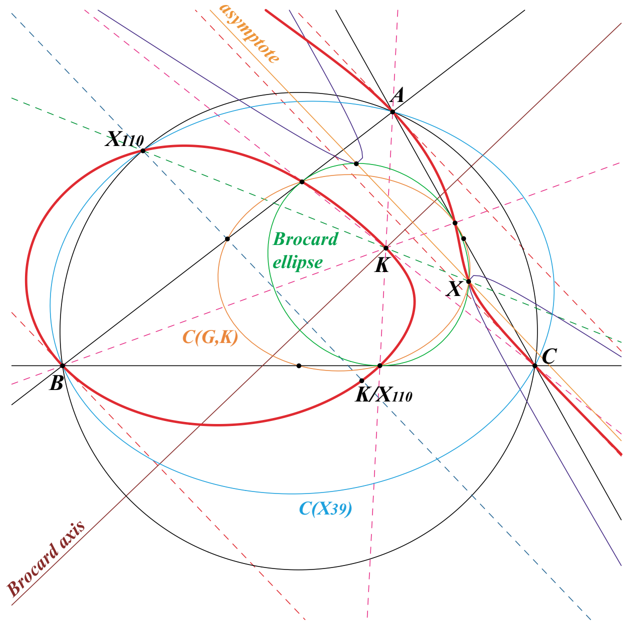

Figure 12. $K367 = \mathrm{p}\mathcal{K}(X_{669}, X_6)$

$\mathrm{p}\mathcal{K}(X_{669}, X_6)$ is the isogonal transform of $\mathrm{p}\mathcal{K}(X_{99}, X_{99})$, a member of the class CL007 in [4]. These are the $\mathrm{p}\mathcal{K}(W, W)$ cubics or parallel tripolars cubics.

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Forum Geometricorum
Volume 7 (2007) 99–102.

FORUM GEOM
ISSN 1534-1178

A Visual Proof of the Erdős-Mordell Inequality

Claudi Alsina and Roger B. Nelsen

Abstract. We present a visual proof of a lemma that reduces the proof of the Erdős-Mordell inequality to elementary algebra.

In 1935, the following problem proposal appeared in the "Advanced Problems" section of the *American Mathematical Monthly* [5]:

> **3740**. *Proposed by Paul Erdős, The University, Manchester, England*.
> From a point O inside a given triangle ABC the perpendiculars OP, OQ, OR are drawn to its sides. Prove that
> $$OA + OB + OC \geq 2(OP + OQ + OR).$$

Trigonometric solutions by Mordell and Barrow appeared in [11]. The proofs, however, were not elementary. In fact, no "simple and elementary" proof of what had become known as the Erdős-Mordell theorem was known as late as 1956 [13]. Since then a variety of proofs have appeared, each one in some sense simpler or more elementary than the preceding ones. In 1957 Kazarinoff published a proof [7] based upon a theorem in Pappus of Alexandria's *Mathematical Collection*; and a year later Bankoff published a proof [2] using orthogonal projections and similar triangles. Proofs using area inequalities appeared in 1997 and 2004 [4, 9]. Proofs employing Ptolemy's theorem appeared in 1993 and 2001 [1, 10]. A trigonometric proof of a generalization of the inequality in 2001 [3], subsequently generalized in 2004 [6]. Many of these authors speak glowingly of this result, referring to it as a "beautiful inequality" [9], a "remarkable inequality" [12], "the famous Erdős-Mordell inequality" [4, 6, 10], and "the celebrated Erdős-Mordell inequality . . . a beautiful piece of elementary mathematics" [3].

In this short note we continue the progression towards simpler proofs. First we present a visual proof of a lemma that reduces the proof of the Erdős-Mordell inequality to elementary algebra. The lemma provides three inequalities relating the lengths of the sides of ABC and the distances from O to the vertices and to the sides. While the inequalities in the lemma are not new, we believe our proof of the lemma is. The proof uses nothing more sophisticated than elementary properties of triangles. In Figure 1(a) we see the triangle as described by Erdős, and in Figure

Publication Date: April 30, 2007. Communicating Editor: Paul Yiu.

1(b) we denote the lengths of relevant line segments by lower case letters, whose use will simplify the presentation to follow. In terms of that notation, the Erdős-Mordell inequality becomes

$$x + y + z \geq 2(p + q + r).$$

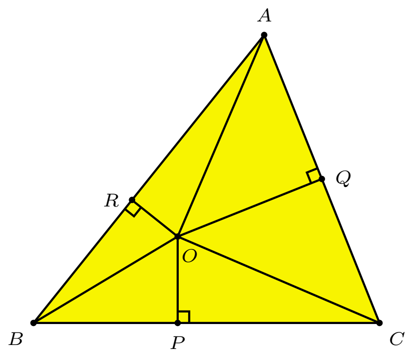

Figure 1(a)

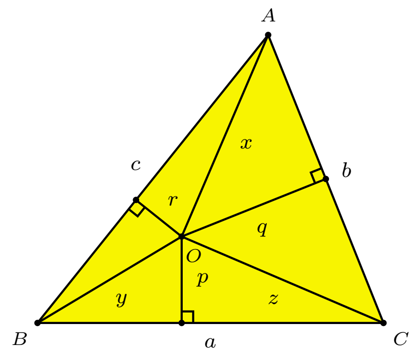

Figure 1(b)

In the proof of the lemma, we construct a trapezoid in Figure 2(b) from three triangles – one similar to ABC, the other two similar to two shaded triangles in Figure 2(a).

Lemma. *For the triangle ABC in Figure 1, we have $ax \geq br + cq$, $by \geq ar + cp$, and $cz \geq aq + bp$.*

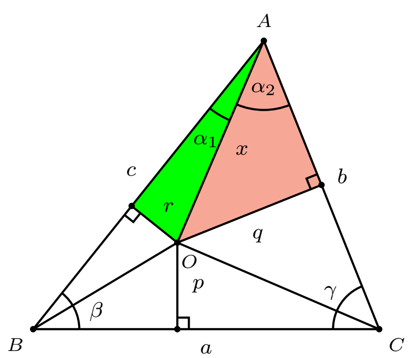

Figure 2(a)

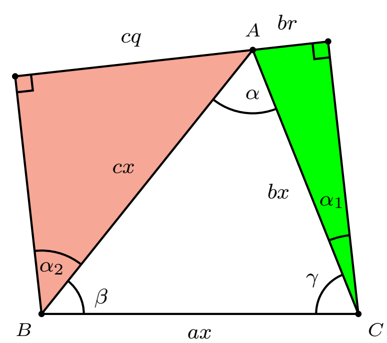

Figure 2(b)

Proof. See Figure 2 for a visual proof that $ax \geq br + cq$. The other two inequalities are established analogously. □

We should note before proceeding that the object in Figure 2(b) really is a trapezoid, since the three angles at the point where the three triangles meet measure $\frac{\pi}{2} - \alpha_2$, $\alpha = \alpha_1 + \alpha_2$, and $\frac{\pi}{2} - \alpha_1$, and thus sum to π.

We now prove

The Erdős-Mordell Inequality. *If O is a point within a triangle ABC whose distances to the vertices are x, y, and z, then*

$$x + y + z \geq 2(p + q + r).$$

Proof. From the lemma we have $x \geq \frac{b}{a}r + \frac{c}{a}q$, $y \geq \frac{a}{b}r + \frac{c}{b}p$, and $z \geq \frac{a}{c}q + \frac{b}{c}p$. Adding these three inequalities yields

$$x + y + z \geq \left(\frac{b}{c} + \frac{c}{b}\right)p + \left(\frac{c}{a} + \frac{a}{c}\right)q + \left(\frac{a}{b} + \frac{b}{a}\right)r. \tag{1}$$

But the arithmetic mean-geometric mean inequality insures that the coefficients of p, q, and r are each at least 2, from which the desired result follows. □

We conclude with several comments about the lemma and the Erdős-Mordell inequality and their relationships to other results.

1. The three inequalities in the lemma are equalities if and only if O is the center of the circumscribed circle of ABC. This follows from the observation that the trapezoid in Figure 2(b) is a rectangle if and only if $\beta + \alpha_2 = \frac{\pi}{2}$ and $\gamma + \alpha_1 = \frac{\pi}{2}$ (and similarly in the other two cases), so that $\angle AOQ = \beta = \angle COQ$. Hence the right triangles AOQ and COQ are congruent, and $x = z$. Similarly one can show that $x = y$. Hence, $x = y = z$ and O must be the circumcenter of ABC. The coefficients of p, q, and r in (1) are equal to 2 if and only if $a = b = c$. Consequently we have equality in the Erdős-Mordell inequality if and only if ABC is equilateral and O is its center.

2. How did Erdős come up with the inequality in his problem proposal? Kazarinoff [8] speculates that he generalized Euler's inequality: if $\overline{r}$ and $\overline{R}$ denote, respectively, the inradius and circumradius of ABC, then $\overline{R} \geq 2\overline{r}$. The Erdős-Mordell inequality implies Euler's inequality for acute triangles. Note that if we take O to be the circumcenter of ABC, then $3\overline{R} \geq 2(p + q + r)$. However, for *any* point O inside ABC, the quantity $p + q + r$ is somewhat surprisingly constant and equal to $\overline{R} + \overline{r}$, a result known as Carnot's theorem. Thus $3\overline{R} \geq 2(\overline{R} + \overline{r})$, or equivalently, $\overline{R} \geq 2\overline{r}$.

3. Many other inequalities relating x, y, and z to p, q, and r can be derived. For example, applying the arithmetic mean-geometric mean inequality to the right side of the inequalities in the lemma yields

$$ax \geq 2\sqrt{bcqr}, \quad by \geq 2\sqrt{carp}, \quad cz \geq 2\sqrt{abpq}.$$

Multiplying these three inequalities together and simplifying yields $xyz \geq 8pqr$. More such inequalities can be found in [8, 12].

4. A different proof of (1) appears in [4].

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Forum Geometricorum
Volume 7 (2007) 103–106.

FORUM GEOM
ISSN 1534-1178

Construction of Triangle from a Vertex and the Feet of Two Angle Bisectors

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Abstract. We give two simple constructions of a triangle given one vertex and the feet of two angle bisectors.

1. Construction from $(A,\ T_a,\ T_b)$

We present two simple solutions of the following construction problem (number 58) in the list compiled by W. Wernick [2]: Given three noncollinear points A, T_a and T_b, to construct a triangle ABC with T_a, T_b on BC, CA respectively such that AT_a and BT_b are bisectors of the triangle. L. E. Meyers [1] has indicated the constructibility of such a triangle. Let ℓ be the half line AT_b. Both solutions we present here make use of the reflection ℓ' of ℓ in AT_a. The vertex B necessarily lies on ℓ'. In what follows $P(Q)$ denotes the circle, center P, passing through the point Q.

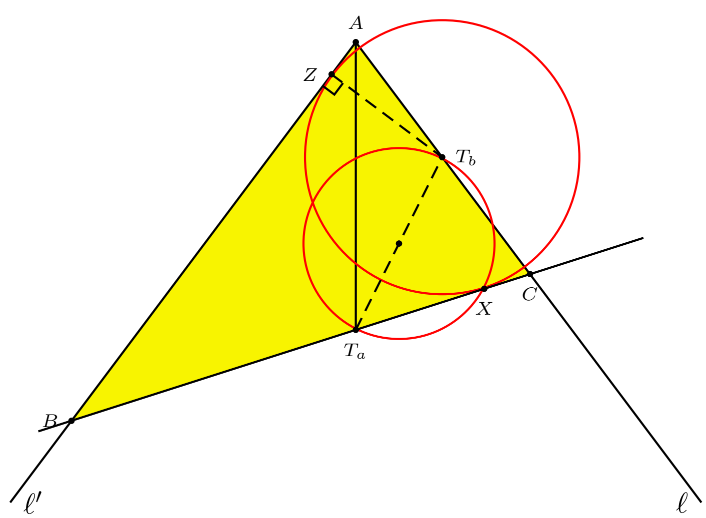

Figure 1

Construction 1. *Let Z be the pedal of T_b on ℓ'. Construct two circles, one $T_b(Z)$, and the other with T_aT_b as diameter. Let X be an intersection, if any, of these two circles. If the line XT_a intersects the half lines ℓ' at B and ℓ at C, then ABC is a desired triangle.*

Publication Date: May 7, 2007. Communicating Editor: Paul Yiu.

Construction 2. *Let P be an intersection, if any, of the circle $T_b(T_a)$ with the half line ℓ'. Construct the perpendicular bisector of PT_a. If this intersects ℓ' at a point B, and if the half line BT_a intersects ℓ at C, then ABC is a desired triangle.*

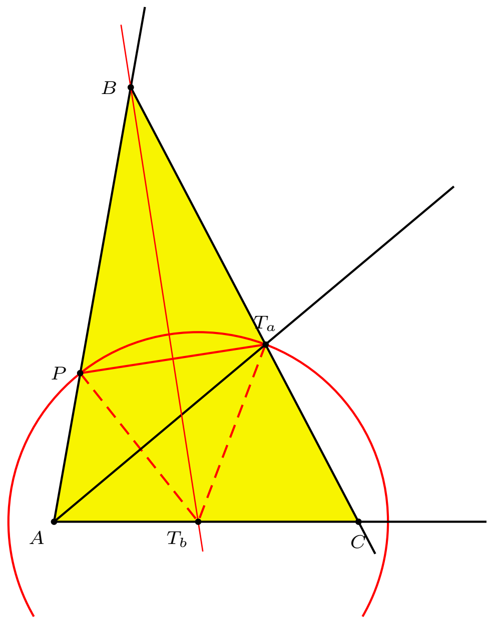

Figure 2

We study the number of solutions for various relative positions of A, T_a and T_b. Set up a polar coordinate system with A at the pole and T_b at $(1, 0)$. Suppose T_a has polar coordinates $(\rho,\ \theta)$ for $\rho > 0$ and $0 < \theta < \frac{\pi}{2}$. The half line ℓ' has polar angle 2θ. The circle $T_b(T_a)$ intersects ℓ' if the equation

$$\sigma^2 - 2\sigma\cos 2\theta = \rho^2 - 2\rho\cos\theta \tag{1}$$

has a positive root σ. This is the case when
(i) $\rho > 2\cos\theta$, or
(ii) $\rho \le 2\cos\theta$, $\cos 2\theta > 0$ and $4\cos^2 2\theta + 4\rho(\rho - 2\cos\theta) \ge 0$. Equivalently, $\rho_+ \le \rho \le 2\cos\theta$, where

$$\rho_\pm = \cos\theta \pm \sqrt{\sin\theta\sin 3\theta}$$

are the roots of the equation $\rho^2 - 2\rho\cos\theta + \cos^2 2\theta = 0$ for $0 < \theta < \frac{\pi}{3}$.

Now, the perpendicular bisector of PT_a intersects the line ℓ' at the point B with polar coordinates $(\beta,\ 2\theta)$, where

$$\beta = \frac{\rho\cos\theta - \sigma\cos 2\theta}{\rho\cos\theta - \sigma}.$$

The requirement $\beta > 0$ is equivalent to $\sigma < \rho\cos\theta$. From (1), this is equivalent to $\rho < 4\cos\theta$.

For $0 < \theta < \frac{\pi}{3}$, let $P_\pm$ be the points with polar coordinates $(\rho_\pm,\ \theta)$. These points bound a closed curve $\mathscr{C}$ as shown in Figure 3. If T_a lies inside the curve $\mathscr{C}$, then the circle $T_b(T_a)$ does not intersect the half line ℓ'. We summarize the results with reference to Figure 3.

The construction problem of ABC from $(A,\ T_a,\ T_b)$ has
(1) a unique solution if T_a lies in the region between the two semicircles $\rho = 2\cos\theta$ and $\rho = 4\cos\theta$,
(2) two solutions if T_a lies between the semicircle $\rho = 2\cos\theta$ and the curve $\mathscr{C}$ for $\theta < \frac{\pi}{4}$.

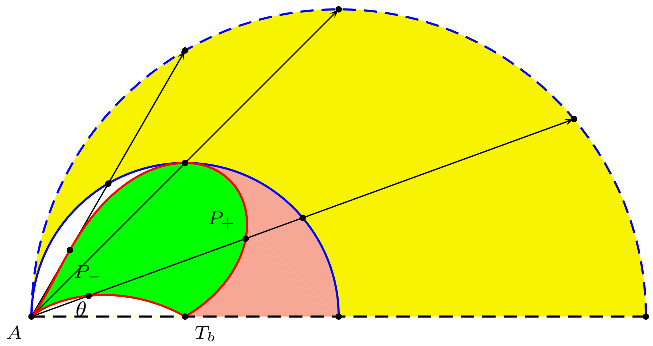

Figure 3.

2. Construction from $(A,\ T_b,\ T_c)$

The construction of triangle ABC from $(A,\ T_a,\ T_b)$ is Problem 60 in Wernick's list [2]. Wernick has indicated constructibility. We present two simple solutions.

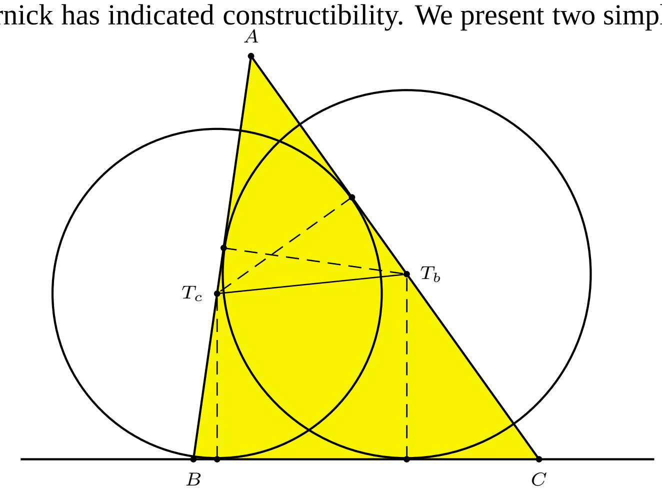

Figure 4.

Construction 3. *Given A, T_b, T_c, construct the circles with centers T_b and T_b, tangent to AT_c and AT_b respectively. The common tangent of these circles that lies opposite to A with respect to the line T_bT_c is the line BC of the required triangle ABC. The construction of the vertices B, C is obvious.*

Construction 4. *Given A, T_b, T_c, construct*
(i) *the circle through the three points*
(ii) *the bisector of angle T_bAT_c to intersect the circle at M,*
(iii) *the reflection M' of M in the line T_bT_c,*
(iv) *the circle $M'(T_b)$ to intersect the bisector at I (so that A and I are on opposite sides of T_bT_c),*
(v) *the half line T_bI to intersect the half line AT_c at B,*
(vi) *the half line T_cI to intersect the half line AT_b at C.*

ABC is the required triangle with incenter I.

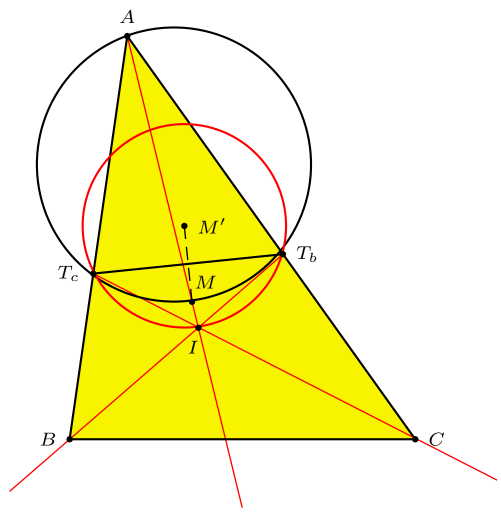

Figure 5.

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Forum Geometricorum
Volume 7 (2007) 107–109.

FORUM GEOM
ISSN 1534-1178

Three Pappus Chains Inside the Arbelos: Some Identities

Giovanni Lucca

Abstract. We consider the three different Pappus chains that can be constructed inside the arbelos and we deduce some identities involving the radii of the circles of n-th order and the incircle radius.

1. Introduction

The Pappus chain [1] is an infinite series of circles constructed starting from the Archimedean figure named arbelos (also said shoemaker knife) so that the generic circle $\mathcal{C}_i$, $(i = 1,\ 2,\ \dots)$ of the chain is tangent to the the circles $\mathcal{C}_{i-1}$ and $\mathcal{C}_{i+1}$ and to two of the three semicircles $\mathcal{C}_{\mathrm{a}}$, $\mathcal{C}_{\mathrm{b}}$ and $\mathcal{C}_{\mathrm{r}}$ forming the arbelos. In a generic arbelos three different Pappus chains can be drawn (see Figure 1).

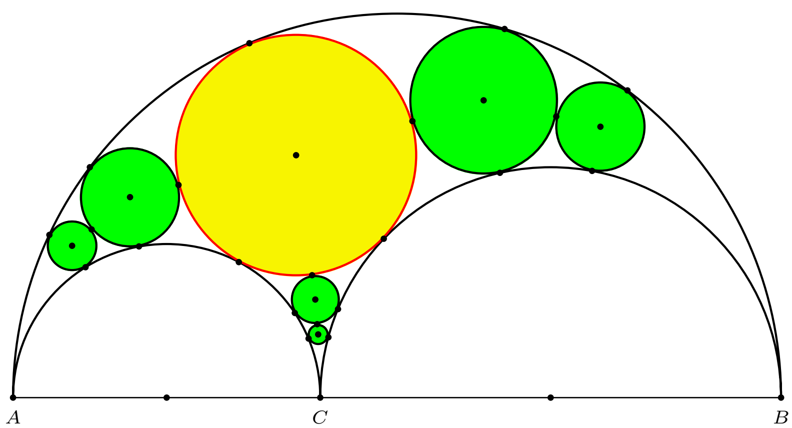

Figure 1.

In Figure 1, the diameter AC of the left semicircle $\mathcal{C}_{\mathrm{a}}$ is $2a$, the diameter CB of the right semicircle $\mathcal{C}_{\mathrm{b}}$ is $2b$, and the diameter AB of the outer semicircle $\mathcal{C}_{\mathrm{r}}$ is $2r$, $r = a + b$. The first circle $\mathcal{C}_1$ is common to all three chains and is named the incircle of the arbelos. By applying the circular inversion technique, it is possible to determine the center coordinates and radius of each chain; the radii are expressed by the formulas reported in Table I. The chain tending to point C is named Γ_{r}, the chain tending to point B is named Γ_{a} and the chain tending to point A is named Γ_{b}. As far as chains Γ_{a} and Γ_{b} are concerned, the expressions for the radii are given in [2] while for Γ_{r}, we give an inductive proof below.

Publication Date: June 4, 2007. Communicating Editor: Floor van Lamoen.

Table I: Radii of the circles forming the three Pappus chains

Chain	Γ_{r}	Γ_{a}	Γ_{b}
Radius of n-th circle	$\rho_{\mathrm{r}n} = \frac{rab}{n^2r^2-ab}$	$\rho_{\mathrm{a}n} = \frac{rab}{n^2a^2+rb}$	$\rho_{\mathrm{b}n} = \frac{rab}{n^2b^2+ra}$

For integers $n \geq 1$, consider the statement

$$P(n) \qquad \rho_{\mathrm{r}n} = \frac{rab}{n^2r^2 - ab}.$$

$P(1)$ is true since the first circle of the chain is the arbelos incircle having radius given by formula (3).

We show that $P(n) \Rightarrow P(n+1)$.

Let us consider the circles $\mathcal{C}_{\mathrm{rn}}$ and $\mathcal{C}_{\mathrm{rn+1}}$ in the chain Γ_{r}, together the inner semicircles $\mathcal{C}_{\mathrm{a}}$ and $\mathcal{C}_{\mathrm{b}}$ inside the arbelos. Applying Descartes' theorem we have

$$2(\varepsilon_{\mathrm{r}n}^2 + \varepsilon_{\mathrm{r}n+1}^2 + \varepsilon_{\mathrm{a}}^2 + \varepsilon_{\mathrm{b}}^2) = (\varepsilon_{\mathrm{r}n} + \varepsilon_{\mathrm{r}n+1} + \varepsilon_{\mathrm{a}} + \varepsilon_{\mathrm{b}})^2, \tag{1}$$

where $\varepsilon_{\mathrm{r}n}$, $\varepsilon_{\mathrm{r}n+1}$, ε_{a} and ε_{b} are the curvatures, *i.e.*, reciprocals of the radii of the circles. Rewriting this as

$$\varepsilon_{\mathrm{r}n+1}^2 - 2\varepsilon_{\mathrm{r}n+1}(\varepsilon_{\mathrm{r}n} + \varepsilon_{\mathrm{a}} + \varepsilon_{\mathrm{b}}) + \varepsilon_{\mathrm{r}n}^2 + \varepsilon_{\mathrm{a}}^2 + \varepsilon_{\mathrm{b}}^2 - 2(\varepsilon_{\mathrm{r}n}\varepsilon_{\mathrm{a}} + \varepsilon_{\mathrm{a}}\varepsilon_{\mathrm{b}} + \varepsilon_{\mathrm{b}}\varepsilon_{\mathrm{r}n}) = 0,$$

we have

$$\varepsilon_{\mathrm{r}n+1} = \varepsilon_{\mathrm{r}n} + \varepsilon_{\mathrm{a}} + \varepsilon_{\mathrm{b}} \pm 2\sqrt{\varepsilon_{\mathrm{r}n}\varepsilon_{\mathrm{a}} + \varepsilon_{\mathrm{a}}\varepsilon_{\mathrm{b}} + \varepsilon_{\mathrm{b}}\varepsilon_{\mathrm{r}n}}. \tag{2}$$

Substituting into (2) $\varepsilon_{\mathrm{a}} = \frac{1}{a}$, $\varepsilon_{\mathrm{b}} = \frac{1}{b}$ and $\varepsilon_{\mathrm{r}n} = \frac{rab}{n^2r^2-ab}$, we obtain, after a few steps of simple algebraic calculations,

$$\rho_{\mathrm{r}n+1} = \frac{1}{\varepsilon_{\mathrm{r}n+1}} = \frac{rab}{(n+1)^2r^2 - ab}.$$

This proves that $P(n) \Rightarrow P(n+1)$, and by induction, $P(n)$ is true for every integer $n \geq 1$.

2. Relationships among the n-th circles radii and incircle radius

For the following, it is useful to write explicitly the incircle radius ρ_{inc} that is given by:

$$\rho_{\mathrm{inc}} = \frac{rab}{a^2 + ab + b^2} \tag{3}$$

Formula (3) is directly obtained by each one of the three formulas for the radius in Table I for $n = 1$. It is useful too to write the square of the incircle radius that is:

$$\rho_{\mathrm{inc}}^2 = \frac{r^2a^2b^2}{a^4 + 2a^3b + 2a^2b^2 + 2ab^3 + b^4}. \tag{4}$$

We enunciate now the following proposition related to three different identities among the circles chains radii and the incircle radius.

Proposition. *Given a generic arbelos with its three Pappus chains, the following identities hold for each integer n:*

$$\rho_{\mathrm{inc}}\left(\frac{1}{\rho_{\mathrm{r}n}}+\frac{1}{\rho_{\mathrm{a}n}}+\frac{1}{\rho_{\mathrm{b}n}}\right)=2n^2+1, \tag{5}$$

$$\rho^2_{\mathrm{inc}}\left(\frac{1}{\rho^2_{\mathrm{r}n}}+\frac{1}{\rho^2_{\mathrm{a}n}}+\frac{1}{\rho^2_{\mathrm{b}n}}\right)=2n^4+1, \tag{6}$$

$$\rho^2_{\mathrm{inc}}\left(\frac{1}{\rho_{\mathrm{r}n}}\cdot\frac{1}{\rho_{\mathrm{a}n}}+\frac{1}{\rho_{\mathrm{a}n}}\cdot\frac{1}{\rho_{\mathrm{b}n}}+\frac{1}{\rho_{\mathrm{b}n}}\cdot\frac{1}{\rho_{\mathrm{r}n}}\right)=n^4+2n^2. \tag{7}$$

Proof. To demonstrate (5), one has to substitute in it the expression for the radius incircle given by (3) and the expressions for the radii of n-th circles chain given in Table I. Using the fact that $r=a+b$, one obtains

$$\frac{rab}{a^2+ab+b^2}\left(\frac{n^2r^2-ab}{rab}+\frac{n^2a^2+rb}{rab}+\frac{n^2b^2+ra}{rab}\right)=2n^2+1.$$

For (6), one has to substitute in it the expression for the square of the radius incircle given by (4) and to take the squares of the radii of n-th circles chain given in Table I. Using the fact that $r=a+b$, one obtains

$$\frac{r^2a^2b^2}{(a^2+ab+b^2)^2}\left(\left(\frac{n^2r^2-ab}{rab}\right)^2+\left(\frac{n^2a^2+rb}{rab}\right)^2+\left(\frac{n^2b^2+ra}{rab}\right)^2\right)=2n^4+1.$$

For (7), one has to substitute in it the expression for the square of the incircle radius given by (4) and the expressions for the radii of the n-th circles given in Table I. This leads to $\frac{r^2a^2b^2}{(a^2+ab+b^2)^2}\cdot\frac{D}{r^2a^2b^2}$, where

$$\begin{aligned} D=&(n^2r^2-ab)(n^2a^2+rb)+(n^2a^2+rb)(n^2b^2+ra)+(n^2b^2+ra)(n^2r^2-ab)\\ =&(n^4+2n^2)(a^2+ab+b^2)^2, \end{aligned}$$

by using the fact that $r=a+b$. Finally, this leads to (7). □

3. Conclusion

Considering the three Pappus chains that can be drawn inside a generic arbelos, some identities involving the incircle radius and the n-th circles chain radii have been shown. All these identities generate sequences of integers.

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Forum Geometricorum
Volume 7 (2007) 111–113.

FORUM GEOM
ISSN 1534-1178

Some Powerian Pairs in the Arbelos

Floor van Lamoen

Abstract. Frank Power has presented two pairs of Archimedean circles in the arbelos. In each case the two Archimedean circles are tangent to each other and tangent to a given circle. We give some more of these Powerian pairs.

1. Introduction

We consider an arbelos with greater semicircle (O) of radius r and smaller semicircle (O_1) and (O_2) of radii r_1 and r_2 respectively. The semicircles (O_1) and (O) meet in A, (O_2) and (O) in B, (O_1) and (O_2) in C and the line through C perpendicular to AB meets (O) in D. Beginning with Leon Bankoff [1], a number of interesting circles congruent to the Archimedean twin circles has been found associated with the arbelos. These have radii $\frac{r_1 r_2}{r}$. See [2]. Frank Power [5] has presented two pairs of Archimedean circles in the Arbelos with a definition unlike the other known ones given for instance in [2, 3, 4].[1]

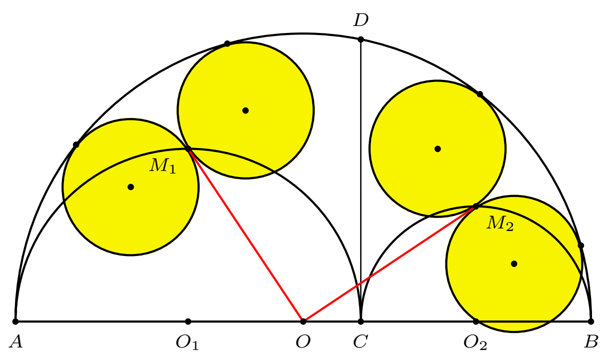

Figure 1

Proposition 1 (Power [5]). *Let M_1 and M_2 be the 'highest' points of (O_1) and (O_2) respectively. Then the pairs of congruent circles tangent to (O) and tangent to each other at M_1 and M_2 respectively, are pairs of Archimedean circles.*

To pairs of Archimedean circles tangent to a given circle and to each other at a given point we will give the name *Powerian pairs*.

Publication Date: June 12, 2007. Communicating Editor: Paul Yiu.

[1]The pair of Archimedean circles (A_{5a}) and (A_{5b}), with numbering as in [4], qualifies for what we will later in the paper refer to as *Powerian pair*, as they are tangent to each other at C and to the circular hull of Archimedes' twin circles. This however is not how they were originally defined.

2. Three double Powerian pairs

2.1. Let M be the midpoint of CD. Consider the endpoints U_1 and U_2 of the diameter of (CD) perpendicular to OM.

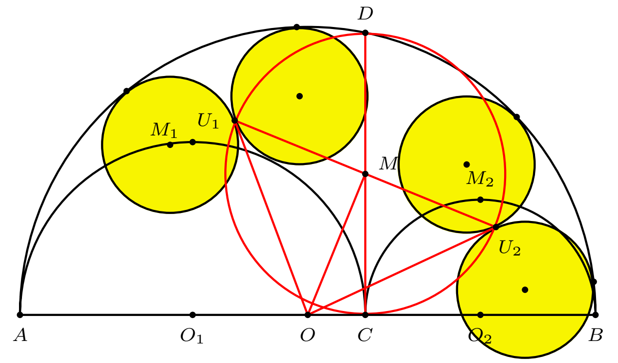

Figure 2

Note that $OC^2 = (r_1 - r_2)^2$ and as $CD = 2\sqrt{r_1 r_2}$ that $OD^2 = r_1^2 - r_1 r_2 + r_2^2$ and $OU_1^2 = r_1^2 + r_2^2$.

Now consider the pairs of congruent circles tangent to each other at U_1 and U_2 and tangent to (O). The radii ρ of these circles satisfy

$$(r_1 + r_2 - \rho)^2 = OU_1^2 + \rho^2$$

from which we see that $\rho = \frac{r_1 r_2}{r}$. This pair is thus Powerian. By symmetry the other pair is Powerian as well.

2.2. Let T_1 and T_2 be the points of tangency of the common tangent of (O_1) and (O_2) not through C. Now consider the midpoint O' of O_1O_2, also the center of the semicircle (O_1O_2), which is tangent to segment T_1T_2 at its midpoint.

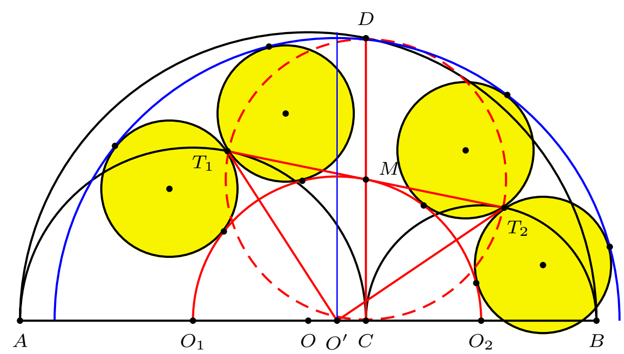

Figure 3

As $T_1T_2 = 2\sqrt{r_1 r_2}$ we see that $O'T_1^2 = \left(\frac{r_1+r_2}{2}\right)^2 + r_1 r_2$. Now consider the pairs of congruent circles tangent to each other at T_1 and tangent to (O_1O_2). The radii ρ of these circles satisfy

$$\left(\frac{r_1+r_2}{2}\right)+\rho)^2-\rho^2=O'T_1^2$$

from which we see that $\rho=\frac{r_1r_2}{r}$ and this pair is Powerian. By symmetry the pair of congruent circles tangent to each other at T_2 and to (O_1O_2) is Powerian.

Remark: These pairs are also tangent to the circle with center O' through the point where the Schoch line meets (O).

2.3. Note that $AD=2\sqrt{rr_1}$, hence

$$AT_1=\frac{r_1}{r}AD=\frac{2r_1\sqrt{r_1}}{\sqrt{r}}.$$

Now consider the pair of congruent circles tangent to each other at T_1 and to the circle with center A through C. The radii of these circles satisfy

$$AT_1^2+\rho^2=(2r_1-\rho)^2$$

from which we see that $\rho=\frac{r_1r_2}{r}$ and this pair is Powerian. In the same way the pair of congruent circles tangent to each other at T_2 and to the circle with center B through C is Powerian.

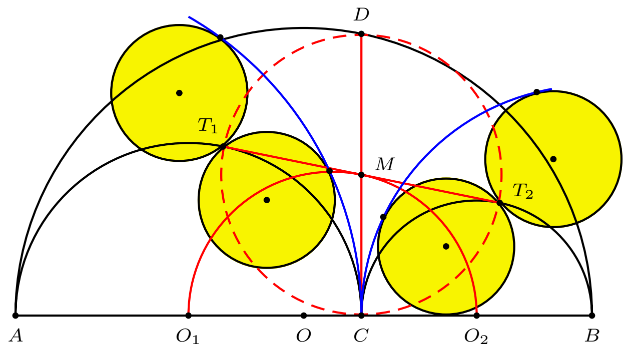

Figure 4

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Forum Geometricorum
Volume 7 (2007) 115–120.

FORUM GEOM
ISSN 1534-1178

The Arbelos and Nine-Point Circles

Quang Tuan Bui

Abstract. We construct some new Archimedean circles in an arbelos in connection with the nine-point circles of some appropriate triangles. We also construct two new pairs of Archimedes circles analogous to those of Frank Power, and one pair of Archimedean circles related to the tangents of the arbelos.

1. Introduction

We consider an arbelos consisting of three semicircles (O_1), (O_2), (O), with points of tangency A, B, P. Denote by r_1, r_2 the radii of (O_1), (O_2) respectively. Archimedes has shown that the two circles, each tangent to (O), the common tangent PQ of (O_1), (O_2), and one of (O_1), (O_2), have congruent radius $r = \frac{r_1 r_2}{r_1+r_2}$. See [1, 2]. Let C be a point on the half line PQ such that $PC = h$. We consider the nine-point circle (N) of triangle ABC. This clearly passes through O, the midpoint of AB, and P, the altitude foot of C on AB. Let AC intersect (O_1) again at A', and BC intersect (O_2) again at B'. Let O_e and H be the circumcenter and orthocenter of triangle ABC. Note that C and H are on opposite sides of the semicircular arc (O), and the triangles ABC and ABH have the same nine-point circle. We shall therefore assume C beyond the point Q on the half line PQ. See Figure 1. In this paper the labeling of knowing Archimedean circles follows [2].

2. Archimedean circles with centers on the nine-point circle

Let the perpendicular bisector of AB cut (N) at O and M_e, and the altitude CP cut (N) at P and M_h. See Figure 1.

2.1. It is easy to show that $POM_\mathrm{e}M_\mathrm{h}$ is a rectangle so M_e is the reflection of P in N. Because O_e is also the reflection of H in N, $HPO_\mathrm{e}M_\mathrm{e}$ is a parallelogram, and we have

$$O_\mathrm{e}M_\mathrm{e} = PH. \tag{1}$$

Furthermore, from the similarity of triangles HPB and APC, we have $\frac{PH}{PB} = \frac{PA}{PC}$. Hence,

$$PH = \frac{4r_1r_2}{h}. \tag{2}$$

Publication Date: June 18, 2007. Communicating Editor: Paul Yiu.

The author thanks Paul Yiu for his help in the preparation of this paper.

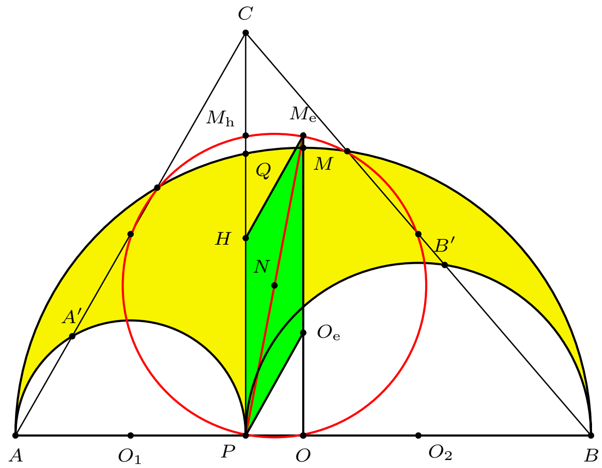

Figure 1.

2.2. Since C is beyond Q on the half line PQ, the intersection F of $A'O_1$ and $B'O_2$ is a point F below the arbelos. Denote by (I) the incircle of triangle FO_1O_2. See Figure 2. The line IO_1 bisects both angles O_2O_1F and $A'O_1A$. Because $O_1A' = O_1A$, IO_1 is perpendicular to AC, and therefore is parallel to BH. Similarly, IO_2 is parallel to AH. From these, two triangles AHB and O_2IO_1 are homothetic with ratio $\frac{AB}{O_2O_1} = 2$. It is easy to show that O is the touch point of (I) with AB and that the inradius is

$$IO = \frac{1}{2} \cdot PH. \tag{3}$$

In fact, if F' is the reflection of F in the midpoint of O_1O_2 then O_1FO_2F' is a parallelogram and the circle (PH) (with PH as diameter) is the incircle of $F'O_1O_2$. It is the reflection of (I) in midpoint of O_1O_2.

2.3. Now we apply these results to the arbelos. From (2), $\frac{1}{2} \cdot PH = \frac{2r_1r_2}{h} =$ Archimedean radius $\frac{r_1r_2}{r_1+r_2}$ if and only if

$$CP = h = 2(r_1 + r_2) = AB.$$

In this case, point C and the orthocenter H of ABC are easy constructed and the circle with diameter PH is the Bankoff triplet circle (W_3). From this we can also construct also the incircle of the arbelos. In this case $F' =$ incenter of the arbelos. From (3) we can show that when $CP = AB$, the incircle of FO_1O_2 is also Archimedean. See Figure 3.

Let M be the intersection of OO_e and the semicircle (O), *i.e.*, the highest point of (O). When $CP = h = 2(r_1 + r_2) = AB$,

$$OO_e = M_hH = M_hC = \frac{h - PH}{2} = (r_1 + r_2) - \frac{r_1r_2}{r_1 + r_2}.$$

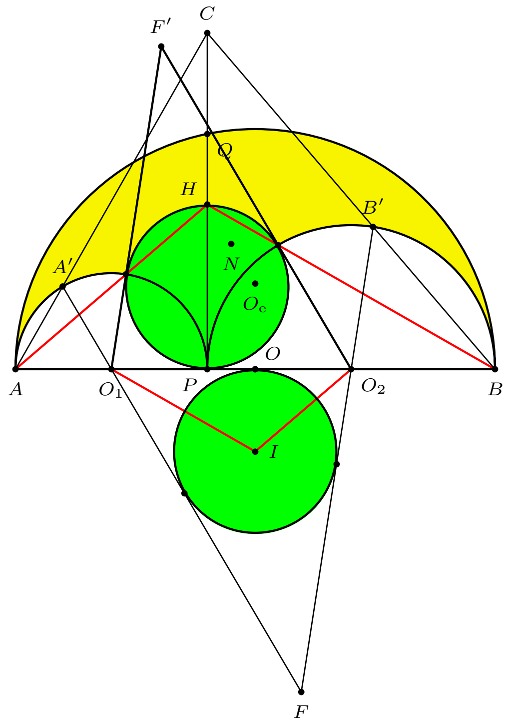

Figure 2.

Therefore,

$$O_eM = (r_1 + r_2) - \left(r_1 + r_2 - \frac{r_1r_2}{r_1 + r_2}\right) = \frac{r_1r_2}{r_1 + r_2}.$$

From (1), $O_eM_e = PH = \frac{2r_1r_2}{r_1+r_2}$. This means that M is the midpoint of O_eM_e, or the two circles centered at O_e and M_e and touching (O) at M are also Archimedean circles. See Figure 3.

We summarize the results as follows.

Proposition 1. *In the arbelos (O_1), (O_2), (O), if C is any point on the half line PQ beyond Q and H is orthocenter of ABC, then the circle (PH) is Archimedean if and only if $CP = AB = 2(r_1 + r_2)$. In this case, we have the following results.*
(1). *The orthocenter H of ABC is the intersection point of Bankoff triplet circle (W_3) with PQ (other than P).*
(2). *The incircle of triangle FO_1O_2 is an Archimedean circle touching AB at O; it is reflection of (W_3) in the midpoint of O_1O_2.*
(3). *The circle centered at circumcenter O_e of ABC and touching (O) at its highest point M is an Archimedean circle. This circle is (W_{20}).*
(4). *The circle centered on nine point circle of ABC and touching (O) at M is an Archimedean circle; it is the reflection of (W_{20}) in M.*
(5). *The reflection F' of F in midpoint of O_1O_2 is the incenter of the arbelos.*

Remarks. (a) The Archimedean circles in (2) and (4) above are new.

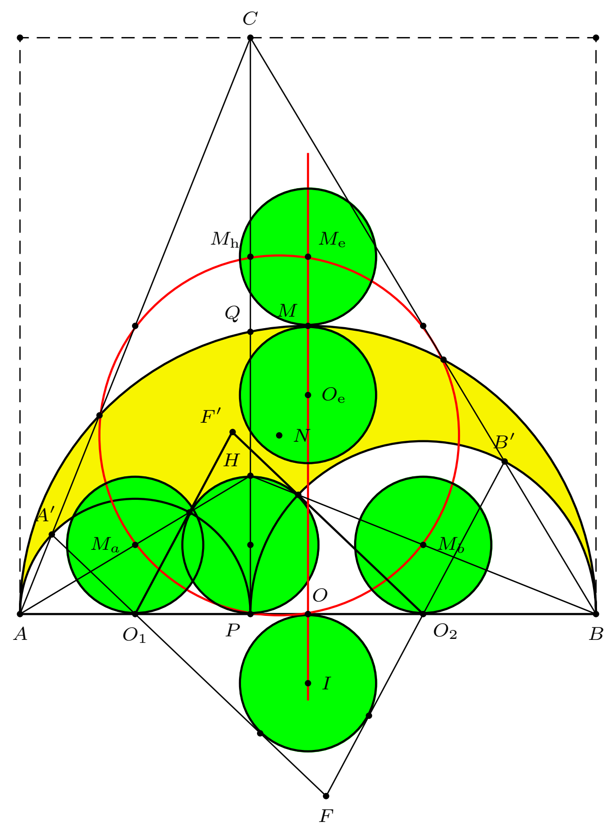

Figure 3.

(b) There are two more obvious Archimedean circles with centers on the nine-point circle. These are (M_a) and (M_b), where M_a and M_b are the midpoints of AH and BH respectively. See Figure 3.

(c) The midpoints M_a, M_b of HA, HB are on nine point circle of ABC and are two vertices of Eulerian triangle of ABC. Two circles centered at M_a, M_b and touch AB at O_1, O_2 respectively are congruent with (W_3) so they are also Archimedean circles (see [2]).

3. Two new pairs of Archimedean circles

If T is a point such that $OT^2 = r_1^2 + r_2^2$, then there is a pair of Archimedean circles mutually tangent at T, and each tangent internally to (O). Frank Power [5]. constructed two such pairs with $T = M_1$, M_2, the highest points of (O_1) and (O_2) respectively. Allowing tangency with other circles, Floor van Lamoen [4] called such a pair Powerian. We construct two new Powerian pairs.

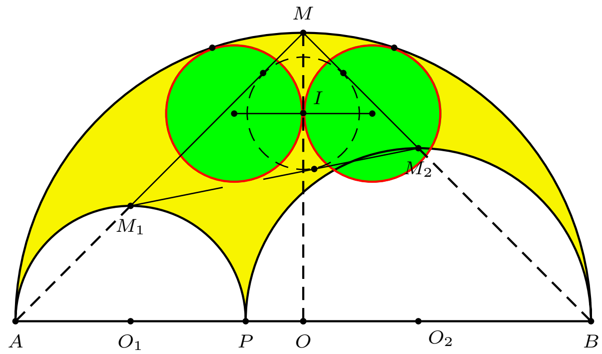

Figure 4.

3.1. The triangle MM_1M_2 has $MM_1 = \sqrt{2} \cdot r_2$, $MM_2 = \sqrt{2} \cdot r_1$, and a right angle at M. Its incenter is the point I on OM such that

$$MI = \sqrt{2} \cdot \frac{1}{2}(MM_1 + MM_2 - M_1M_2) = (r_1 + r_2) - \sqrt{r_1^2 + r_2^2}.$$

Therefore, $OI^2 = r_1^2 + r_2^2$, and we have a Powerian pair. See Figure 4.

3.2. Consider also the semicircles (T_1) and (T_2) with diameters AO_2 and BO_1. The intersection J of (T_1) and (T_2) satisfies

$$OJ^2 = OP^2 + PJ^2 = (r_1 - r_2)^2 + 2r_1r_2 = r_1^2 + r_2^2.$$

Therefore, we have another Powerian pair. See Figure 5.

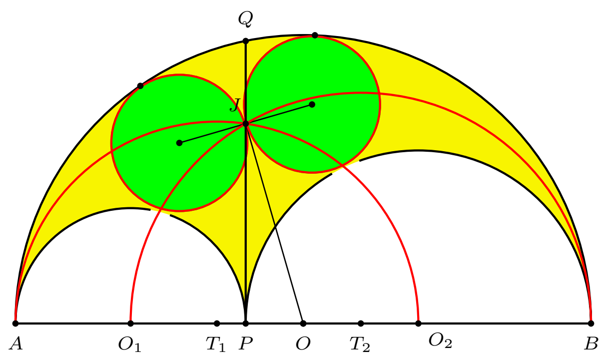

Figure 5.

4. Two Archimedean circles related to the tangents of the arbelos

We give two more Archimedean circles related to the tangents of the arbelos.

Let $\mathcal{L}$ be the tangent of (O) at Q, and Q_1, Q_2 the orthogonal projections of O_1, O_2 on $\mathcal{L}$. The lines O_1Q_1 and O_2Q_2 intersect the semicircles (O_1) and (O_2) at R_1 and R_2 respectively. Note that R_1R_2 is a common tangent of the semicircles (O_1) and (O_2). The circles (N_1), (N_2) with diameters Q_1R_1 and Q_2R_2 are Archimedean. Indeed, if (W_6) and (W_7) are the two Archimedean circles through

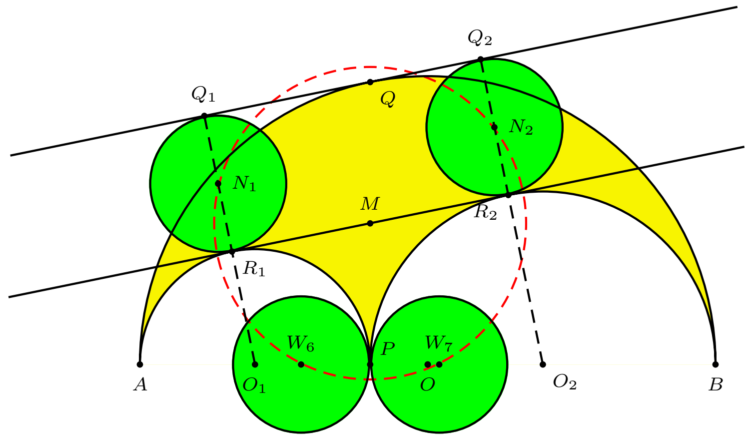

Figure 6.

P with centers on AB (see [2]), then N_1, N_2, W_6, W_7 lie on the same circle with center the midpoint M of PQ. See Figure 6. We leave the details to the reader.

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Forum Geometricorum
Volume 7 (2007) 121–123.

FORUM GEOM
ISSN 1534-1178

Characterizations of an Infinite Set of Archimedean Circles

Hiroshi Okumura and Masayuki Watanabe

Abstract. For an arbelos with the two inner circles touching at a point O, we give necessary and sufficient conditions that a circle passing through O is Archimedean.

Consider an arbelos with two inner circles α and β with radii a and b respectively touching externally at a point O. A circle of radius $r_A = ab/(a+b)$ is called Archimedean. In [3], we have constructed three infinite sets of Archimedean circles. One of these consists of circles passing through the point O. In this note we give some characterizations of Archimedean circles passing through O. We set up a rectangular coordinate system with origin O and the positive x-axis along a diameter OA of α (see Figure 1).

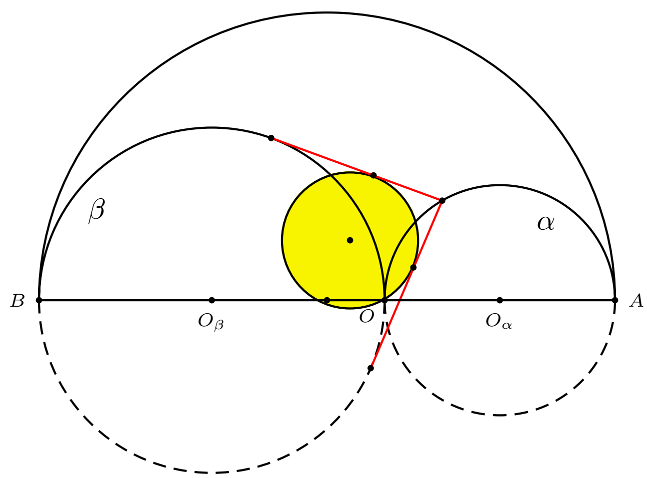

Figure 1

Theorem 1. *A circle through O (not tangent internally to β) is Archimedean if and only if its external common tangents with β intersect at a point on α.*

Proof. Consider a circle δ with radius $r \neq b$ and center $(r\cos\theta, r\sin\theta)$ for some real number θ with $\cos\theta \neq -1$. The intersection of the common external tangents

Publication Date: July 2, 2007. Communicating Editor: Paul Yiu.

of β and δ is the external center of similitude of the two circles, which divides the segment joining their centers externally in the ratio $b : r$. This is the point

$$\left(\frac{br(1+\cos\theta)}{b-r}, \frac{br\sin\theta}{b-r}\right). \tag{1}$$

The theorem follows from

$$\left(\frac{br(1+\cos\theta)}{b-r} - a\right)^2 + \left(\frac{br\sin\theta}{b-r}\right)^2 - a^2 = \frac{2br(a+b)(1+\cos\theta)}{(b-r)^2}(r - r_A).$$

$\square$

Let O_α and O_β be the centers of the circles α and β respectively.

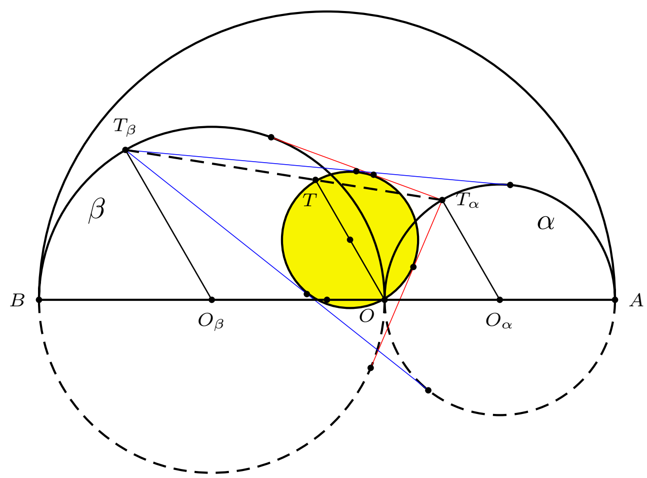

Figure 2

Corollary 2. *Let δ be an Archimedean circle with a diameter OT, and T_α the intersection of the external common tangents of the circles δ and β; similarly define T_β.*
(i) *The vectors $\overrightarrow{OT}$ and $\overrightarrow{O_\alpha T_\alpha}$ are parallel with the same direction.*
(ii) *The point T divides the segment $T_\alpha T_\beta$ internally in the ratio $a : b$.*

Proof. We describe the center of δ by $(r_A\cos\theta, r_A\sin\theta)$ for some real number θ (see Figure 2). Then the point T_α is described by

$$\left(\frac{br_A(1+\cos\theta)}{b-r_A}, \frac{br_A\sin\theta}{b-r_A}\right) = (a(1+\cos\theta), a\sin\theta)$$

by (1). This implies $\overrightarrow{O_\alpha T_\alpha} = a(\cos\theta, \sin\theta)$. (ii) is obtained directly, since T_β is expressed by $(b(-1+\cos\theta), b\sin\theta)$. $\square$

In Theorem 1, we exclude the Archimedean circle which touches β internally at the point O. But this corollary holds even if the circle δ touches β internally. If δ is the Bankoff circle touching the line OA at the origin O [1], then T_α is the highest point on α. If δ is the Archimedean circle touching β externally at the point O, then T_α obviously coincides with the point A. This fact is referred in [2] using the circle labeled W_6. Another notable Archimedean circle passing through O is that having center on the Schoch line $x = \frac{b-a}{b+a}r_A$, which is labeled as U_0 in [2]. We have showed that the intersection of the external common tangents of β and this circle is the intersection of the line $x = 2r_A$ and the circle α [3].

By the uniqueness of the figure, we get the following characterizations of the Archimedean circles passing through the point O.

Corollary 3. *Let δ be a circle with a diameter OT, and let T_α and T_β be points on α and β respectively such that $\overrightarrow{O_\alpha T_\alpha}$ and $\overrightarrow{O_\beta T_\beta}$ are parallel to $\overrightarrow{OT}$ with the same direction.* (i) *The circle δ is Archimedean if and only if the points T divides the line segment $T_\alpha T_\beta$ internally in the ratio $a : b$.* (ii) *If the center of δ does not lie on the line OA, then δ is Archimedean if and only if the three points T_α, T_β and T are collinear.*

The statement (i) in this corollary also holds when δ touches β internally.

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Forum Geometricorum
Volume 7 (2007) 125–128.

FORUM GEOM
ISSN 1534-1178

Remarks on Woo's Archimedean Circles

Hiroshi Okumura and Masayuki Watanabe

Abstract. The property of Woo's Archimedean circles does not hold only for Archimedean circles but circles with any radii. The exceptional case of this has a close connection to Archimedean circles.

1. Introduction

Let A and B be points with coordinates $(2a, 0)$ and $(-2b, 0)$ on the x-axis with the origin O and positive real numbers a and b. Let α, β and γ be semicircles forming an arbelos with diameters OA, OB and AB respectively. We follow the notations in [4]. For a real number n, let $\alpha(n)$ and $\beta(n)$ be the semicircles in the upper half-plane with centers $(n, 0)$ and $(-n, 0)$ respectively and passing through the origin O. A circle with radius $r = \frac{ab}{a+b}$ is called an Archimedean circle. Thomas Schoch has found that the circle touching the circles $\alpha(2a)$ and $\beta(2b)$ externally and γ internally is Archimedean [2] (see Figure 1). Peter Woo called the Schoch line the one passing through the center of this circle and perpendicular to the x-axis, and found that the circle U_n touching the circles $\alpha(na)$ and $\beta(nb)$ externally with center on the Schoch line is Archimedean for a nonnegative real number n. In this note we consider the property of Woo's Archimedean circles in a general way.

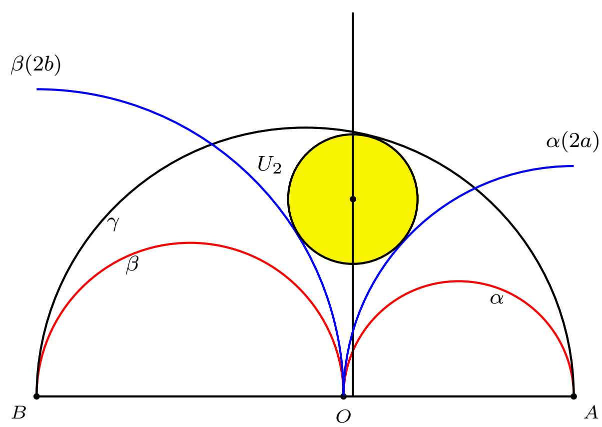

Figure 1.

Publication Date: September 4, 2007. Communicating Editor: Paul Yiu.

2. A generalization of Woo's Archimedean circles

We show that the property of Woo's Archimedean circles does not only hold for Archimedean circles. Indeed circles with any radii can be obtained in a similar way. We say that a circle touches $\alpha(na)$ *appropriately* if they touch externally (respectively internally) for a positive (respectively negative) number n. If one of the two circles is a point circle and lies on the other, we also say that the circle touches $\alpha(na)$ appropriately. The same notion of appropriate tangency applies to $\beta(nb)$.

Theorem 1. *Let s and t be nonzero real numbers such that $tb \pm sa \neq 0$. If there is a circle of radius ρ touching the circles $\alpha(nsa)$ and $\beta(ntb)$ appropriately for a real number n, then its center lies on the line*

$$x = \frac{tb - sa}{tb + sa}\rho. \tag{1}$$

Proof. Consider the center (x, y) of the circle with radius ρ touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately. The distance between (x, y) and the centers of $\alpha(nsa)$ and $\beta(ntb)$ are $|\rho + nsa|$ and $|\rho + ntb|$ respectively. Therefore by the Pythagorean theorem,

$$y^2 = (\rho + nsa)^2 - (x - nsa)^2 = (\rho + ntb)^2 - (x + ntb)^2.$$

Solving the equations, we get (1) above. □

For a real number k different from 0 and $\pm\rho$, we can choose the real numbers s and t so that (1) expresses the line $x = k$. Let us assume $st > 0$. Then the circles $\alpha(nsa)$ and $\beta(ntb)$ lie on opposite sides of the y-axis. If $sz > 0$ and $tz > 0$, there is always a circle of radius ρ touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately. If $ns < 0$ and $nt < 0$, such a circle exists when $-2n(sa + tb) \leq 2\rho$. Hence in the case $st > 0$, the tangent circle exists if $n(sa + tb) + \rho \geq 0$. Now let us assume $st < 0$. Then circles $\alpha(nsa)$ and $\beta(ntb)$ lie on the same side of the y-axis. The circle of radius ρ touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately exists if $-2n(sa + tb) \geq 2\rho$. Hence in the case $st < 0$, the tangent circle exists if $n(sa+tb)+\rho \leq 0$. In any case the center of the circle with radius ρ touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately is

$$\left(\frac{tb - sa}{tb + sa}\rho,\ \pm\frac{2\sqrt{nabst((sa + tb) + \rho)\rho}}{|sa + tb|}\right).$$

Therefore, for every point P not on the lines $x = 0, \pm\rho$, we can choose real numbers s, t and n so that the circle, center P, radius ρ, is touching $\alpha(nsa)$ and $\beta(tzb)$ appropriately.

The Schoch line is the line $x = \frac{b-a}{b+a}r$ (see [4]). Therefore Woo's Archimedean circles and the Schoch line are obtained when $s = t$ and $\rho = r$ in Theorem 1. If $st > 0$, then $-1 < \frac{tb-sa}{tb+sa} < 1$. Hence the line (1) lies in the region $-\rho < x < \rho$ in this case.

The external center of similitude of β and a circle with radius ρ and center on the line (1) lies on the line

$$x = \frac{2tb^2\rho}{(b-\rho)(sa+tb)}$$

by similarity. In particular, the external centers of similitude of Woo's Archimedean circles and β lie on the line $x = 2r$. See [4].

3. Circles with centers on the y-axis

We have excluded the cases $tb \pm sa \neq 0$ in Theorem 1. The case $tb + sa = 0$ is indeed trivial since the circles $\alpha(nsa)$ and $\beta(ntb)$ coincide. By Theorem 1, for $k \neq 0$, the circle touching $\alpha(nsa)$ and $\beta(ntb)$ appropriately and with center on the line $x = k$ has radius $\frac{tb+sa}{tb-sa}k$. On the other hand, if $tb = sa$, the circles $\alpha(nsa)$ and $\beta(ntb)$ are congruent and lie on opposite sides of the y-axis, and the line (1) coincides with the y-axis. Therefore the radii of circles touching the two circles appropriately and having the center on this line cannot be determined uniquely.

We show that this exceptional case $(tb = sa)$ has a close connection with Archimedean circles. Since $\alpha(nsa)$ and $\beta(ntb)$ are congruent, we now define $\alpha[n] = \alpha(n(a+b))$ and $\beta[n] = \beta(n(a+b))$. The circles $\alpha[n]$ and $\beta[n]$ are congruent, and their radii are n times of the radius of γ. For two circles of radii ρ_1, ρ_2 and with distance d between their centers, consider their inclination [3] given by

$$\frac{\rho_1^2 + \rho_2^2 - d^2}{2\rho_1\rho_2}.$$

This is the cosine of the angle between the circles if they intersect, and is 0, $+1$, -1 according as they are orthogonal or tangent internally or externally.

Theorem 2. *If a circle $\mathcal{C}$ of radius ρ touches $\alpha[n]$ and $\beta[n]$ appropriately for a real number n, then the inclination of $\mathcal{C}$ and γ is $\frac{2r}{\rho} - n$.*

Proof. The square of the distance between the centers of the circles $\mathcal{C}$ and γ is $(\rho + n(a+b))^2 - (n(a+b))^2 + (a-b)^2$ by the Pythagorean theorem. Therefore their inclination is

$$\frac{\rho^2 + (a+b)^2 - (\rho + n(a+b))^2 + (n(a+b))^2 - (a-b)^2}{2\rho(a+b)} = \frac{2r}{\rho} - n.$$

□

Let k be a positive real number. The radius of a circle touching $\alpha[n]$ and $\beta[n]$ appropriately is kr if and only if the inclination of the circle and γ is $\frac{2}{k} - n$ for a real number n.

Corollary 3. *A circle touching $\alpha[n]$ and $\beta[n]$ appropriately for a real number n is Archimedean if and only if the inclination of this circle and γ is $2 - n$.*

This gives an infinite set of Archimedean circles δ_n with centers on the positive y-axis. The circle δ_n exists if $n \geq \frac{-r}{2(a+b)}$, and the maximal value of the inclination of γ and δ_n is $2 + \frac{r}{2(a+b)}$. The circle δ_1 touches γ internally, δ_2 is orthogonal to

γ, and δ_3 touches γ externally by the corollary (see Figure 2). The circle δ_0 is the Bankoff circle [1], whose inclination with γ is 2.

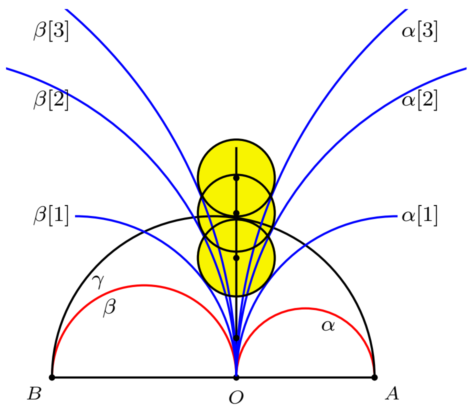

Figure 2

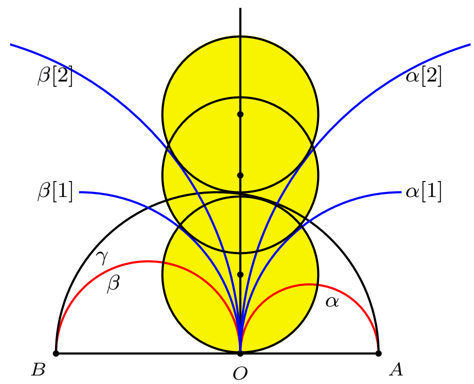

Figure 3

By the remark preceding Corollary 3 we can get circles with various radii and centers on the y-axis tangent or orthogonal to γ. Figure 3 shows some such examples. The three circles all have radii $2r$. One touches the degenerate circles $\alpha[0]$ and $\beta[0]$ (and the line AB) at O, and γ internally. A second circle touches $\alpha[1]$ and $\beta[1]$ externally and are orthogonal to γ. Finally, a third circle touches $\alpha[2]$, $\beta[2]$, and γ externally.

From [4], the center of the Woo circle U_n is the point

$$\left(\frac{b-a}{b+a}r,\ 2r\sqrt{n+\frac{r}{a+b}}\right).$$

The inclination of U_n and γ is $1+\frac{2(2-n)r}{a+b}$. This depends on the radii of α and β except the case $n=2$. In contrast to this, Corollary 3 shows that the inclination of δ_n and γ does not depend on the radii of α and β.

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Forum Geometricorum
Volume 7 (2007) 129–135.

FORUM GEOM
ISSN 1534-1178

Heronian Triangles Whose Areas Are Integer Multiples of Their Perimeters

Lubomir Markov

Abstract. We present an improved algorithm for finding all solutions to Goehl's problem $A = mP$ for triangles, *i.e.*, the problem of finding all Heronian triangles whose area (A) is an integer multiple (m) of the perimeter (P). The new algorithm does not involve elimination of extraneous rational triangles, and is a true extension of Goehl's original method.

1. Introduction and main result

In a recent paper [3], we presented a solution to the problem of finding all Heronian triangles (triangles with integer sides and area) for which the area A is a multiple m of the perimeter P, where $m \in \mathbb{N}$. The problem was introduced by Goehl [2] and is of interest because although its solution is exceedingly simple in the special case of right triangles, the general case remained unsolved for about 20 years despite considerable effort. It is also remarkable and somewhat contrary to intuition that for each m there are only finitely many triangles with the property $A = mP$; for instance, the triangles $(6, 8, 10)$, $(5, 12, 13)$, $(6, 25, 29)$, $(7, 15, 20)$ and $(9, 10, 17)$ are the only ones whose area equals their perimeter (the case $m = 1$). Reproducing Goehl's solution to the problem in the special case of right triangles is a simple matter: Suppose that a and b are the legs of a right triangle and $c = \sqrt{a^2 + b^2}$ is the hypotenuse. Setting the area equal to a multiple m of the perimeter and manipulating, one immediately obtains the identities $8m^2 = (a - 4m)(b - 4m)$ and $c = a + b - 4m$. These allow us to determine a, b and c after finding all possible factorizations of the left-hand side of the form $8m^2 = d_1 \cdot d_2$ and matching d_1 and d_2 with $(a - 4m)$ and $(b - 4m)$, respectively; restricting d_1 to those integers that do not exceed $\sqrt{8m^2} = 2\sqrt{2}m$ assures $a < b$ and avoids repetitions. We state Goehl's result in the following form:

Publication Date: September 10, 2007. Communicating Editor: Paul Yiu.

The author expresses his sincerest thanks to Dr. John Goehl, Jr., for the meticulous care with which he reviewed this paper, and for the inspirational enthusiasm for integer number theory he has conveyed to him. The author also wishes to thank Barry University for granting him a sabbatical leave during the fall semester of 2006.

Theorem 1. *For a given m, the right-triangle solutions (a,b,c) to the problem $A = mP$ are determined from the relations*

$$8m^2 = (a-4m)(b-4m), \tag{1}$$
$$c = a+b-4m. \tag{2}$$

Each factorization

$$8m^2 = d_1 \cdot d_2, \tag{3}$$

where

$$d_1 \leq \lfloor 2m\sqrt{2} \rfloor, \tag{4}$$

generates a solution triangle with sides given by the formulas

$$\begin{cases} a = d_1 + 4m, \\ b = d_2 + 4m, \\ c = d_1 + d_2 + 4m. \end{cases} \tag{5}$$

Our paper [3] extended Goehl's result to general triangles, but the solution involved extraneous rational triangles, which then had to be eliminated. The aim of this work is to present a radical simplification of our previous solution, which does not introduce extraneous triangles and is a direct generalization of Goehl's method. Our main goal is to prove the following theorem:

Theorem 2. *For a given m, all solutions (a,b,c) to the problem $A = mP$ are determined as follows: Find all divisors u of $2m$; for each u, find all numbers v relatively prime to u and such that $1 \leq v \leq \lfloor \sqrt{3}u \rfloor$; to each pair u and v, there correspond a factorization identity*

$$4m^2(u^2+v^2) = \left[v\Big(a - \frac{2m}{u}v\Big) - 2mu\right]\left[v\Big(b - \frac{2m}{u}v\Big) - 2mu\right], \tag{6}$$

and a relation

$$c = a + b - \frac{4mv}{u}. \tag{7}$$

Each factorization

$$4m^2(u^2+v^2) = \delta_1 \cdot \ \delta_2, \tag{8}$$

where

$$\delta_1 \leq \left\lfloor 2m\sqrt{u^2+v^2} \right\rfloor \tag{9}$$

and only those factors δ_1, δ_2 for which $v \mid \delta_1 + 2mu$ and $v \mid \delta_2 + 2mu$ are considered, generates a solution triangle with sides given by the formulas

$$\begin{cases} a = \frac{\delta_1 + 2mu}{v} + \frac{2mv}{u}, \\ b = \frac{\delta_2 + 2mu}{v} + \frac{2mv}{u}, \\ c = \frac{\delta_1 + \delta_2 + 4mu}{v}. \end{cases} \tag{10}$$

Furthermore, for each fixed u, one concludes from the corresponding v's that
(1) *the obtuse-triangle solutions are obtained exactly when $v < u$;*
(2) *the acute-triangle solutions are obtained exactly when $u < v \leq \lfloor \sqrt{3}u \rfloor$, with*

the further restriction $\dfrac{2m}{u}(v^2-u^2) \le \delta_1 \le \left\lfloor 2m\sqrt{u^2+v^2} \right\rfloor$;
(3) *the right-triangle solutions are obtained exactly when* $u = v = 1$.

Note that Theorem 1 is a special case of Theorem 2 and that the substitution $u = v = 1$ transforms relations (6) through (10) into relations (1) through (5), respectively.

2. Summary of preliminary facts

Let A be the area and P the perimeter of a triangle with sides a, b, c, with the agreement that c shall always denote the largest side. Our problem (we call it $A = mP$ for short) is to find all Heronian triangles whose area equals an integer multiple m of the perimeter. We state all preliminaries as a sequence of lemmas whose proofs can either be easily reproduced by the reader, or can be found (except for Lemma 5) in [3].

First we note that Heron's formula

$$4A = \sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$$

and simple trigonometry easily imply the following lemma:

Lemma 3. *Assume that the triple* (a, b, c) *solves the problem* $A = mP$.
(1) $a + b - c$ *is an even integer.*
(2) $a + b - c < 4m\sqrt{3}$.
(3) *The resulting triangle is* $\begin{Bmatrix} obtuse \\ acute \\ right \end{Bmatrix}$ *if and only if* $a + b - c \begin{Bmatrix} < 4m \\ > 4m \\ = 4m \end{Bmatrix}$.

Next, we need a crucial rearrangement of Heron's formula:

Lemma 4. *The following doubly-Pythagorean form of Heron's formula holds:*

$$\left[c^2 - (a^2+b^2)\right]^2 + (4A)^2 = (2ab)^2. \tag{11}$$

This representation allows the problem $A = mP$ to be reduced to a problem about Pythagorean triples; for our purposes, a Pythagorean triple (x, y, z) shall consist of nonnegative integers such that z (the "hypotenuse") shall always represent the largest number, whereas x and y (the "legs") need not appear in any particular order. The following parametric representation of *primitive* Pythagorean triples (*i.e.*, such that the components do not have a common factor greater than 1) is the only preliminary statement not proved in [3]; a self-contained proof can be found in [1]:

Lemma 5. *Depending on whether the first leg* x *is odd or even, every primitive Pythagorean triple* (x, y, z) *is uniquely expressed as* $(u^2 - v^2, 2uv, u^2 + v^2)$ *where* u *and* v *are relatively prime of opposite parity, or* $\left(\dfrac{u^2-v^2}{2}, uv, \dfrac{u^2+v^2}{2}\right)$ *where* u *and* v *are relatively prime and odd.*

A combination of Lemmas 4 and 5 easily yields

Lemma 6. *For a fixed m, solving the problem $A = mP$ is equivalent to determining all integer a, b, c that satisfy the equation*

$$\left[c^2 - (a^2 + b^2)\right]^2 + \left[4m(a + b + c)\right]^2 = (2ab)^2, \tag{12}$$

or equivalently, to solving in positive integers the following system of three equations in six unknowns:

$$\begin{cases} \pm\left[c^2 - (a^2 + b^2)\right] &= k(u^2 - v^2); \\ 4m(a + b + c) &= 2kuv; \\ 2ab &= k(u^2 + v^2). \end{cases} \tag{13}$$

It is easy to see that the first equation in (13) can be interpreted as follows.

Lemma 7. *Assume that, corresponding to certain values of u and v, there is a triple (a, b, c) which solves the problem $A = mP$. Then the triangle (a, b, c) is* $\begin{Bmatrix} obtuse \\ acute \\ right \end{Bmatrix}$ *if and only if* $\begin{Bmatrix} u > v \\ u < v \\ u = v = 1 \end{Bmatrix}$.

3. Proof of Theorem 2

Let us first investigate the case of an obtuse triangle (the case $u > v$); thus, the system (13) is $c^2 - (a^2 + b^2) = k(u^2 - v^2)$, $4m(a + b + c) = 2kuv$, $2ab = k(u^2 + v^2)$. For completeness, we reproduce the crucial proof of the main factorization identity from [3] (equation (17) below), which in essence solves the problem $A = mP$. Indeed, from the first and the third equations in (13) we get $(a + b)^2 - c^2 = 2kv^2$, and after factoring the left-hand side and using the second equation we get $a + b - c = \dfrac{4mv}{u}$. This implies that u must divide $2m$ because $a + b - c$ is even, and u, v are relatively prime. Combining the last relation with $a + b + c = \dfrac{kuv}{2m}$ and solving the resulting system yields

$$b + a = \frac{ku^2v + 8m^2v}{4mu}, \quad c = \frac{ku^2v - 8m^2v}{4mu}.$$

Similarly, adding the first and second equations and rearranging terms gives $(a - b)^2 = c^2 - 2ku^2$. Let us assume for a moment that $b \geq a$; then we have $b - a = \sqrt{c^2 - 2ku^2}$, and it is clear that the radicand must be a square. Put $Q = \dfrac{2m}{u}$ and substitute it in the expressions for c, $b + a$ and $b - a$. After simplification, one gets

$$c = \frac{kv - 2Q^2v}{2Q}, \; b + a = \frac{kv + 2Q^2v}{2Q}, \; b - a = \frac{1}{2Q}\sqrt{\left(kv - 2Q^2v\right)^2 - 32km^2}, \tag{14}$$

where the radicand must be a square. Put $\left(kv - 2Q^2v\right)^2 - 32km^2 = X^2$, and get

$$c = \frac{kv - 2Q^2v}{2Q}, \qquad b + a = \frac{kv + 2Q^2v}{2Q}, \qquad b - a = \frac{1}{2Q}X. \tag{15}$$

On the other hand, consider $\left(kv - 2Q^2v\right)^2 - 32km^2 = X^2$ as an equation in the variables X and k. Expanding the square and rearranging yields

$$k^2v^2 - 4k(v^2Q^2 + 8m^2) + 4Q^4v^2 = X^2.$$

The last equation is a Diophantine equation solvable by factoring: subtract the quantity $\left(kv - \dfrac{2(v^2Q^2 + 8m^2)}{v}\right)^2$ from both sides, simplify and rearrange terms; the result is

$$\left[2(v^2Q^2 + 8m^2)\right]^2 - \left(2v^2Q^2\right)^2 = \left(v^2k - 2v^2Q^2 - 16m^2\right)^2 - (Xv)^2. \quad (16)$$

In (16), factor both sides, substitute $Q = \dfrac{2m}{u}$ and simplify. This gives

$$\left(\frac{16m^2}{u}\right)^2(u^2+v^2) = \left[v^2(k - 2Q^2) - 16m^2 - Xv\right]\left[v^2(k - 2Q^2) - 16m^2 + Xv\right] \quad (17)$$

which is the main factorization identity mentioned above.

Now, the new idea is to eliminate k and X in (17), using (15) and the crucial fact that $a + b - c = \dfrac{4mv}{u}$. Indeed, from (15) we immediately obtain

$$X = 2Q(b - a), \quad k = \frac{2Qc + 2Q^2v}{v}, \quad (18)$$

which we substitute in (17) and simplify to get

$$16m^2(u^2 + v^2) = \left[v(c - b + a) - 4mu\right]\left[v(c + b - a) - 4mu\right]. \quad (19)$$

In the last relation, substitute $c = a + b - \dfrac{4mv}{u}$ and simplify again. The result is

$$4m^2(u^2 + v^2) = \left[v\left(a - \frac{2m}{u}v\right) - 2mu\right]\left[v\left(b - \frac{2m}{u}v\right) - 2mu\right],$$

which is exactly (6). This identity allows us to find sides a and b by directly matching factors of the left-hand side to respective quantities on the right; then c will be determined from $c = a + b - \dfrac{4mv}{u}$. Suppose $4m^2(u^2 + v^2) = \delta_1 \cdot \delta_2$. Since we want $\delta_1 = v\left(a - \dfrac{2m}{u}v\right) - 2mu$, it is clear that for a to be an integer, we necessarily must have $v \mid \delta_1 + 2mu$. Similarly, the requirement $v \mid \delta_2 + 2mu$ will ensure that b is an integer. Imposing these additional restrictions will produce *only* the integer solutions to the problem. Furthermore, choosing $\delta_1 \leq \delta_2$ (or equivalently, $\delta_1 \leq 2m\sqrt{u^2 + v^2}$) will guarantee that $a \leq b$.

Next, solve

$$\delta_1 = v\left(a - \frac{2m}{u}v\right) - 2mu, \quad \delta_2 = v\left(a - \frac{2m}{u}v\right) - 2mu$$

for a and b, express c in terms of them and thus obtain formulas for the sides:

$$\begin{cases} a = \dfrac{\delta_1 + 2mu}{v} + \dfrac{2mv}{u}, \\ b = \dfrac{\delta_2 + 2mu}{v} + \dfrac{2mv}{u}, \\ c = \dfrac{\delta_1 + \delta_2 + 4mu}{v}; \end{cases}$$

these are exactly the formulas (10). To ensure $c \geq b$, we solve the inequality

$$\frac{\delta_1 + \delta_2 + 4mu}{v} \geq \frac{\delta_2 + 2mu}{v} + \frac{2mv}{u}$$

and obtain, after simplification,

$$\delta_1 \geq \frac{2m}{u}(v^2 - u^2). \tag{20}$$

The last relation will always be true if $u > v$, and thus the proof of the obtuse-case part of the theorem is concluded. Now, consider the acute case; i.e., the case $v > u$. The first equation in (13) is again $c^2 - (a^2 + b^2) = k(u^2 - v^2)$ (both sides are negative), and all the above derivations continue to hold true; it is now crucial to use the important bound $a + b - c < 4m\sqrt{3}$ which, combined with $a + b - c = \dfrac{4mv}{u}$, implies that $u < v < \sqrt{3}u$. The only difference from the obtuse case is that the bound (20) does not hold automatically; now it must be imposed to avoid repetitions and guarantee that $b \leq c$. Since the right-triangle case is obviously incorporated in the theorem, the proof is complete.

4. An example

We again examine the case $m = 2$ (cf. [3]). Let $m = 2$ in the algorithm suggested by Theorem 2; then $2m = 4$ and thus u could be 4, 2 or 1. For each u, determine the corresponding v's:

(A) $u = 4 \Rightarrow v = 1, 3;\ 5$
(B) $u = 2 \Rightarrow v = 1;\ 3$
(C) $u = 1 \Rightarrow v = 1$.

Now observe how the case $u = 4,\ v = 5$ has to be discarded since we have $4m^2(u^2 + v^2) = 656 = 2^4 \cdot 41,\ 9 \leq \delta_1 \leq 25$, the only factor in that range is 16, and it must be thrown out because $v = 5$ does not divide $\delta_1 + 2mu = 32$. The working factorizations are shown in the table below.

u	v	type of triangle	δ_1 range	$4m^2(u^2+v^2)$	$\delta_1 \cdot \delta_2$	(a,b,c)
4	1	obtuse	$\delta_1 \leq 16$	272	$1 \cdot 272$	$(18, 289, 305)$
					$2 \cdot 136$	$(19, 153, 170)$
					$4 \cdot 68$	$(21, 85, 104)$
					$8 \cdot 34$	$(25, 51, 74)$
					$16 \cdot 17$	$(33, 34, 65)$
4	3	obtuse	$\delta_1 \leq 20$	400	$2 \cdot 200$	$(9, 75, 78)$
					$5 \cdot 80$	$(10, 35, 39)$
					$8 \cdot 50$	$(11, 25, 30)$
					$20 \cdot 20$	$(15, 15, 24)$
2	1	obtuse	$\delta_1 \leq 8$	80	$1 \cdot 80$	$(11, 90, 97)$
					$2 \cdot 40$	$(12, 50, 58)$
					$4 \cdot 20$	$(14, 30, 40)$
					$5 \cdot 16$	$(15, 26, 37)$
					$8 \cdot 10$	$(18, 20, 34)$
2	3	acute	$10 \leq \delta_1 \leq 14$	208	$13 \cdot 16$	$(13, 14, 15)$
1	1	right	$\delta_1 \leq 5$	32	$1 \cdot 32$	$(9, 40, 41)$
					$2 \cdot 16$	$(10, 24, 26)$
					$4 \cdot 8$	$(12, 16, 20)$

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Forum Geometricorum
Volume 7 (2007) 137–146.

FORUM GEOM
ISSN 1534-1178

Coincidence of Centers for Scalene Triangles

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Abstract. A *center function* is a function $\mathcal{Z}$ that assigns to every triangle T in a Euclidean plane $\mathbf{E}$ a point $\mathcal{Z}(T)$ in $\mathbf{E}$ in a manner that is symmetric and that respects isometries and dilations. A family $\mathbf{F}$ of center functions is said to be *complete* if for every scalene triangle ABC and every point P in its plane, there is $\mathcal{Z} \in \mathbf{F}$ such that $\mathcal{Z}(ABC) = P$. It is said to be *separating* if no two center functions in $\mathbf{F}$ coincide for any scalene triangle. In this note, we give simple examples of complete separating families of continuous triangle center functions. Regarding the impression that no two different center functions can coincide on a scalene triangle, we show that for every center function $\mathcal{Z}$ and every scalene triangle T, there is another center function $\mathcal{Z}'$, of a simple type, such that $\mathcal{Z}(T) = \mathcal{Z}'(T)$.

1. Introduction

Exercise 1 of [33, p. 37] states that if any two of the four classical centers coincide for a triangle, then it is equilateral. This can be seen by proving each of the 6 substatements involved, as is done for example in [26, pp. 78–79], and it also follows from more interesting considerations as described in Remark 5 below. The statement is still true if one adds the Gergonne, the Nagel, and the Fermat-Torricelli centers to the list. Here again, one proves each of the relevant 21 substatements; see [15], where variants of these 21 substatements are proved. If one wishes to extend the above statement to include the hundreds of centers catalogued in Kimberling's encyclopaedic work [25], then one must be prepared to test the tens of thousands of relevant substatements. This raises the question whether it is possible to design a definition of the term *triangle center* that encompasses the well-known centers and that allows one to prove in one stroke that no two centers coincide for a scalene triangle. We do not attempt to answer this expectedly very difficult question. Instead, we adhere to the standard definition of what a center is, and we look at maximal families of centers within which no two centers coincide for a scalene triangle.

In Section 2, we review the standard definition of triangle centers and introduce the necessary terminology pertaining to them. Sections 3 and 4 are independent.

Publication Date: September 17, 2007. Communicating Editor: Paul Yiu.
This work is supported by a research grant from Yarmouk University.
We would like to thank the referee who, besides being responsible for Remark 5, has made many valuable suggestions that helped improve the paper.

In Section 3, we examine the family of polynomial centers of degree 1. Noting the similarity between the line that these centers form and the Euler line, we digress to discuss issues related to these two lines. In Section 4, we exhibit maximal families of continuous, in fact polynomial, centers within which no two centers coincide for a scalene triangle. We also show that for every scalene triangle T and for every center function $\mathcal{Z}$, there is another center function of a fairly simple type that coincides with $\mathcal{Z}$ on T.

2. Terminology

By a *non-degenerate* triangle ABC, we mean an ordered triple (A, B, C) of non-collinear points in a fixed Euclidean plane $\mathbf{E}$. Non-degenerate triangles form a subset of $\mathbf{E}^3$ that we denote by $\mathbf{T}$. For a subset $\mathbf{U}$ of $\mathbf{T}$, the set of triples $(a, b, c) \in \mathbf{R}^3$ that occur as the side-lengths of a triangle in $\mathbf{U}$ is denoted by $\mathbf{U}_0$. Thus

$$\begin{aligned}\mathbf{U}_0 &= \{(a,b,c) \in \mathbf{R}^3 : a, b, c \text{ are the side-lengths of some triangle } ABC \text{ in } \mathbf{U}\},\\ \mathbf{T}_0 &= \{(a,b,c) \in \mathbf{R}^3 : 0 < a < b+c,\ 0 < b < c+a,\ 0 < c < a+b\}.\end{aligned}$$

In the spirit of [23] – [25], a *symmetric triangle center function* (or simply, a *center function*, or a *center*) is defined as a function that assigns to every triangle in $\mathbf{T}$ (or more generally in some subset $\mathbf{U}$ of $\mathbf{T}$) a point in its plane in a manner that is symmetric and that respects isometries and dilations. Writing $\mathcal{Z}(A, B, C)$ as a barycentric combination of the position vectors A, B, and C, and letting a, b, and c denote the side-lengths of ABC in the standard order, we see that a center function $\mathcal{Z}$ on $\mathbf{U}$ is of the form

$$\mathcal{Z}(A,B,C) = f(a,b,c)A + f(c,a,b)B + f(b,c,a)C, \tag{1}$$

where f is a real-valued function on $\mathbf{U}_0$ having the following properties:

$$f(a,b,c) = f(a,c,b), \tag{2}$$

$$f(a,b,c) + f(b,c,a) + f(c,a,b) = 1, \tag{3}$$

$$f(\lambda a, \lambda b, \lambda c) = f(a,b,c)\ \forall\ \lambda > 0. \tag{4}$$

Here, we have treated the points in our plane $\mathbf{E}$ as position vectors relative to a fixed but arbitrary origin. We will refer to the center $\mathcal{Z}$ defined by (1) as *the center function defined by* f without referring explicitly to (1). The function f may be an explicit function of other elements of the triangle (such as its angles) that are themselves functions of a, b and c.

Also, we will always assume that the domain $\mathbf{U}$ of $\mathcal{Z}$ is closed under permutations, isometries and dilations, and has non-empty interior. In other words, we assume that $\mathbf{U}_0$ is closed under permutations and multiplication by a positive number, and that it has a non-empty interior.

According to this definition of a center $\mathcal{Z}$, one need only define $\mathcal{Z}$ on the similarity classes of triangles. On the other hand, the values that $\mathcal{Z}$ assigns to two triangles in different similarity classes are completely independent of each other. To reflect more faithfully our intuitive picture of centers, one must impose the condition that a center function be continuous. Thus a center function $\mathcal{Z}$ on $\mathbf{U}$ is called *continuous* if it is defined by a function f that is continuous on $\mathbf{U}_0$. If f can be chosen to be a rational function, then $\mathcal{Z}$ is called a *polynomial center function*. Since two rational functions cannot coincide on a non-empty open set, it follows that the rational function that defines a polynomial center function is unique. Also, a rational function $f(x, y, z)$ that satisfies (4) is necessarily of the form $f = g/h$, where g and h are d-forms, i.e., homogeneous polynomials of the same degree d. If $d = 1$, f is called a *projective linear function*. *Projective quadratic functions* correspond to $d = 2$, and so on. Thus a polynomial center $\mathcal{Z}$ is a center defined by a projective function.

A family $\mathbf{F}$ of center functions on $\mathbf{U}$ is said to be *separating* if no two elements in $\mathbf{F}$ coincide on any scalene triangle. It is said to be *complete* if for every scalene triangle T in $\mathbf{U}$, $\{\mathcal{Z}(T) : \mathcal{Z} \in \mathbf{F}\}$ is all of $\mathbf{E}$. The assumption that T is scalene is necessary here. In fact, if a triangle $T = ABC$ is such that $AB = AC$, then $\{\mathcal{Z}(T) : \mathcal{Z} \in \mathbf{F}\}$ will be contained in the line that bisects angle A, being a line of symmetry of ABC, and thus cannot cover $\mathbf{E}$.

3. Polynomial centers of degree 1

We start by characterizing the simplest polynomial center functions, i.e., those defined by projective linear functions. We note the similarity between the line these centers form and the Euler line and we discuss issues related to these two lines.

Theorem 1. *A projective linear function $f(x, y, z)$ satisfies (2), (3), and (4) if and only if*

$$f(x, y, z) = \frac{(1 - 2t)x + t(y + z)}{x + y + z} \tag{5}$$

for some t. If $\mathcal{S}_t$ is the center function defined by (5) (and(1)), then $\mathcal{S}_0$, $\mathcal{S}_{1/3}$, $\mathcal{S}_{1/2}$, and $\mathcal{S}_1$ are the incenter, centroid, Spieker center, and Nagel center, respectively. Also, the centers $\{\mathcal{S}_t(ABC) : t \in \mathbf{R}\}$ of a non-equilateral triangle ABC in $\mathbf{T}$ form the straight line whose trilinear equation is

$$a(b - c)\alpha + b(c - a)\beta + c(a - b)\gamma = 0.$$

Furthermore, the distance $|\mathcal{S}_t\mathcal{S}_u|$ between $\mathcal{S}_t$ and $\mathcal{S}_u$ is given by

$$|\mathcal{S}_t\mathcal{S}_u| = \frac{|t - u|\sqrt{H}}{a + b + c}, \tag{6}$$

where

$$\begin{aligned} H &= (-a + b + c)(a - b + c)(a + b - c) + (a + b)(b + c)(c + a) - 9abc \\ &= -(a^3 + b^3 + c^3) + 2(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2) - 9abc. \end{aligned} \tag{7}$$

Proof. Let $f(x, y, z) = L_0/M_0$, where L_0 and M_0 are linear forms in x, y, and z, and suppose that f satisfies (2), (3), and (4). Let σ be the cycle $(x\ y\ z)$, and let $L_i = \sigma^i(L_0)$ and $M_i = \sigma^i(M_0)$. Since f satisfies (3), it follows that $L_0M_1M_2 + L_1M_0M_2 + L_2M_0M_1 - M_0M_1M_2$ vanishes on $\mathbf{U}_0$ and hence vanishes identically. Thus M_0 divides $L_0M_1M_2$. If M_0 divides L_0, then f is a constant, and hence of the desired form, with $t = 1/3$. If M_0 divides M_1, then it follows easily that M_1 is a constant multiple of M_0 and that M_0 is a constant multiple of $x + y + z$. The same holds if M_0 divides M_2. Finally, we use (3) and (4) to see that L_0 is of the desired form.

Let $\mathcal{S}_t$ be as given. The barycentric coordinates of $\mathcal{S}_t(ABC)$ are given by

$$f(a,b,c) : f(b,c,a) : f(c,a,b)$$

and therefore the trilinear coordinates $\alpha : \beta : \gamma$ of $\mathcal{S}_t(ABC)$ are given by

$$\begin{aligned} \alpha a : \beta b : \gamma c &= (1-2t)a + t(b+c) : (1-2t)b + t(c+a) : (1-2t)c + t(a+b) \\ &= a + t(b+c-2a) : b + t(c+a-2b) : c + t(a+b-2c) \end{aligned}$$

Therefore there exists non-zero λ such that

$$\lambda\alpha a - a = t(b+c-2a),\ \lambda\beta b - b = t(c+a-2b),\ \lambda\gamma c - c = t(a+b-2c).$$

It is clear that the value $t = 0$ corresponds to the incenter. Thus we assume $t \neq 0$. Eliminating t, we obtain

$$\begin{aligned} (\lambda\alpha - 1)a(c+a-2b) &= (\lambda\beta - 1)b(b+c-2a), \\ (\lambda\beta - 1)b(a+b-2c) &= (\lambda\gamma - 1)c(c+a-2b). \end{aligned}$$

Eliminating λ and simplifying, we obtain

$$(a-2b+c)\left[a(b-c)\alpha + b(c-a)\beta + c(a-b)\gamma\right] = 0.$$

Dividing by $a - 2b + c$, we get the desired equation.

Finally, the last statement follows after routine, though tedious, calculations. We simply note that the actual trilinear coordinates of $\mathcal{S}_t$ are given by

$$\frac{2K((1-2t)a + t(b+c))}{a(a+b+c)} : \frac{2K((1-2t)b + t(c+a))}{b(a+b+c)} : \frac{2K((1-2t)c + t(a+b))}{c(a+b+c)},$$

where K is the area of the triangle, and we use the fact that the distance $|PP'|$ between the points P and P' whose actual trilinear coordinates are $\alpha : \beta : \gamma$ and $\alpha' : \beta' : \gamma'$ is given by

$$|PP'| = \frac{1}{2K}\sqrt{-abc[a(\beta-\beta')(\gamma-\gamma') + b(\gamma-\gamma')(\alpha-\alpha') + c(\alpha-\alpha')(\beta-\beta')]};$$

see [25, Theorem 1B, p. 31]. □

4. The Euler-like line $L(\mathcal{I}, \mathcal{G})$

The straight line $\{\mathcal{S}_t : t \in \mathbf{R}\}$ in Theorem 1 is the first central line in the list of [25, p. 128], where it is denoted by $L(1,2,8,10)$. The notation $L(1,2,8,10)$ reflects the fact that it passes through the centers catalogued in [25] as X_1, X_2, X_8, and X_{10}. These are the incenter, centroid, Nagel center, and Spieker center, and they correspond in $\{\mathcal{S}_t : t \in \mathbf{R}\}$ to the values $t = 0, 1/3, 1$, and $1/2$, respectively. We shall denote this line by $L(\mathcal{I}, \mathcal{G})$ to indicate that it is the line joining the incenter $\mathcal{I}$ and the centroid $\mathcal{G}$. Letting $\mathcal{O}$ be the circumcenter, the line $L(\mathcal{G}, \mathcal{O})$ is then nothing but the Euler line. In this section, we survey similarities between these lines. For the third line $L(\mathcal{O}, \mathcal{I})$ and a natural context in which it occurs, we refer the reader to [17].

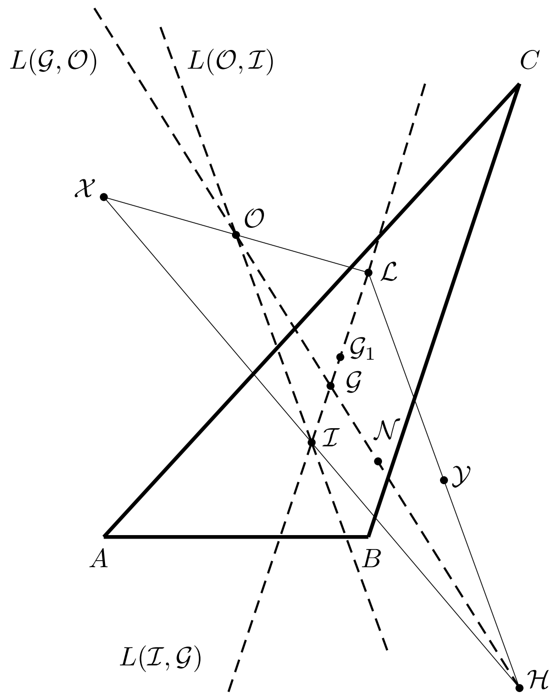

Figure 1

It follows from (6) that the Spieker center $\mathcal{G}_1 = \mathcal{S}_{1/2}$ and the centroid $\mathcal{G} = \mathcal{S}_{1/3}$ of a triangle are at distances in the ratio $3 : 2$ from the incenter $\mathcal{I} = \mathcal{S}_0$. This is shown in Figure 1 which is taken from [17]. The collinearity of $\mathcal{G}_1$, $\mathcal{I}$, and $\mathcal{G}$ and the ratio $3 : 2$ are highlighted in [6, pp. 137–138] and [27], and they also appear in [22, pp. 225–227] and the first row in [25, Table 5.5, p. 143]. In spite of this, we feel that these elegant facts and the striking similarity between the line $L(\mathcal{I}, \mathcal{G})$ and the Euler line $L(\mathcal{O}, \mathcal{G})$ deserve to be better known. Unaware of the aforementioned references, the authors of [3] rediscovered the collinearity of the incenter, the Spieker center, and the centroid and the ratio $3 : 2$, and they proved, in Theorems 6 and 7, that the same thing holds for any polygon that admits an incircle, i.e., a circle that touches the sides of the polygon internally. Here, the centroid of a polygon is the center of mass of a lamina of uniform density that is laid on the polygon, the Spieker center is the centroid of wires of uniform density placed on the sides, and the incenter is the center of the incircle. Later, the same authors, again unaware of [8, p. 69], rediscovered (in [4]) similar properties of $L(\mathcal{I}, \mathcal{G})$ in dimension 3 and made interesting generalizations to solids admitting inspheres. For a deeper explanation of the similarity between the Euler line and its rival $L(\mathcal{I}, \mathcal{G})$ and for affine and other generalizations, see [29] and [28].

We should also mention that the special case of (6) pertaining to the distance between the incenter and the centroid appeared in [7]. Also, the fact that the Spieker center $\mathcal{G}_1$ is the midpoint of the segment joining the incenter $\mathcal{I}$ and the Nagel point $\mathcal{L}$ is the subject matter of [12], [30], and [31]. In each of these references, $\mathcal{L}$ (respectively, $\mathcal{G}_1$) is described as the point of intersection of the lines that bisect the perimeter and that pass through the vertices (respectively, the midpoints of the sides). It is not apparent that the authors of these references are aware that $\mathcal{L}$ and $\mathcal{G}_1$ are the Nagel and Spieker centers. For the interesting part that $\mathcal{G}_1$ is indeed the Spieker center, see [5] and [20, pp. 1–14]. One may also expect that the Euler line and the line $L(\mathcal{I},\mathcal{G})$ cannot coincide unless the triangle is isosceles. This is indeed so, as is proved in [21, Problem 4, Section 11, pp. 142–144]. It also follows from the fact that the area of the triangle $\mathcal{GOI}$ is given by the elegant formula

$$[\mathcal{GOI}] = \left|\frac{s(b-c)(c-a)(a-b)}{24K}\right|,$$

where s is the semiperimeter and K the area of ABC; see [34, Exercise 5.7].

We also note that the Euler line consists of the centers $\mathcal{T}_t$ defined by the function

$$g = \frac{(1-2t)\tan A + t(\tan B + \tan C)}{\tan A + \tan B + \tan C} \tag{8}$$

obtained from f of (5) by replacing a, b, and c by $\tan A$, $\tan B$, and $\tan C$, respectively. Then $\mathcal{T}_0$, $\mathcal{T}_{1/3}$, $\mathcal{T}_{1/2}$, and $\mathcal{T}_1$ are nothing but the circumcenter, centroid, the center of the nine-point circle, and the orthocenter, respectively. The distance $|\mathcal{T}_t\mathcal{T}_u|$ between $\mathcal{T}_t$ and $\mathcal{T}_u$ is given by

$$|\mathcal{T}_t\mathcal{T}_u| = \frac{|t-u|\sqrt{H^*}}{a+b+c},$$

where H^* is obtained from H in (7) by replacing a, b, and c with $\tan A$, $\tan B$, and $\tan C$, respectively. Letting K be the area of the triangle with side-lengths a, b, and c, and using the identity $\tan A = 4K/(b^2+c^2-a^2)$ and its iterates, H^* reduces to a rational function in a, b, and c. In view of the formula $144K^2r^2 = E$ given in [32], where

$$E = a^2b^2c^2 - (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2), \tag{9}$$

and where r is the distance between the circumcenter $\mathcal{T}_0$ and the centroid $\mathcal{T}_{1/3}$, H^* is expected to simplify into

$$H^* = \frac{(a+b+c)^2E}{16K^2},$$

where E is as given in (9), and where $16K^2$ is given by Heron's formula

$$16K^2 = 2(a^2b^2+b^2c^2+c^2a^2) - (a^4+b^4+c^4). \tag{10}$$

Referring to Figure 1, let $\mathcal{X}$ be the point where the lines $\mathcal{LO}$ and $\mathcal{HI}$ meet, and let $\mathcal{Y}$ be the midpoint of $\mathcal{HL}$. Then the Euler line and the line $L(\mathcal{I},\mathcal{G})$ are medians of both triangles $\mathcal{XHL}$ and $\mathcal{OIY}$. The points $\mathcal{X}$ and $\mathcal{Y}$ do not seem to be catalogued in [25]. Also, of the many lines that can be formed in Figure 1, the line $\mathcal{IN}$ is catalogued in [25] as the line joining $\mathcal{I}$, $\mathcal{N}$, and the Feuerbach point. As for distances between various points in Figure 1, formulas for the distances $\mathcal{IN}$, $\mathcal{IO}$, $\mathcal{IH}$, and $\mathcal{OH}$ can be found in [9, pp. 6–7]. The first two are quite well-known and they are associated with Euler, Steiner, Chapple and Feuerbach. Also, formulas for the distances $\mathcal{GI}$ and $\mathcal{GO}$ appeared in [7] and [32], as mentioned earlier. These formulas, as well as other formulas for distances between several other pairs of centers, had already been found by Euler [35, Section XIB, pp. 88–90].

5. Complete separating families of polynomial centers

In the next theorem, we exhibit a complete separating family of polynomial center functions that contains the functions used to define the line $L(\mathcal{I},\mathcal{G})$ encountered in Theorem 1.

Theorem 2. *Let ABC be a scalene triangle and let V be any point in its plane. Then there exist unique real numbers t and v such that V is the center of ABC with respect to the center function $\mathcal{Q}_{t,v}$ defined by the projective quadratic function f given by*

$$f(x,y,z) \;=\; \frac{(1-2t)x^2+t(y^2+z^2)+2(1-v)yz+vx(y+z)}{(x+y+z)^2}. \tag{11}$$

Consequently, the family $\mathbf{F}=\{\mathcal{Q}_{t,v}: t,v\in\mathbf{R}\}$ is a complete separating family. Also, $\mathbf{F}$ contains the line $L(\mathcal{I},\mathcal{G})$ described in Theorem 1.

Proof. Clearly f satisfies the conditions (2), (3), and (4). Since V is in the plane of ABC, it follows that $V=\xi A+\eta B+\zeta C$ for some ξ, η, and ζ with $\xi+\eta+\zeta=1$. Let a, b, and c be the side-lengths of ABC as usual. The system $f(a,b,c)=\xi$, $f(b,c,a)=\eta$, $f(c,a,b)=\zeta$ of equations is equivalent to the system

$$\begin{aligned}(b^2+c^2-2a^2)t+(-2bc+ca+ab)v&=\xi(a+b+c)^2-a^2-2bc,\\ (a^2+b^2-2c^2)t+(-2ab+bc+ca)v&=\zeta(a+b+c)^2-c^2-2ab.\end{aligned}$$

The existence of a (unique) solution (t,v) to this system now follows from the fact that its determinant $-3(a-b)(b-c)(c-a)(a+b+c)$ is not zero.

The last statement follows from the observation that if $v=1-t$, then the expression of $f(x,y,z)$ in (11) reduces to the projective linear function $f(x,y,z)$ given in (5). $\square$

Remarks. (1) According to [25, p. 46], the Fermat-Torricelli point is not a polynomial center. Therefore it does not belong to the family $\mathbf{F}$ defined in Theorem 2. Also, the circumcenter, the orthocenter, and the Gergonne point do not belong to $\mathbf{F}$, although they are polynomial centers. In fact, these centers are defined by the functions f given by

$$\frac{x^2(y^2+z^2-x^2)}{16K^2},\ \frac{y^2+z^2-x^2}{x^2+y^2+z^2},\ \frac{(x-y+z)(x+y-z)}{2(xy+yz+zx)-(x^2+y^2+z^2)},$$

respectively, where K is the area of the triangle whose side-lengths are x, y, and z, and is given by Heron's formula as in (10); see [24, pp. 172–173].

(2) One may replace the denominator of f in (11) by an arbitrary symmetric quadratic form that does not vanish on any point in $\mathbf{T}_0$, and obtain a different

separating complete family of center functions. Thus if we replace f by the similar function

$$g(x,y,z) \;=\; \frac{(-1-2t)x^2+t(y^2+z^2)+2vyz+(1-v)x(y+z)}{2(xy+yz+zx)-(x^2+y^2+z^2)},$$

then we would obtain a complete separating family **G** of center functions that contains the centroid, the Gergonne center and the Mittenpunkt, but not any of the other well known traditional centers. Here the Mittenpunkt is the center defined by the function

$$g(x,y,z)=\frac{xy+xz-x^2}{2(xy+yz+zx)-(x^2+y^2+z^2)}.$$

(3) It is clear that complete families are maximal separating families. However, it is not clear whether the converse is true. It also follows from Zorn's Lemma that every separating family of center functions can be imbedded in a maximal separating family. Thus the seven centers mentioned at the beginning of this note belong to some maximal separating family of centers. The question is whether such a family can be defined in a natural way.

The next theorem shows that pairs of center functions that coincide on scalene triangles exist in abundance. However, it does not answer the question whether such a pair can be chosen from the hundreds of centers that are catalogued in [25]. In case this is not possible, the question arises whether this is due to certain intrinsic properties of the centers in [25].

Theorem 3. *Let $\mathcal{Z}$ be a center function, and let ABC be any scalene triangle in the domain of $\mathcal{Z}$. Then there exists another center function $\mathcal{Z}'$ defined by a projective function f such that $\mathcal{Z}(A,B,C)=\mathcal{Z}'(A,B,C)$.*

Moreover, if $\mathcal{Z}$ is not the centroid, then f can be chosen to be quadratic. If $\mathcal{Z}$ is the centroid, then f can be chosen to be quartic.

Proof. Let **F** and **G** be the families of centers defined in Theorem 2 and in Remark 4. Clearly, the centroid is the only center function that these two families have in common.

If $\mathcal{Z}\notin \mathbf{F}$, then we use Theorem 2 to produce the center $\mathcal{Z}'=\mathcal{Z}_{t,v}$ for which $\mathcal{Z}_{t,v}(A,B,C)=\mathcal{Z}(A,B,C)$, and we take $\mathcal{Z}'=\mathcal{Z}_{t,v}$. If $\mathcal{Z}\notin \mathbf{G}$, then we argue similarly as indicated in Remark 2 to produce the desired center function.

It remains to deal with the case when $\mathcal{Z}$ is the centroid. In this case, we let $f(x,y,z)=g(x,y,z)/h(x,y,z)$, where

$$\begin{aligned}
h(x,y,z) &= (x^4+y^4+z^4)+(x^3y+y^3z+z^3x+x^3z+y^3x+z^3y)\\
&\quad +(x^2y^2+y^2z^2+z^2x^2)\\
g(x,y,z) &= (1-2t)x^4+t(y^4+z^4)+vx^3(y+z)+wx(y^3+z^3)\\
&\quad +(1-v-w)x(y^3+z^3)+sx^2(y^2+z^2)+(1-2s)y^2z^2,
\end{aligned}$$

and we consider the equations

$$f(a,b,c) = f(b,c,a) = f(c,a,b) = \frac{1}{3}.$$

These are linear equations in the variables t, v, w, and s that have an obvious solution $(t,v,w,s) = (1/3,1/3,1/3,1/3)$. Hence they have infinitely many other solutions. Choose any of these solutions and let $\mathscr{Z}$ be the center defined by the function f that corresponds to that choice. Then for the given triangle ABC, $\mathscr{Z}$ is the centroid, as desired. $\square$

Remarks. (4) The question that underlies this paper is whether two centers can coincide for a scalene triangle. The analogous question, for higher dimensional simplices, of how much regularity is implied by the coincidence of two or more centers has led to various interesting results in [18], [19], [10], [11], and [16].

(5, due to the referee) Let $\mathcal{O}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{I}$ be the circumcenter, centroid, orthocenter, and incenter of a non-equilateral triangle. Euler's theorem states that $\mathcal{O}$, $\mathcal{G}$, and $\mathcal{H}$ are collinear with $\mathcal{OG} : \mathcal{GH} = 1 : 2$. A theorem of Guinand in [13] shows that $\mathcal{I}$ ranges freely over the interior of the centroidal disk (with diameter $\mathcal{GH}$) punctured at the nine-point center $\mathcal{N}$. It follows that no two of the centers $\mathcal{O}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{I}$ coincide for a non-equilateral triangle, thus providing a proof, other than case by case chasing, of the very first statement made in the introduction.

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Forum Geometricorum
Volume 7 (2007) 147–149.

FORUM GEOM
ISSN 1534-1178

On the Diagonals of a Cyclic Quadrilateral

Claudi Alsina and Roger B. Nelsen

Abstract. We present visual proofs of two lemmas that reduce the proofs of expressions for the lengths of the diagonals and the area of a cyclic quadrilateral in terms of the lengths of its sides to elementary algebra.

The purpose of this short note is to give a new proof of the following well-known results of Brahmagupta and Parameśhvara [4, 5].

Theorem. If a, b, c, d denote the lengths of the sides; p, q the lengths of the diagonals, R the circumradius, and Q the area of a cyclic quadrilateral, then

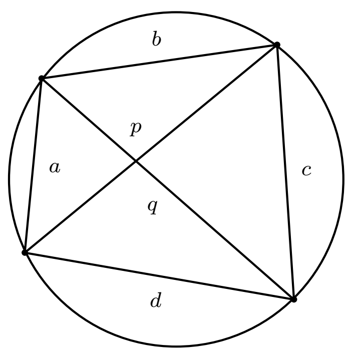

Figure 1

$$p = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}, \qquad q = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}},$$

and

$$Q = \frac{1}{4R}\sqrt{(ab+cd)(ac+bd)(ad+bc)}.$$

We begin with visual proofs of two lemmas, which will reduce the proof of the theorem to elementary algebra. Lemma 1 is the well-known relationship for the area of a triangle in terms of its circumradius and three side lengths; and Lemma 2 expresses the ratio of the diagonals of a cyclic quadrilateral in terms of the lengths of the sides.

Lemma 1. *If a, b, c denote the lengths of the sides, R the circumradius, and K the area of a triangle, then $K = \frac{abc}{4R}$.*

Publication Date: September 24, 2007. Communicating Editor: Paul Yiu.

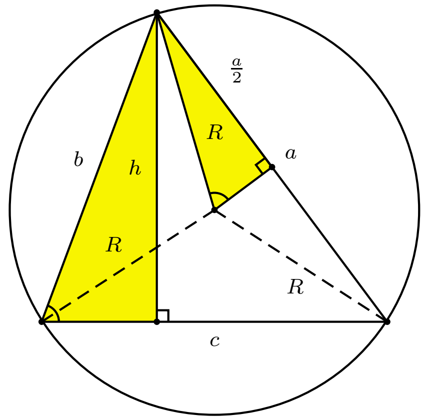

Figure 2

Proof. From Figure 2,

$$\frac{h}{b} = \frac{\frac{a}{2}}{R} \Rightarrow h = \frac{ab}{2R} \Rightarrow K = \frac{1}{2}hc = \frac{abc}{4R}.$$

□

Lemma 2 ([2])**.** *Under the hypotheses of the Theorem,* $\frac{p}{q} = \frac{ad+bc}{ab+cd}$.

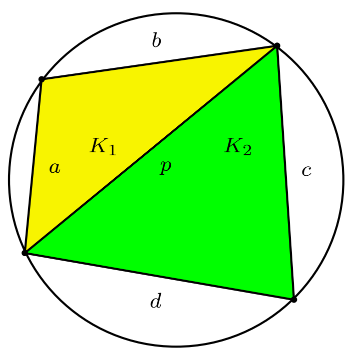

Figure 3

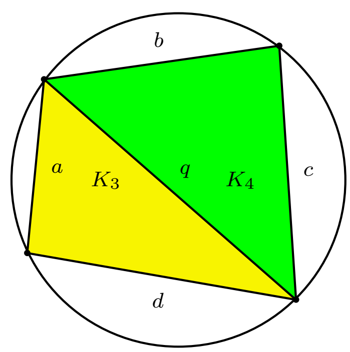

Figure 4

Proof. From Figures 3 and 4 respectively,

$$\begin{aligned} Q =& K_1 + K_2 = \frac{pab}{4R} + \frac{pcd}{4R} = \frac{p(ab+cd)}{4R}, \\ Q =& K_3 + K_4 = \frac{qad}{4R} + \frac{qbc}{4R} = \frac{q(ad+bc)}{4R}. \end{aligned}$$

Therefore,

$$\begin{aligned} p(ab+cd) =& q(ad+bc), \\ \frac{p}{q} =& \frac{ad+bc}{ab+cd}. \end{aligned}$$

□

In the proof of our theorem, we use Lemma 2 and Ptolemy's theorem: Under the hypotheses of our theorem,

$$pq = ac + bd.$$

For proofs of Ptolemy's theorem, see [1, 3].

Proof of the Theorem.

$$\begin{aligned} p^2 =& pq \cdot \frac{p}{q} = \frac{(ac+bd)(ad+bc)}{ab+cd}, \\ q^2 =& pq \cdot \frac{q}{p} = \frac{(ac+bd)(ab+cd)}{ad+bc}; \\ Q^2 =& \frac{pq(ab+cd)(ad+bc)}{(4R)^2} = \frac{(ac+bd)(ab+cd)(ad+bc)}{(4R)^2}. \end{aligned}$$

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Forum Geometricorum
Volume 7 (2007) 151–158.

FORUM GEOM
ISSN 1534-1178

Some Triangle Centers Associated with the Excircles

Tibor Dosa

Abstract. We construct a few new triangle centers associated with the excircles of a triangle.

1. Introduction

Consider a triangle ABC with its excircles. We study a triad of extouch triangles and construct some new triangle centers associated with them. By the A-extouch triangle, we mean the triangle with vertices the points of tangency of the A-excircle with the sidelines of ABC. This is triangle $A_aB_aC_a$ in Figure 1. Similarly, the B- and C-extouch triangles are respectively $A_bB_bC_b$, and $A_cB_cC_c$. Consider also the incircles of these extouch triangles, with centers I_1, I_2, I_3 respectively, and points of tangency X of (I_1) with B_aC_a, Y of (I_2) with C_bA_b, and Z of (I_3) with A_cB_c.

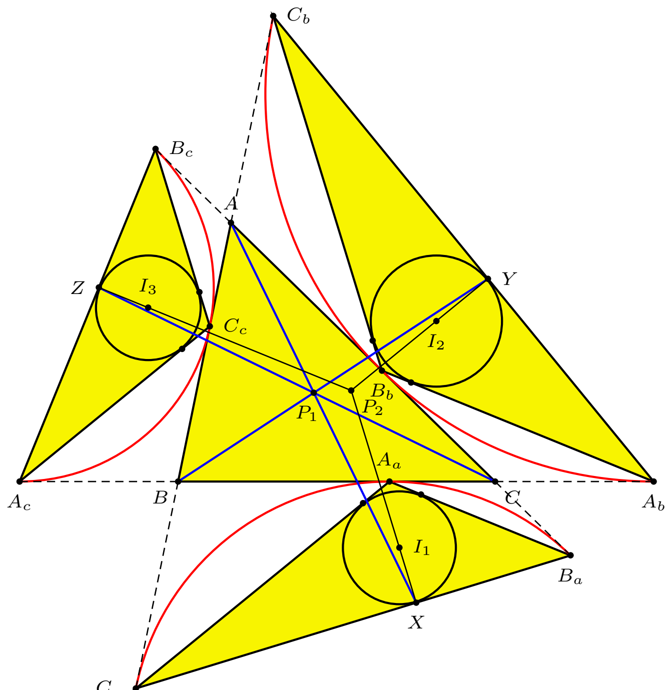

Figure 1.

In this paper, we adopt the usual notations of triangle geometry as in [3] and work with homogeneous barycentric coordinates with reference to triangle ABC.

Publication Date: October 1, 2007. Communicating Editor: Paul Yiu.

Theorem 1. (1) *The lines AX, BY, CZ are concurrent at*

$$P_1 = \left(\cos\frac{A}{2}\cos^2\frac{A}{4} : \cos\frac{B}{2}\cos^2\frac{B}{4} : \cos\frac{C}{2}\cos^2\frac{C}{4}\right).$$

(2) *The lines I_1X, I_2Y, I_3Z are concurrent at*

$$\begin{aligned} P_2 = \Bigg(& a\left(1-\cos\frac{B}{2}-\cos\frac{C}{2}\right)+(b+c)\cos\frac{A}{2} \\ & : b\left(1-\cos\frac{C}{2}-\cos\frac{A}{2}\right)+(c+a)\cos\frac{B}{2} \\ & : c\left(1-\cos\frac{A}{2}-\cos\frac{B}{2}\right)+(a+b)\cos\frac{C}{2}\Bigg). \end{aligned}$$

2. Some preliminary results

Let s and R be the semiperimeter and circumradius respectively of triangle ABC. The following homogeneous barycentric coordinates are well known.

$$\begin{array}{lll} A_a=(0:s-b:s-c), & B_a=(-(s-b):0:s), & C_a=(-(s-c):s:0); \\ A_b=(0:-(s-a):s), & B_b=(s-a:0:s-c), & C_b=(s:-(s-c):0); \\ A_c=(0:s:-(s-c)), & B_c=(s:0:-(s-a)), & C_c=(s-a:s-b:0). \end{array}$$

The lengths of the sides of the A-extouch triangle are as follows:

$$B_aC_a = 2s\cdot\sin\frac{A}{2},\quad C_aA_a = 2(s-c)\cos\frac{B}{2},\quad A_aB_a = 2(s-b)\cos\frac{C}{2}. \tag{1}$$

Lemma 2.

$$\begin{aligned} s &= 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}, \\ s-a &= 4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, \\ s-b &= 4R\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}, \\ s-c &= 4R\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}. \end{aligned}$$

We omit the proof of this lemma. It follows easily from, for example, [1, §293].

3. Proof of Theorem 1

$$\begin{aligned}
B_aX =&\frac{1}{2}(B_aC_a - A_aC_a + A_aB_a)\\
=&s\cdot\sin\frac{A}{2} - (s-c)\cos\frac{B}{2} + (s-b)\cos\frac{C}{2}\\
=&4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left(\cos\frac{A}{2} - \sin\frac{B}{2} + \sin\frac{C}{2}\right)\\
=&4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left(\sin\frac{B+C}{2} - \sin\frac{B}{2} + \sin\frac{C}{2}\right)\\
=&4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left(2\sin\frac{B+C}{4}\cos\frac{B+C}{4} - 2\sin\frac{B-C}{4}\cos\frac{B+C}{4}\right)\\
=&16R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{B+C}{4}\cdot\cos\frac{B}{4}\sin\frac{C}{4}.
\end{aligned}$$

Similarly, $XC_a = 16R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{B+C}{4}\cdot\sin\frac{B}{4}\cos\frac{C}{4}$. The point X therefore divides B_aC_a in the ratio

$$B_aX : XC_a = \cos\frac{B}{4}\sin\frac{C}{4} : \sin\frac{B}{4}\cos\frac{C}{4}.$$

This allows us to compute its absolute barycentric coordinate in terms of B_a and C_a. Note that

$$B_a = \frac{\left(-\sin\frac{A}{2}\sin\frac{C}{2},\ 0,\ \cos\frac{A}{2}\cos\frac{C}{2}\right)}{\sin\frac{B}{2}}, \qquad C_a = \frac{\left(-\sin\frac{A}{2}\sin\frac{B}{2},\ \cos\frac{A}{2}\cos\frac{B}{2},\ 0\right)}{\sin\frac{C}{2}}.$$

From these we have

$$\begin{aligned}
X =&\frac{\sin\frac{B}{4}\cos\frac{C}{4}\cdot B_a + \cos\frac{B}{4}\sin\frac{C}{4}\cdot C_a}{\sin\frac{B+C}{4}}\\
=&\frac{\sin\frac{B}{4}\cos\frac{C}{4}\cdot\frac{\left(-\sin\frac{A}{2}\sin\frac{C}{2},0,\cos\frac{A}{2}\cos\frac{C}{2}\right)}{\sin\frac{B}{2}} + \cos\frac{B}{4}\sin\frac{C}{4}\cdot\frac{\left(-\sin\frac{A}{2}\sin\frac{B}{2},\cos\frac{A}{2}\cos\frac{B}{2},\ 0\right)}{\sin\frac{C}{2}}}{\sin\frac{B+C}{4}}\\
=&\frac{\cos\frac{C}{4}\cdot\frac{\left(-\sin\frac{A}{2}\sin\frac{C}{2},0,\cos\frac{A}{2}\cos\frac{C}{2}\right)}{2\cos\frac{B}{4}} + \cos\frac{B}{4}\cdot\frac{\left(-\sin\frac{A}{2}\sin\frac{B}{2},\cos\frac{A}{2}\cos\frac{B}{2},\ 0\right)}{2\cos\frac{C}{4}}}{\sin\frac{B+C}{4}}\\
=&\frac{\cos^2\frac{C}{4}\left(-\sin\frac{A}{2}\sin\frac{C}{2},0,\cos\frac{A}{2}\cos\frac{C}{2}\right) + \cos^2\frac{B}{4}\left(-\sin\frac{A}{2}\sin\frac{B}{2},\cos\frac{A}{2}\cos\frac{B}{2},\ 0\right)}{2\cos\frac{B}{4}\cos\frac{C}{4}\sin\frac{B+C}{4}}\\
=&\frac{\left(-\sin\frac{A}{2}\left(\sin\frac{B}{2}\cos^2\frac{B}{4} + \sin\frac{C}{2}\cos^2\frac{C}{4}\right),\cos\frac{A}{2}\cos\frac{B}{2}\cos^2\frac{B}{4},\ \cos\frac{A}{2}\cos\frac{C}{2}\cos^2\frac{C}{4}\right)}{2\cos\frac{B}{4}\cos\frac{C}{4}\sin\frac{B+C}{4}}.
\end{aligned}$$

From this we obtain the homogeneous barycentric coordinates of X, and those of Y and Z by cyclic permutations of A, B, C:

$$X = \left(-\sin\frac{A}{2}\left(\sin\frac{B}{2}\cos^2\frac{B}{4}+\sin\frac{C}{2}\cos^2\frac{C}{4}\right) : \cos\frac{A}{2}\cos\frac{B}{2}\cos^2\frac{B}{4} : \cos\frac{A}{2}\cos\frac{C}{2}\cos^2\frac{C}{4}\right),$$
$$Y = \left(\cos\frac{B}{2}\cos\frac{A}{2}\cos^2\frac{A}{4} : -\sin\frac{B}{2}\left(\sin\frac{C}{2}\cos^2\frac{C}{4}+\sin\frac{A}{2}\cos^2\frac{A}{4}\right) : \cos\frac{B}{2}\cos\frac{C}{2}\cos^2\frac{C}{4}\right),$$
$$Z = \left(\cos\frac{C}{2}\cos\frac{A}{2}\cos^2\frac{A}{4} : \cos\frac{C}{2}\cos\frac{B}{2}\cos^2\frac{B}{4} : -\sin\frac{C}{2}\left(\sin\frac{A}{2}\cos^2\frac{A}{4}+\sin\frac{B}{2}\cos^2\frac{B}{4}\right)\right).$$

Equivalently,

$$X = \left(-\tan\frac{A}{2}\left(\sin\frac{B}{2}\cos^2\frac{B}{4}+\sin\frac{C}{2}\cos^2\frac{C}{4}\right) : \cos\frac{B}{2}\cos^2\frac{B}{4} : \cos\frac{C}{2}\cos^2\frac{C}{4}\right),$$
$$Y = \left(\cos\frac{A}{2}\cos^2\frac{A}{4} : -\tan\frac{B}{2}\left(\sin\frac{C}{2}\cos^2\frac{C}{4}+\sin\frac{A}{2}\cos^2\frac{A}{4}\right) : \cos\frac{C}{2}\cos^2\frac{C}{4}\right),$$
$$Z = \left(\cos\frac{A}{2}\cos^2\frac{A}{4} : \cos\frac{B}{2}\cos^2\frac{B}{4} : -\tan\frac{C}{2}\left(\sin\frac{A}{2}\cos^2\frac{A}{4}+\sin\frac{B}{2}\cos^2\frac{B}{4}\right)\right).$$

It is clear that the lines AX, BY, CZ intersect at a point P_1 with coordinates

$$\left(\cos\frac{A}{2}\cos^2\frac{A}{4} : \cos\frac{B}{2}\cos^2\frac{B}{4} : \cos\frac{C}{2}\cos^2\frac{C}{4}\right).$$

This completes the proof of Theorem 1(1).

For (2), note that the line I_1X is parallel to the bisector of angle A. Its has barycentric equation

$$\begin{vmatrix} -\sin\frac{A}{2}\left(\sin\frac{B}{2}\cos^2\frac{B}{4}+\sin\frac{C}{2}\cos^2\frac{C}{4}\right) & \cos\frac{A}{2}\cos\frac{B}{2}\cos^2\frac{B}{4} & \cos\frac{A}{2}\cos\frac{C}{2}\cos^2\frac{C}{4} \\ -(b+c) & b & c \\ x & y & z \end{vmatrix} = 0.$$

A routine calculation, making use of the fact that the sum of the entries in the first row is $\sin\frac{C}{2}\cos^2\frac{B}{4}+\sin\frac{B}{2}\cos^2\frac{C}{4}$, gives

$$-(x+y+z)\left(b\cos\frac{C}{2}-c\cos\frac{B}{2}\right)+bz-cy=0.$$

Similarly, the lines I_2Y and I_3Z have equations

$$-(x+y+z)\left(c\cos\frac{A}{2}-a\cos\frac{C}{2}\right)+cx-az=0,$$
$$-(x+y+z)\left(a\cos\frac{B}{2}-b\cos\frac{A}{2}\right)+ay-bx=0.$$

These three lines intersect at

$$P_2 = \left(a\left(1-\cos\frac{B}{2}-\cos\frac{C}{2}\right)+(b+c)\cos\frac{A}{2}\right.$$
$$: b\left(1-\cos\frac{C}{2}-\cos\frac{A}{2}\right)+(c+a)\cos\frac{B}{2}$$
$$\left.: c\left(1-\cos\frac{A}{2}-\cos\frac{B}{2}\right)+(a+b)\cos\frac{C}{2}\right).$$

This completes the proof of Theorem 1(2).

Remark. The barycentric coordinates of the incenter I_1 of the A-extouch triangle are

$$\left(-\sin\frac{A}{2}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right) : \cos\frac{B}{2}\left(\sin\frac{C}{2}+\cos\frac{A}{2}\right) : \cos\frac{C}{2}\left(\cos\frac{A}{2}+\sin\frac{B}{2}\right)\right).$$

4. Some collinearities

The homogeneous barycentric coordinates of P_1 can be rewritten as

$$\left(\cos^2\frac{A}{2}+\cos\frac{A}{2} \ : \cos^2\frac{B}{2}+\cos\frac{B}{2} \ : \cos^2\frac{C}{2}+\cos\frac{C}{2}\right).$$

From this it is clear that the point P_1 lies on the line joining the two points with coordinates $\left(\cos^2\frac{A}{2} : \cos^2\frac{B}{2} : \cos^2\frac{C}{2}\right)$ and $\left(\cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}\right)$. We briefly recall their definitions.

(i) The point $M = \left(\cos^2\frac{A}{2} : \cos^2\frac{B}{2} : \cos^2\frac{C}{2}\right) = (a(s-a) : b(s-b) : c(s-c))$ is the Mittenpunkt. It is the perspector of the excentral triangle and the medial triangle. It is the triangle center X_9 of [2].

(ii) The point $Q = \left(\cos\frac{A}{2} \ : \cos\frac{B}{2} \ : \cos\frac{C}{2}\right)$ appears as X_{188} in [2], and is named the second mid-arc point. Here is an explicit description. Consider the anticomplementary triangle $A'B'C'$ of ABC, with its incircle (I'). If the segments $I'A'$, $I'B'$, $I'C'$ intersect the incircle (I') at A'', B'', C'', then the lines AA'', BB'', CC'' are concurrent at Q. See Figure 2.

Proposition 3. (1) *The point P_1 lies on the line MQ.*
(2) *The point P_2 lies on the line joining the incenter to Q.*

Proof. We need only prove (2). This is clear from

$$P_2 = \left(1-\cos\frac{A}{2}-\cos\frac{B}{2}-\cos\frac{C}{2}\right)I+\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)Q.$$

In fact,

$$P_2 = I+\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)\overrightarrow{IQ}. \tag{2}$$

□

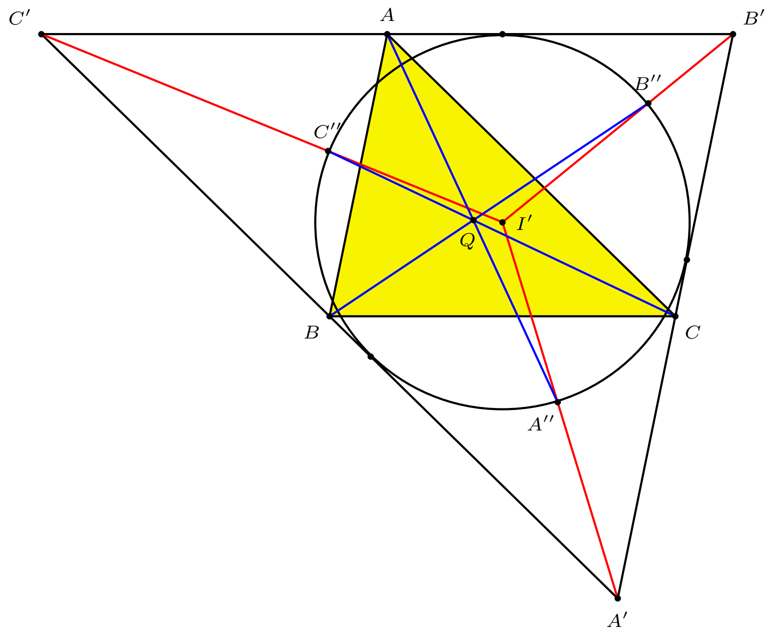

Figure 2.

5. The excircles of the extouch triangles

Consider the excircle of triangle $A_aB_aC_a$ tangent to the side B_aC_a at X'. It is clear that X' and X are symmetric with respect to the midpoint of B_aC_a. Since triangle AB_aC_a is isosceles, the lines AX' and AX are isogonal with respect to AB_a and AC_a. As such, they are isogonal with respect to AB and AC. Likewise, if we consider the excircle of $A_bB_bC_b$ tangent to C_bA_b at Y', and that of $A_cB_cC_c$ tangent to A_cB_c at Z', then the lines AX', BY', CZ', being respectively isogonal to AX, BY, CZ, intersect at the isogonal conjugate of P_1.

Proposition 4. *The barycentric coordinates of P_1^* are*

$$\left(\cos\frac{A}{2}\sin^2\frac{A}{4} : \cos\frac{B}{2}\sin^2\frac{B}{4} : \cos\frac{C}{2}\sin^2\frac{C}{4}\right).$$

Proof. This follows from

$$\begin{aligned} P_1^* &= \left(\frac{\sin^2 A}{\cos\frac{A}{2}\cos^2\frac{A}{4}} : \frac{\sin^2 B}{\cos\frac{B}{2}\cos^2\frac{B}{4}} : \frac{\sin^2 C}{\cos\frac{C}{2}\cos^2\frac{C}{4}}\right) \\ &= \left(\frac{\sin^2\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{4}} : \frac{\sin^2\frac{B}{2}\cos\frac{B}{2}}{\cos^2\frac{B}{4}} : \frac{\sin^2\frac{C}{2}\cos\frac{C}{2}}{\cos^2\frac{C}{4}}\right) \\ &= \left(\cos\frac{A}{2}\sin^2\frac{A}{4} : \cos\frac{B}{2}\sin^2\frac{B}{4} : \cos\frac{C}{2}\sin^2\frac{C}{4}\right). : \end{aligned}$$

□

Corollary 5. *The points P_1, P_1^* and Q are collinear.*

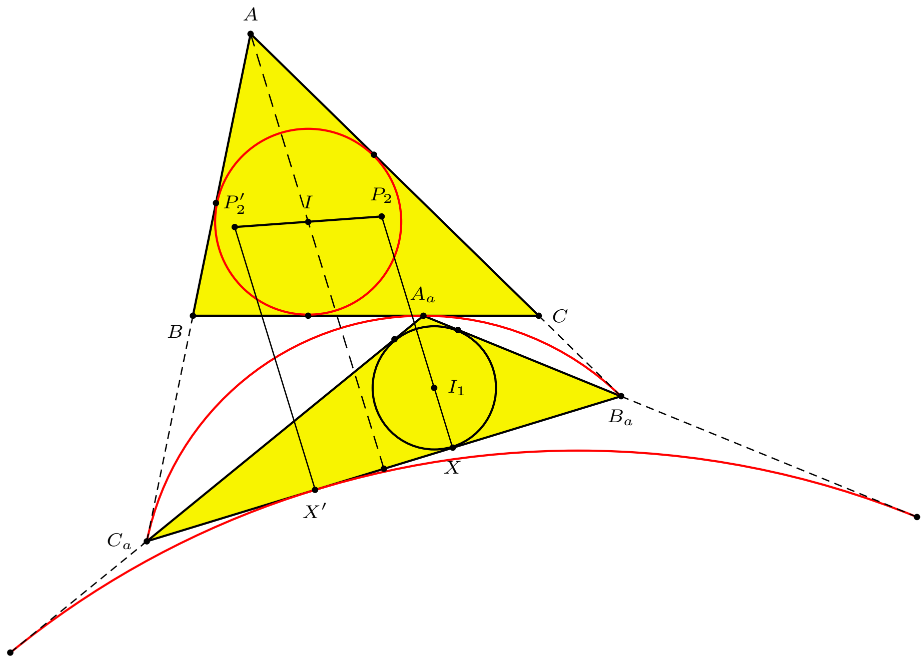

Figure 3.

Proposition 6. *The perpendiculars to B_aC_a at X', to C_bA_b at Y', and to A_cB_c at Z' are concurrent at the reflection of P_2 in I, which is the point*

$$\begin{aligned} P_2' =& a\left(1+\cos\frac{B}{2}+\cos\frac{C}{2}\right)-(b+c)\cos\frac{A}{2} \\ &:b\left(1+\cos\frac{C}{2}+\cos\frac{A}{2}\right)-(c+a)\cos\frac{B}{2} \\ &:c\left(1+\cos\frac{A}{2}+\cos\frac{B}{2}\right)-(a+b)\cos\frac{C}{2}. \end{aligned}$$

Proof. Let P_2' be the reflection of P_2 in I. Since X and X' are symmetric in the midpoint of B_aC_a, and P_2X is perpendicular to B_aC_a, it follows that $P_2'X'$ is also perpendicular to B_aC_a. The same reasoning shows that $P_2'Y'$ and $P_2'Z'$ are perpendicular to C_bA_b and A_cB_c respectively. It follows from (2) that

$$P_2' = I - \left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)\overrightarrow{IQ}.$$

From this, we easily obtain the homogeneous barycentric coordinates as given above. $\square$

We conclude this paper with the construction of another triangle center. It is known that the perpendiculars from A_a to B_aC_a, B_b to C_bA_b, and C_c to A_cB_c

intersect at

$$P_3 = ((b+c)\cos A \ : (c+a)\cos B \ : (a+b)\cos C). \tag{3}$$

This is the triangle center X_{72} in [2].

If we let X_0, Y_0, Z_0 be these pedals, then it is also known that AX_0, BY_0, CZ_0 intersect at the Mittenpunkt X_9. Now, let X_1, Y_1, Z_1 be the reflections of X_0, Y_0, Z_0 in the midpoints of B_aC_a, C_bA_b, A_cB_c respectively. The lines AX_1, BY_1, CZ_1 clearly intersect at the reflection of X_{72} in I. This is the point

$$P_3' = ((b+c)\cos A - 2a \ : (c+a)\cos B - 2b \ : (a+b)\cos C - 2c)\,.$$

These coordinates are particularly simple since the sum of the coordinates of P_3 given in (3) is $a+b+c$.

The triangle centers P_1, P_1^*, P_2, P_2' and P_3' do not appear in [2].

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Forum Geometricorum
Volume 7 (2007) 159–168.

FORUM GEOM
ISSN 1534-1178

Fixed Points and Fixed Lines of Ceva Collineations

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Abstract. In the plane of a triangle ABC, the U-Ceva collineation maps points to points and lines to lines. If U is a triangle center other than the incenter, then the U-Ceva collineation has three distinct fixed points F_1, F_2, F_3 and three distinct fixed lines F_2F_3, F_3F_1, F_1F_2, these being the trilinear polars of F_1, F_2, F_3. When U is the circumcenter, the fixed points are the symmedian point and the isogonal conjugates of the points in which the Euler line intersects the circumcircle.

1. Introduction

This note is a sequel to [3], in which the notion of a U-Ceva collineation is introduced. In this introduction, we briefly summarize the main results of [3].

We use homogeneous trilinear coordinates and denote the isogonal conjugate of a point X by X^{-1}. The X-Ceva conjugate of $U = u : v : w$ and $X = x : y : z$ is given by

$$X©U = u(-uyz + vzx + wxy) : v(uyz - vzx + wxy) : w(uyz + vzx - wxy),$$

and if $P = p : q : r$ is a point, then the equation $P = X©U$ is equivalent to

$$\begin{aligned} X &= (ru + pw)(pv + qu) : (pv + qu)(qw + rv) : (qw + rv)(ru + pw) \qquad (1) \\ &= \text{cevapoint}(P, U). \end{aligned}$$

If $\mathcal{L}_1$ is a line $l_1\alpha + m_1\beta + n_1\gamma = 0$ and $\mathcal{L}_2$ is a line $l_2\alpha + m_2\beta + n_2\gamma = 0$, then there exists a unique point U such that if $X \in \mathcal{L}_1$, then $X^{-1}©U \in \mathcal{L}_2$, and the mapping $X \to X^{-1}©U$ is surjective. This mapping is written as $\mathcal{C}_U(X) = X^{-1}©U$, and $\mathcal{C}_U$ is called the U-Ceva collineation. Explicitly,

$$\mathcal{C}_U(X) = u(-ux + vy + wz) : v(ux - vy + wz) : w(ux + vy - wz).$$

The inverse mapping is given by

$$\begin{aligned} \mathcal{C}_U^{-1}(X) &= wy + vz : uz + wx : vx + uy \\ &= (\text{cevapoint}(X, U))^{-1}. \end{aligned}$$

The collineation $\mathcal{C}_U$ maps the vertices A, B, C to the vertices of the anticevian triangle of U and maps U^{-1} to U. The collineation $\mathcal{C}_U^{-1}$ maps A, B, C to the vertices of the cevian triangle of U^{-1} and maps U to U^{-1}.

Publication Date: October 15, 2007. Communicating Editor: Paul Yiu.

2. Fixed points

The fixed points of the $\mathcal{C}_U$-collineation are also the fixed points of the inverse collineation, $\mathcal{C}_U^{-1}$. In this section, we seek all points X satisfying $\mathcal{C}_U^{-1}(X) = X$; *i.e.*, we wish to solve the equation

$$\mathcal{C}_U^{-1}(X) = \begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = MX$$

for the vector X. Writing $(M - tI)X = 0$, where I denotes the 3×3 identity matrix, we have the characteristic equation $\det(M - tI) = 0$ of M, which can be written

$$\begin{vmatrix} -t & w & v \\ w & -t & u \\ v & u & -t \end{vmatrix} = 0.$$

Expanding the determinant gives

$$t^3 - gt - h = 0, \tag{2}$$

where $g = u^2+v^2+w^2$ and $h = 2uvw$. Now suppose t is a root, *i.e.*, an eigenvalue of M. The equation $(M - tI)X = 0$ is equivalent to the system

$$\begin{aligned} -tx + wy + vz &= 0 \\ wx - ty + uz &= 0 \\ vx + uy - tz &= 0. \end{aligned}$$

For any z, the first two of the three equations can be written as

$$\begin{pmatrix} -t & w \\ w & -t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -vz \\ -uz \end{pmatrix},$$

and if $t^2 \neq w^2$, then

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -t & w \\ w & -t \end{pmatrix}^{-1} \begin{pmatrix} -vz \\ -uz \end{pmatrix} \\ &= \frac{1}{t^2 - w^2} \begin{pmatrix} tvz + uwz \\ tuz + vwz \end{pmatrix}, \end{aligned}$$

Thus, for each z,

$$x = \frac{1}{t^2 - w^2}(tvz + uwz) \quad \text{and} \quad y = \frac{1}{t^2 - w^2}(tuz + vwz),$$

so that

$$x : y = tv + uw : tu + vw \quad \text{and} \quad \frac{y}{z} = \frac{1}{t^2 - w^2}(tu + vw),$$

and $x : y : z$ is as shown in (6) below.

Continuing with the case $t^2 \neq w^2$, let $f(t)$ be the polynomial in (2), and let

$$r = \sqrt{(u^2 + v^2 + w^2)/3},$$

so that

$$f(-r) = -2uvw + \frac{2}{3}(u^2 + v^2 + w^2)r; \tag{3}$$
$$f(r) = -2uvw - \frac{2}{3}(u^2 + v^2 + w^2)r.$$

Clearly, $f(r) < 0$. To see that $f(-r) \geq 0$, we shall use the inequality of the geometric and arithmetic means, stated here for $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$:

$$(x_1x_2x_3)^{1/3} \leq \frac{x_1 + x_2 + x_3}{3}. \tag{4}$$

Taking $x_1 = u^2$, $x_2 = v^2$, $x_3 = w^2$ gives

$$27u^2v^2w^2 \leq \left(u^2 + v^2 + w^2\right)^3,$$

or equivalently,

$$3uvw \leq \left(u^2 + v^2 + w^2\right) r,$$

so that by (3), we have $f(-r) \geq 0$. We consider two cases: $f(-r) > 0$ and $f(r) = 0$. In the first case, there is a root t in the interval $(-\infty, -r)$. Since $f(0) < 0$, there is a root in $(-r, 0)$, and since $f(r) < 0$, there is a root in (r, ∞). For each of the three roots, or eigenvalues, there is an eigenvector, or point X, such that $\mathcal{C}_U^{-1}(X) = X$.

In the second case, that $f(r) = 0$, we have $(u^2 + v^2 + w^2)r = -3uvw$, so that $(u^2 + v^2 + w^2)^3 = 27u^2v^2w^2$, which implies that equality holds in (4). This is known to occur if and only if if $x_1 = x_2 = x_3$, or equivalently, $u^2 = v^2 = w^2$, which is to say that U is the incenter or one of the excenters; *i.e.*, that U is a member of the set

$$\{1 : 1 : 1, \quad -1 : 1 : 1, \quad 1 : -1 : 1, \quad 1 : 1 : -1\}. \tag{5}$$

We consider this case further in Examples 1 and 2 below, and summarize the rest of this section as a theorem.

Theorem 1. *Suppose U is not one of the four points in* (4)*, that t is a root of* (2)*, and that $t^2 \neq w^2$. Then the point*

$$X = tv + uw : tu + vw : t^2 - w^2 \tag{6}$$

is a fixed point of $\mathcal{C}_U^{-1}$, hence also a fixed point of $\mathcal{C}_U$. There are three distinct roots t, hence three distinct fixed points X.

3. Examples

As a first example, we address the possibility that the hypothesis $t^2 \neq w^2$ in Theorem 1 does not hold.

Example 1. $U = 1 : 1 : 1$. The characteristic polynomial is

$$\begin{vmatrix} -t & 1 & 1 \\ 1 & -t & 1 \\ 1 & 1 & -t \end{vmatrix} = (2 - t)(t + 1)^2.$$

We have two cases: $t = 2$ and $t = -1$. For $t = 2$, we easily find the fixed point $1 : 1 : 1$. For $t = -1$, the method of proof of Theorem 1 does not apply because $t^2 = w^2$. Instead, the system to be solved degenerates to the single equation $z = -x - y$. The solutions, all fixed points, are many; for example, let $f : g : h$ be any point, and let

$$x = g - h, \quad y = h - f, \quad z = f - g$$

(e.g., $x : y : z = b - c : c - a : a - b$, which is the triangle center[1] X_{512}.). Geometrically, $x : y : z$ are coefficients of the line joining $1 : 1 : 1$ and $f : g : h$.

Example 2. $U = -1 : 1 : 1$, the A-excenter. The characteristic polynomial is

$$\begin{vmatrix} -t & 1 & 1 \\ -1 & -t & 1 \\ -1 & 1 & -t \end{vmatrix} = -(t+1)\left(t^2 - t + 2\right).$$

For $t = -1$, we find that every point on the line $x + y + z = 0$ is a fixed point. If $t^2 - t + 2 = 0$, then $t = (1 \pm \sqrt{-7})/2$, and the (nonreal) fixed point is $1 : 1 : t - 1$. Similar results are obtained for $U \in \{1 : -1 : 1, 1 : 1 : -1\}$.

Example 3. $U = \cos A : \cos B : \cos C$. It can be checked using a computer algebra system that X_6, X_{2574}, and X_{2575} are fixed points. The first of these corresponds to the eigenvalue $t = 1$, as shown here:

$$\begin{aligned} x : y : z &= tv + uw : tu + vw : t^2 - w^2 \\ &= \cos B + \cos A \cos C : \cos A + \cos B \cos C : 1 - \cos^2 C \\ &= \sin A \sin C : \sin B \sin C : \sin C \sin C \\ &= \sin A : \sin B : \sin C \\ &= X_6. \end{aligned}$$

See also Example 6.

Example 4. $U = a(b^2 + c^2) : b(c^2 + a^2) : c(a^2 + b^2) = X_{39}$. The three roots of $t^3 - gt - h = 0$ are

$$-2abc, \quad abc - \sqrt{3a^2b^2c^2 + S(2,4)}, \quad abc + \sqrt{3a^2b^2c^2 + S(2,4)},$$

where

$$S(2,4) = a^2b^4 + a^4b^2 + a^2c^4 + a^4c^2 + b^2c^4 + b^4c^2.$$

The solution $t = -2abc$ easily leads to the fixed point

$$X_{512} = (b^2 - c^2)/a : (c^2 - a^2)/b : (a^2 - b^2)/c.$$

[1] We use the indexing of triangle centers in the *Encyclopedia of Triangle Centers* [3].

Example 5. For arbitrary real n, let $u = \cos nA$, $v = \cos nB$, $w = \cos nC$. A fixed point is $X = \sin nA : \sin nB : \sin nC$, as shown here:

$$\begin{aligned}
\mathcal{C}_U^{-1}(X) = &\ \sin nB \cos nC + \sin nC \cos nB \\
: &\ \sin nC \cos nA + \sin nA \cos nC \\
: &\ \sin nA \cos nB + \sin nB \cos nA \\
= &\ \sin(nB + nC) : \sin(nC + nA) : \sin(nA + nB) \\
= &\ \sin nA : \sin nB : \sin nC.
\end{aligned}$$

4. Images of lines

Let $\mathcal{L}$ be the line $l\alpha + m\beta + n\gamma = 0$ and let L the point[2] $l : m : n$. We shall determine coefficients of the line $\mathcal{C}_U^{-1}(\mathcal{L})$. Two points on $\mathcal{L}$ are

$$P = cm - bn : an - cl : bl - am \quad \text{and} \quad Q = m - n : n - l : l - m$$

Their images on $\mathcal{C}_U^{-1}(\mathcal{L})$ are given by

$$\begin{aligned}
P' = \mathcal{C}_U^{-1}(P) &= \begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix} \begin{pmatrix} cm - bn \\ an - cl \\ bl - am \end{pmatrix}, \\
Q' = \mathcal{C}_U^{-1}(Q) &= \begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix} \begin{pmatrix} m - n \\ n - l \\ l - m \end{pmatrix}.
\end{aligned}$$

We expand these products and use the resulting trilinears as rows 2 and 3 of the following determinant:

$$\begin{aligned}
&\begin{vmatrix} \alpha & \beta & \gamma \\ w(an - cl) + v(bl - am) & w(cm - bn) + u(bl - am) & v(cm - bn) + u(an - cl) \\ w(n - l) + v(l - m) & w(m - n) + u(l - m) & v(m - n) + u(n - l) \end{vmatrix} \\
= &\ - ((b - c)l + (c - a)m + (a - b)n) \\
&\ \cdot (u(-ul + vm + wn)\alpha + b(ul - vm + wn)\beta + c(ul + vm - wn)\gamma).
\end{aligned}$$

If the first factor is not 0, then the required line $\mathcal{C}_U^{-1}(\mathcal{L})$ is given by

$$u(-ul + vm + wn)\alpha + v(ul - vm + wn)\beta + w(ul + vm - wn)\gamma = 0, \quad (7)$$

of which the coefficients are the trilinears of the point

$$L^{-1} © U = u(-ul + vm + wn) : v(ul - vm + wn) : w(ul + vm - wn).$$

Even if the first factor is 0, the points P' and Q' are easily checked to lie on the line (7).

[2]Geometrically, $\mathcal{L}$ is the trilinear polar of L^{-1}. However, the methods in this paper are algebraic rather than geometric, and the results extend beyond the boundaries of euclidean geometry. For example, in this paper, a, b, c are unrestricted positive real numbers; *i.e.*, they need not be sidelengths of a triangle.

The same method shows that the coefficients of the line $\mathcal{C}_U(\mathcal{L})$ are the trilinears of $(\text{cevapoint}(L,U))^{-1}$; that is, $\mathcal{C}_U(\mathcal{L})$ is the line

$$(wm+vn)\,\alpha+(un+wl)\,\beta+w\,(vl+um)\,\gamma=0.$$

5. Fixed lines

The line $\mathcal{L}$ is a fixed line of $\mathcal{C}_U$ (and of $\mathcal{C}_U^{-1}$) if $\mathcal{C}_U(\mathcal{L})=\mathcal{L}$, that is, if

$$(\text{cevapoint}(U,L))^{-1}=L,$$

or, equivalently,

$$\begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix}\begin{pmatrix} l \\ m \\ n \end{pmatrix}=\begin{pmatrix} l \\ m \\ n \end{pmatrix}.$$

This is the same equation as already solved (with L in place of X) in Section 2. For each of the three roots of (2), there is an eigenvector, or point L, and hence a line $\mathcal{L}$, such that $\mathcal{C}_U(\mathcal{L})=\mathcal{L}$, and we have the following theorem.

Theorem 2. *The mapping C_U has three distinct fixed lines, corresponding to the three distinct real roots of $f(t)$ in* (2). *For each root t, the corresponding fixed line $l\alpha+m\beta+n\gamma=0$ is given by*

$$l:m:n=tv+uw:tu+vw:t^2-w^2. \tag{8}$$

6. Iterations and convergence

In this section we examine sequences

$$X,\;\; \mathcal{C}_U^{-1}(X),\;\; \mathcal{C}_U^{-1}(\mathcal{C}_U^{-1}(X)),\ldots \tag{9}$$

of iterates. If X is a fixed point of $\mathcal{C}_U^{-1}$, then the sequence is simply $X,X,X,...$; otherwise, with exceptions to be recognized, the sequence converges to a fixed point. We begin with the case that X lies on a fixed line, so that all the points in (9) lie on that same line. Let the two fixed points on the fixed line be

$$F_1=f_1:g_1:h_1 \quad \text{and} \quad F_2=f_2:g_2:h_2.$$

Then for X on the line F_1F_2, we have

$$X=f_1+tf_2:g_1+tg_2:h_1+th_2$$

for some function t homogeneous in a,b,c, and we wish to show that (9) converges to F_1 or F_2. As a first step,

$$\mathcal{C}_U^{-1}(X)=\begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix}\begin{pmatrix} f_1+tf_2 \\ g_1+tg_2 \\ h_1+th_2 \end{pmatrix}=\begin{pmatrix} wg_1+vh_1+t(wg_2+vh_2) \\ wf_1+uh_1+t(wf_2+uh_2) \\ vf_1+ug_1+t(vf_2+ug_2) \end{pmatrix}.$$

For $i = 1, 2$, because $f_i : g_i : h_i$ is fixed by $\mathcal{C}_U^{-1}$, there exists a homogeneous function t_i such that

$$\begin{aligned} wg_i + vh_i &= t_i f_i, \\ wf_i + vh_i &= t_i g_i, \\ vf_i + ug_i &= t_i h_i, \end{aligned}$$

so that

$$\mathcal{C}_U^{-1}(X) = \begin{pmatrix} t_1 f_1 + t_2 t f_2 \\ t_1 g_1 + t_2 t g_2 \\ t_1 h_1 + t_2 t h_2 \end{pmatrix} = t_1 \begin{pmatrix} f_1 + \dfrac{t_2}{t_1} t f_2 \\ g_1 + \dfrac{t_2}{t_1} t g_2 \\ h_1 + \dfrac{t_2}{t_1} t h_2 \end{pmatrix}.$$

Applying $\mathcal{C}_U^{-1}$ again thus gives

$$\mathcal{C}_U^{-2}(X) = \begin{pmatrix} f_1 + \dfrac{t_4}{t_3}\dfrac{t_2}{t_1} t f_2 \\ g_1 + \dfrac{t_4}{t_3}\dfrac{t_2}{t_1} t g_2 \\ h_1 + \dfrac{t_4}{t_3}\dfrac{t_2}{t_1} t h_2 \end{pmatrix},$$

where t_3 and t_4 satisfy

$$\begin{pmatrix} wg_1 + vh_1 + \dfrac{t_2}{t_1} t(wg_2 + vh_2) \\ wf_1 + uh_1 + \dfrac{t_2}{t_1} t(wf_2 + uh_2) \\ vf_1 + ug_1 + \dfrac{t_2}{t_1} t(vf_2 + ug_2) \end{pmatrix} = \begin{pmatrix} t_3 f_1 + t_4 \dfrac{t_2}{t_1} t f_2 \\ t_3 g_1 + t_4 \dfrac{t_2}{t_1} t f_2 \\ t_3 h_1 + t_4 \dfrac{t_2}{t_1} t f_2 \end{pmatrix} = t_3 \begin{pmatrix} f_1 + \dfrac{t_4}{t_3}\dfrac{t_2}{t_1} t f_2 \\ g_1 + \dfrac{t_4}{t_3}\dfrac{t_2}{t_1} t g_2 \\ h_1 + \dfrac{t_4}{t_3}\dfrac{t_2}{t_1} t h_2 \end{pmatrix}.$$

Now

$$\begin{aligned} t_1 &= \frac{wg_1 + vh_1}{f_1} = \frac{wf_1 + uh_1}{g_1} = \frac{vf_1 + ug_1}{h_1}, \\ t_2 &= \frac{wg_2 + vh_2}{f_2} = \frac{wf_2 + uh_2}{g_2} = \frac{vf_2 + ug_2}{h_2}, \\ t_3 &= \frac{wg_1 + vh_1}{f_1} = \frac{wf_1 + uh_1}{g_1} = \frac{vf_1 + ug_1}{h_1} = t_1, \\ t_4 &= \frac{w(\frac{t_2}{t_1})g_2 + v(\frac{t_2}{t_1})h_2}{(\frac{t_2}{t_1})f_2} = \frac{w(\frac{t_2}{t_1})f_2 + u(\frac{t_2}{t_1})h_2}{(\frac{t_2}{t_1})g_2} = \frac{v(\frac{t_2}{t_1})f_2 + u(\frac{t_2}{t_1})g_2}{(\frac{t_2}{t_1})h_2} = t_2. \end{aligned}$$

Consequently,

$$\mathcal{C}_U^{-2}(X) = \begin{pmatrix} f_1 + (\dfrac{t_2}{t_1})^2 t f_2 \\ g_1 + (\dfrac{t_2}{t_1})^2 t g_2 \\ h_1 + (\dfrac{t_2}{t_1})^2 t h_2 \end{pmatrix},$$

and, by induction,

$$\mathcal{C}_U^{-n}(X) = \begin{pmatrix} f_1 + (\frac{t_2}{t_1})^n t f_2 \\ g_1 + (\frac{t_2}{t_1})^n t g_2 \\ h_1 + (\frac{t_2}{t_1})^n t h_2 \end{pmatrix}. \tag{10}$$

Regarding the quotient $\frac{t_2}{t_1}$ in (10), if $\frac{t_2}{t_1} = 1$ then $\mathcal{C}_U^{-n}(X)$ is invariant of n, which is to say that X is a fixed point. If $\frac{t_2}{t_1} = -1$, then $\mathcal{C}_U^{-2}(X) = X$, which is to say that X is a fixed point of the collineation $\mathcal{C}_U^{-2}$. If $\left|\frac{t_2}{t_1}\right| \neq 1$, we call the line F_1F_2 a *regular fixed line*, and in this case, by (10), $\lim_{n\to\infty} \mathcal{C}_U^{-n}(X)$ is F_1 or F_2, according as $\left|\frac{t_2}{t_1}\right| < 1$ or $\left|\frac{t_2}{t_1}\right| > 1$. We summarize these findings as Lemma 3.

Lemma 3. *If X lies on a regular fixed line of $\mathcal{C}_U^{-1}$ (or equivalently, a regular fixed line of $\mathcal{C}_U$), then the sequence of points $\mathcal{C}_U^{-n}(X)$ (or equivalently, the points $\mathcal{C}_U^{n}(X)$) converges to a fixed point of $\mathcal{C}_U^{-1}$ (and of $\mathcal{C}_U$).*

Next, suppose that P is an arbitrary point in the plane of ABC. We shall show that $\mathcal{C}_U^{-n}(P)$ converges to a fixed point. Let F_1, F_2, F_3 be distinct fixed points. Define

$$P_2 = PF_2 \cap F_1F_3, \quad P_3 = PF_3 \cap F_1F_2 \quad P^{(0)} = \mathcal{C}_U^{-1}(P);$$
$$P_2^{(n)} = \mathcal{C}^{-n}(P_2) \text{ and } P_3^{(n)} = \mathcal{C}^{-n}(P_3) \text{ for } n = 1, 2, 3, \ldots$$

The collineation $\mathcal{C}_U^{-1}$ maps the line F_2P to the line $F_2P^{(0)}$, which is also the line $F_2P_2^{(1)}$ because F_2, P, P_2 are collinear; likewise, $\mathcal{C}_U^{-1}$ maps the line F_3P to the line $F_3P_3^{(1)}$. Consequently,

$$P^{(0)} = F_2P_2^{(1)} \cap F_3P_3^{(1)},$$

and by induction,

$$\mathcal{C}_U^{-n}(P) = F_2P_2^{(n)} \cap F_3P_3^{(n)}. \tag{10}$$

By Lemma 3,

$$\lim_{n\to\infty} \mathcal{C}_U^{-n}(P_2) \quad \text{and} \quad \lim_{n\to\infty} \mathcal{C}_U^{-n}(P_3)$$

are fixed points, so that by (10),

$$\lim_{n\to\infty} \mathcal{C}_U^{-n}(P)$$

must also be a fixed point. This completes a proof of the following theorem.

Theorem 4. *Suppose that the fixed lines of $\mathcal{C}_U^{-1}$ (or, equivalently, of $\mathcal{C}_U$) are regular. Then for every point X, the sequence of points $\mathcal{C}_U^{-n}(X)$ (or equivalently, the sequence $\mathcal{C}_U^{n}(X)$) converges to a fixed point of $\mathcal{C}_U^{-1}$(and of $\mathcal{C}_U$).*

Example 6. Extending Example 3, the three fixed lines, X_6X_{2574}, X_6X_{2575}, $X_{2574}X_{2575}$ are regular. The points X_{2574} and X_{2575} are the isogonal conjugates of the points X_{1113} and X_{1114} in which the Euler line intersects the circumcircle. Thus, the line $X_{2574}X_{2575}$ is the line at infinity. Because X_{1113} and X_{1114} are antipodal points on the circumcircle, the lines X_6X_{2574} are X_6X_{2575} are perpendicular (proof indicated at (x) below).

While visiting the author in February, 2007, Peter Moses analyzed the configuration in Example 6. His findings are given here.

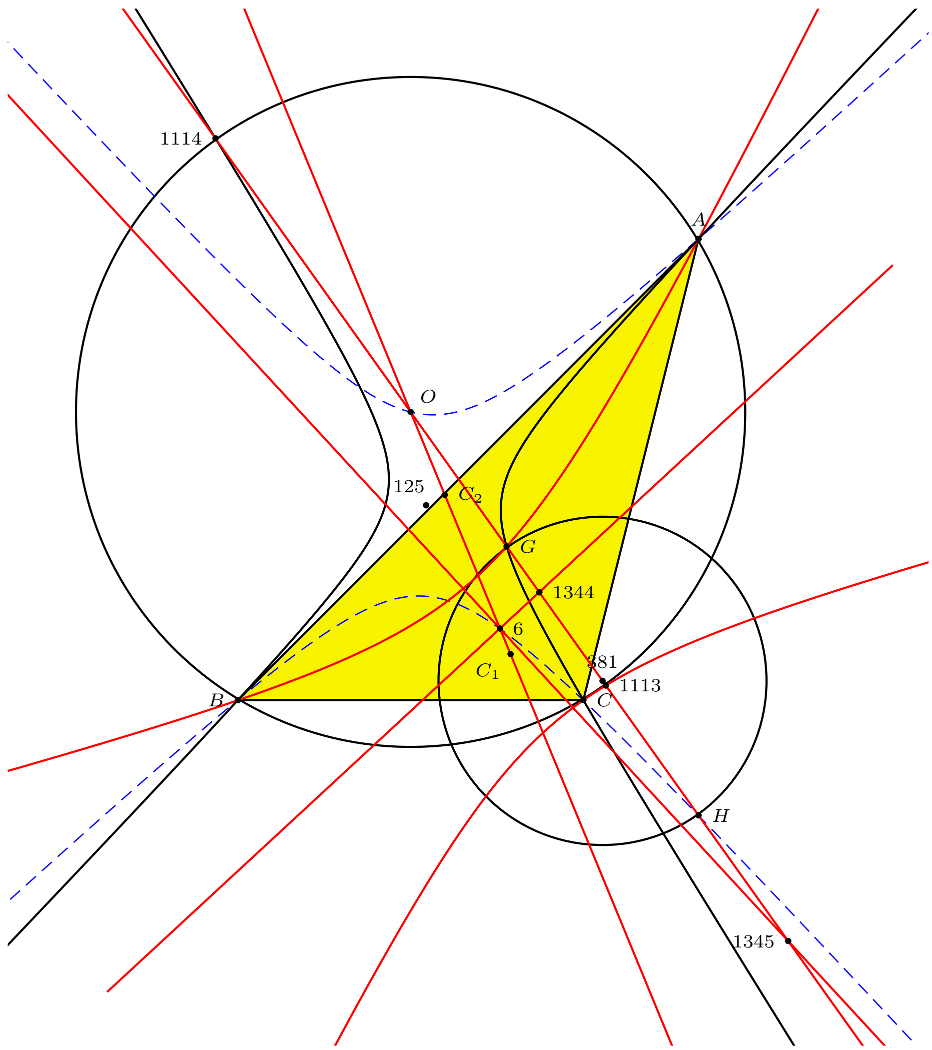

Figure 1.

(i) A point on line X_6X_{2574} is X_{1344}; a point on X_6X_{2575} is X_{1345}.

(ii) Segment GH (in Figure 1) is the diameter of the orthocentroidal circle, with center X_{381}. The points X_{1344} and X_{1345} are the internal and external centers of similitude of the orthocentroidal circle and the circumcircle.

(iii) Line GH, the Euler line, passes through the points

$$O,\ X_{1113},\ X_{1114},\ X_{1344},\ X_{1345}.$$

(iv) X_{125} is the center of the Jerabek hyperbola, which is the isogonal conjugate of the Euler line. (As isogonal conjugacy is a function, one may speak of its image when applied to lines as well as individual points).

(v) The line through X_{125} parallel to line X_6X_{1344} is the Simson line of X_{1114}, and the line through X_{125} parallel to line X_6X_{1345} is the Simson line of X_{1113}.

(vi) Hyperbola $ABCGX_{1113}$, with center C_1, is the isogonal conjugate of the $\mathcal{C}_U$-fixed line X_6X_{2574}, and hyperbola $ABCGX_{1114}$, with center C_2, is the isogonal conjugate of the $\mathcal{C}_U$-fixed line X_6X_{2575}.

(vii) C_1 is the barycentric square of X_{2575}, and C_2 is the barycentric square of X_{2574}.

(viii) The perspectors of the hyperbolas $ABCGX_{1113}$ and $ABCGX_{1114}$ are X_{2575} and X_{2574}, respectively. The fact that these perspectors are at infinity implies that the two conics, $ABCGX_{1113}$ and $ABCGX_{1114}$, are indeed hyperbolas.

(ix) The midpoint of the points C_1 and C_2 is the point $X_3X_6 \cap X_2X_{647}$.

(x) Line X_6X_{2574} is parallel to the Simson line of X_{1114}, and line X_6X_{2575} is parallel to the Simson line of X_{1113}. The two Simson lines are perpendicular ([1, p. 207]), so that the $\mathcal{C}_U$-fixed lines X_6X_{2574} and X_6X_{2575} are perpendicular.

(xi) The circle that passes through the points X_6, X_{1344}, and X_{1345} also passes through the point X_{2453}, which is the reflection of X_6 in the Euler line. This circle is a member of the coaxal family of the circumcircle, the nine-point circle, and the orthocentroidal circle.

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Forum Geometricorum
Volume 7 (2007) 169–180.

FORUM GEOM
ISSN 1534-1178

On a Product of Two Points Induced by Their Cevian Triangles

Cosmin Pohoata and Paul Yiu

Abstract. The intersections of the corresponding sidelines of the cevian triangles of two points P_0 and P_1 form the anticevian triangle of a point $T(P_0,\ P_1)$. We prove a number of interesting results relating the pair of inscribed conics with perspectors (Brianchon points) P_0 and P_1, in particular, a simple description of the fourth common tangent of the conics. We also show that the corresponding sides of the cevian triangles of points are concurrent if and only if the points lie on a circumconic. A characterization is given of circumconics whose centers lie on the cevian circumcircles of points on them (Brianchon-Poncelet theorem). We also construct a number of new triangle centers with very simple coordinates.

1. Introduction

A famous problem in triangle geometry [8] asks to show that the corresponding sidelines of the orthic triangle, the intouch triangle, and the cevian triangle of the incenter are concurrent.

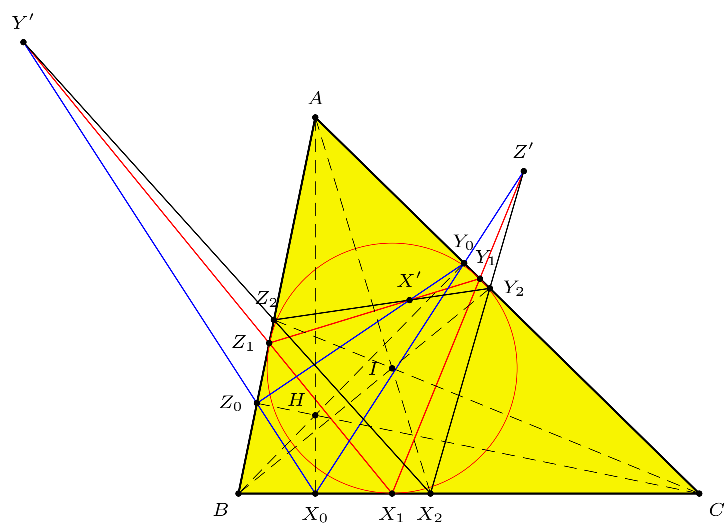

Figure 1.

Given a triangle ABC with orthic triangle $X_0Y_0Z_0$ and intouch triangle $X_1Y_1Z_1$, let

$$X' = Y_0Z_0 \cap Y_1Z_1, \qquad Y' = Z_0X_0 \cap Z_1X_1, \qquad Z' = X_0Y_0 \cap X_1Y_1.$$

Publication Date: November 14, 2007. Communicating Editor: Jean-Pierre Ehrmann.
We thank Jean-Pierre Ehrmann for his excellent comments leading to improvements of this paper, especially in pointing us to the classic references of Brianchon-Poncelet and Gergonne.

Emelyanov and Emelyanova [2] have proved the following interesting theorem. If XYZ is an inscribed triangle (with X, Y, Z on the sidelines BC, CA, AB respectively, and Y' on XZ and Z' on XY), then the circle through X, Y, Z also passes through the Feuerbach point, the point of tangency of the incircle with the nine-point circle of triangle ABC.

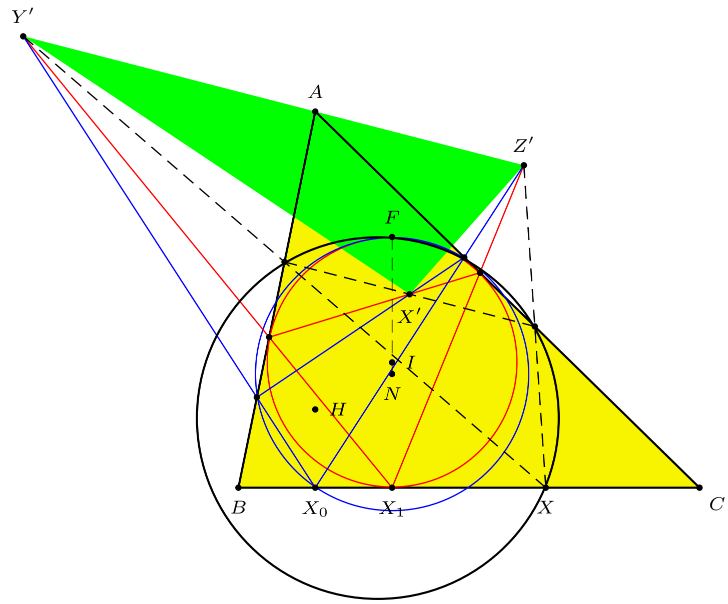

Figure 2.

In this note we study a general situation which reveals more of the nature of these theorems. By showing that the intersections of the corresponding sidelines of the cevian triangles of two points P_0 and P_1 form the anticevian triangle of a point $T(P_0,\ P_1)$, we prove a number of interesting results relating the pair of inscribed conics with perspectors (Brianchon points) P_0 and P_1. Proposition 5 below shows that the corresponding sidelines of the cevian triangles of three points are concurrent if and only if the three points lie on a circumconic. We characterize such circumconics whose centers lie on the cevian circumcircles of points on them (Proposition 9).

We shall work with homogeneous barycentric coordinates with reference to triangle ABC, and make use of standard notations of triangle geometry. A basic reference is [10]. Except for the commonest ones, triangle centers are labeled according to [7].

2. A product induced by two cevian triangles

Let $P_0 = (u_0 : v_0 : w_0)$ and $P_1 = (u_1 : v_1 : w_1)$ be two given points, with cevian triangles $X_0Y_0Z_0$ and $X_1Y_1Z_1$ respectively. The intersections

$$X' = Y_0Z_0 \cap Y_1Z_1, \qquad Y' = Z_0X_0 \cap Z_1X_1, \qquad Z' = X_0Y_0 \cap X_1Y_1$$

are the vertices of the anticevian triangle of a point with homogeneous barycentric coordinates

$$\left(u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right) : v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right) : w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)\right) \tag{1}$$

$$=\left(u_1\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right) : v_1\left(\frac{w_1}{w_0}-\frac{u_1}{u_0}\right) : w_1\left(\frac{u_1}{u_0}-\frac{v_1}{v_0}\right)\right). \tag{2}$$

That these two sets of coordinates should represent the same point is quite clear geometrically. They define a product of P_0 and P_1 which clearly lies on the trilinear polars of P_0 and P_1. This product is therefore the intersection of the trilinear polars of P_0 and P_1. We denote this product by $T(P_0,\ P_1)$.

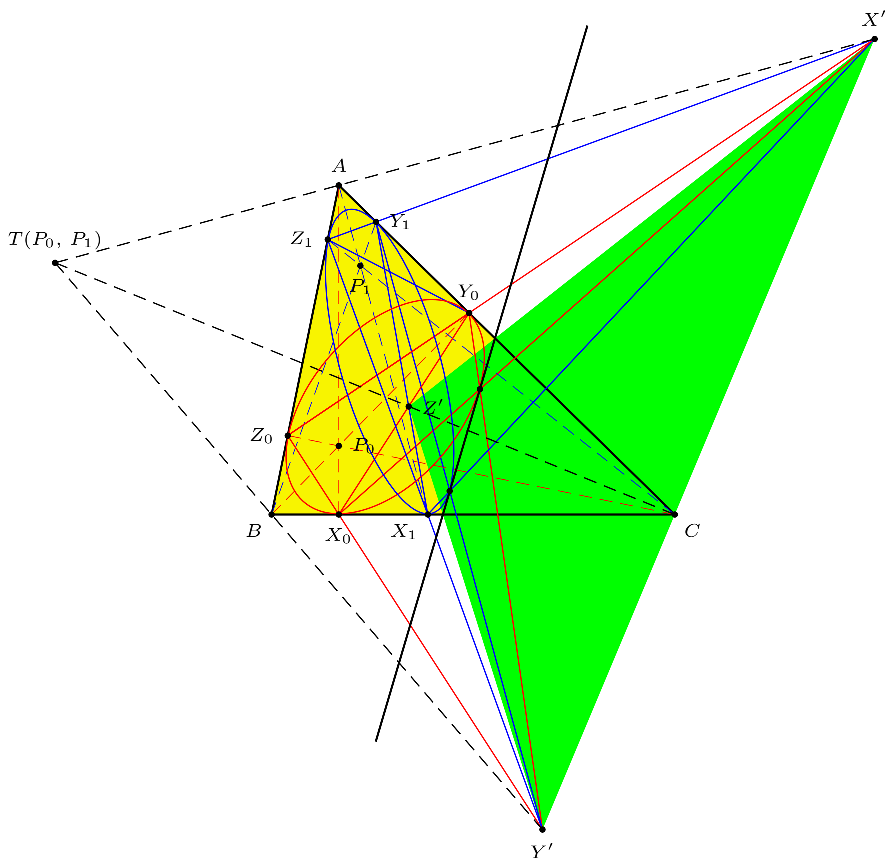

Figure 3.

The point $T(P_0,\ P_1)$ is also the perspector of the circumconic through P_0 and P_1. In particular, if P_0 and P_1 are both on the circumcircle, then $T(P_0,\ P_1) = K$, the symmedian point.

Proposition 1. *Triangle $X'Y'Z'$ is perspective to*
(i) *triangle $X_0Y_0Z_0$ at the point*

$$P_0/(T(P_0,\ P_1)) = \left(u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)^2 : v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)^2 : w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)^2\right),$$

(ii) *triangle $X_1Y_1Z_1$ at the point*

$$P_1/(T(P_0,\ P_1)) = \left(u_1\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)^2 : v_1\left(\frac{w_1}{w_0}-\frac{u_1}{u_0}\right)^2 : w_1\left(\frac{u_1}{u_0}-\frac{v_1}{v_0}\right)^2\right).$$

Proof. Since $X'Y'Z'$ is an anticevian triangle, the perspectivity is clear in each case by the cevian nest theorem (see [10, §8.3] and [4, p.165, Supp. Exercise 7]). The perspectors are the cevian quotients $P_0/(T(P_0,\ P_1))$ and $P_1/(T(P_0,\ P_1))$. We need only consider the first case.

$$\begin{aligned}
&P_0/(T(P_0,\ P_1))\\
=&\left(u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)\left(-\frac{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}{u_0}+\frac{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}{v_0}+\frac{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}{w_0}\right)\right.\\
&\quad : v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)\left(\frac{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}{u_0}-\frac{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}{v_0}+\frac{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}{w_0}\right)\\
&\quad \left. : w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)\left(\frac{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}{u_0}+\frac{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}{v_0}-\frac{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}{w_0}\right)\right)\\
=&\left(u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)^2 :\ v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)^2 :\ w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)^2\right).
\end{aligned}$$

□

Remark. See Proposition 12 for another triangle whose sidelines contain the points X', Y', Z'.

The conic with perspector P_0 has equation

$$\frac{x^2}{u_0^2}+\frac{y^2}{v_0^2}+\frac{z^2}{w_0^2}-\frac{2yz}{v_0w_0}-\frac{2zx}{w_0u_0}-\frac{2xy}{u_0v_0}=0,$$

and each point on the conic is of the form $(u_0p^2 : v_0q^2 : w_0r^2)$ for $p+q+r=0$. From this it is clear that $P_0/(T(P_0,\ P_1))$ lies on the inscribed conic with perspector P_0. Similarly, $P_1/(T(P_0,\ P_1))$ lies on that with perspector P_1.

Proposition 2. *The line joining $P_0/(T(P_0,\ P_1))$ and $P_1/(T(P_0,\ P_1))$ is the trilinear polar of $T(P_0,\ P_1)$ with respect to triangle ABC. It is also the (fourth) common tangent of the two inscribed conics with perspectors P_0 and P_1.* (See Figure 3).

Proof. For the first part it is enough to verify that $P_0/(T(P_0,\ P_1))$ lies on the said trilinear polar:

$$\frac{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)^2}{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}+\frac{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)^2}{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}+\frac{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)^2}{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}=0.$$

Note that the coordinates of $T(P_0,\ P_1)$ are given by both (1) and (2). Interchanging the subscripts 0's and 1's shows that the trilinear polar of $T(P_0,\ P_1)$ also contains the point $P_1/(T(P_0,\ P_1))$.

The inscribed conic with perspector P_0 is represented by the matrix

$$\begin{pmatrix} \frac{1}{u_0^2} & \frac{-1}{u_0v_0} & \frac{-1}{u_0w_0} \\ \frac{-1}{u_0v_0} & \frac{1}{v_0^2} & \frac{-1}{v_0w_0} \\ \frac{-1}{u_0w_0} & \frac{-1}{v_0w_0} & \frac{1}{w_0^2} \end{pmatrix}.$$

The tangent at the point $P_0/(T(P_0,\ P_1))$ has line coordinates

$$\begin{pmatrix} \frac{1}{u_0^2} & \frac{-1}{u_0v_0} & \frac{-1}{u_0w_0} \\ \frac{-1}{u_0v_0} & \frac{1}{v_0^2} & \frac{-1}{v_0w_0} \\ \frac{-1}{u_0w_0} & \frac{-1}{v_0w_0} & \frac{1}{w_0^2} \end{pmatrix}\begin{pmatrix} u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)^2 \\ v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)^2 \\ w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)^2 \end{pmatrix}=\begin{pmatrix} \frac{2}{u_0}\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right) \\ \frac{2}{v_0}\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right) \\ \frac{2}{w_0}\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right). \end{pmatrix}$$

This is the line

$$\frac{x}{u_0\left(\frac{v_0}{v_1}-\frac{w_0}{w_1}\right)}+\frac{y}{v_0\left(\frac{w_0}{w_1}-\frac{u_0}{u_1}\right)}+\frac{z}{w_0\left(\frac{u_0}{u_1}-\frac{v_0}{v_1}\right)}=0,$$

which is the trilinear polar of $T(P_0,\ P_1)$. Interchanging the subscripts 0's and 1's, we note that the same line is also the tangent at the point $P_1/(T(P_0,\ P_1))$of the inscribed conics with perspector P_1. It is therefore the common tangent of the two conics. □

Proposition 3. *The triangle $X'Y'Z'$ is self polar with respect to each of the inscribed conics with perspectors P_0 and P_1.*

Proof. Since $X_1Y_1Z_1$ is a cevian triangle and $X'Y'Z'$ is an anticevian triangle with respect to ABC, we have

$$(Y'Z_0, Y'A, Y'Y_0, Y'C) = (Y'Z_0, Y'A, Y'Y_0, Y'X') = -1.$$

Therefore, Y' lies on the polar of X' with respect to the inscribed conic with perspector P_0. Similarly, Z' also lies on the polar of X'. It follows that $Y'Z'$ is the polar of X'. For the same reason, $Z'X'$ and $X'Y'$ are the polars of Y' and Z' respectively. This shows that triangle $X'Y'Z'$ is self-polar with respect to the inscribed conic with perspector P_0. The same is true with respect to the inscribed conic with perspector P_1. □

In the case of the incircle (with $P_0 = X_7$), we have the following interesting result.

Corollary 4. *For an arbitrary point Q, the anticevian triangle $X_7 * Q$ has orthocenter I.*

We present some examples of $T(P_0,\ P_1)$.

	G	O	H	K	Ge	N_a	E
I	X_{513}	X_{652}	X_{650}	X_{649}	X_{650}	X_{650}	X_{2245}
G		X_{520}	X_{523}	X_{512}	X_{514}	X_{522}	X_{511}
O			X_{647}	X_{647}			
H				X_{647}	X_{650}	X_{650}	X_{3003}
K					X_{665}	X_{187}	
G_e						X_{650}	X_{3002}

Remarks. (1) X_{3002} is the intersection of the Brocard axis and the trilinear polar of the Gergonne point. It has coordinates

$$(a^2(a^3(b^2+c^2)-a^2(b+c)(b-c)^2-a(b^4+c^4)+(b+c)(b-c)^2(b^2+c^2)) : \cdots : \cdots).$$

(2) X_{3003} is the intersection of the orthic and Brocard axes. It has coordinates

$$(a^2(a^4(b^2+c^2)-2a^2(b^4-b^2c^2+c^4)+(b^2-c^2)^2(b^2+c^2)) : \cdots : \cdots).$$

The center of the rectangular hyperbola through E is X_{113}, the inferior of X_{74} on the the circumcircle.

Here are some examples of cevian products with very simple coordinates. They do not appear in the current edition of [7].

P_0	P_1	first barycentric coordinate of $T(P_0,\ P_1)$
G	X_9	$(a(b-c)(b+c-a)^2$
G	X_{56}	$(a^2(b-c)(a(b+c)+b^2+c^2)$
O	X_{21}	$(a^3(b-c)(b+c-a)(b^2+c^2-a^2)^2$
O	X_{55}	$(a^3(b-c)(b+c-a)^2(b^2+c^2-a^2)$
O	X_{56}	$(a^3(b-c)(b^2+c^2-a^2)$
K	N_a	$(a(b-c)(b+c-a)(a(b+c)+b^2+c^2)$
K	X_{99}	$(a^2(a^2(b^2+c^2)-2b^2c^2)$
G_e	X_{56}	$\frac{a^2(b^2-c^2)}{b+c-a}$
G_e	X_{57}	$\frac{a(b-c)}{b+c-a}$
N_a	X_{55}	$(a^2(b^2-c^2)(b+c-a)$
X_{21}	X_{55}	$(a^3(b-c)(b+c-a)$
X_{56}	X_{57}	$\frac{a^2(b-c)}{b+c-a}$

Remark. $T(X_{21},\ X_{55}) = T(X_{21},\ X_{56}) = T(X_{55},\ X_{56})$.

3. Inscribed triangles which circumscribe a given anticevian triangle

Proposition 5. *Let P be a given point with anticevian triangle $X'Y'Z'$. If XYZ is an inscribed triangle of ABC (with X, Y, Z on the sidelines BC, CA, AB respectively) such that X', Y', Z' lie on the lines YZ, ZX, XY respectively,* i.e.,

$X'Y'Z'$ is an inscribed triangle of XYZ, then XYZ is the cevian triangle of a point on the circumconic with perspector P.

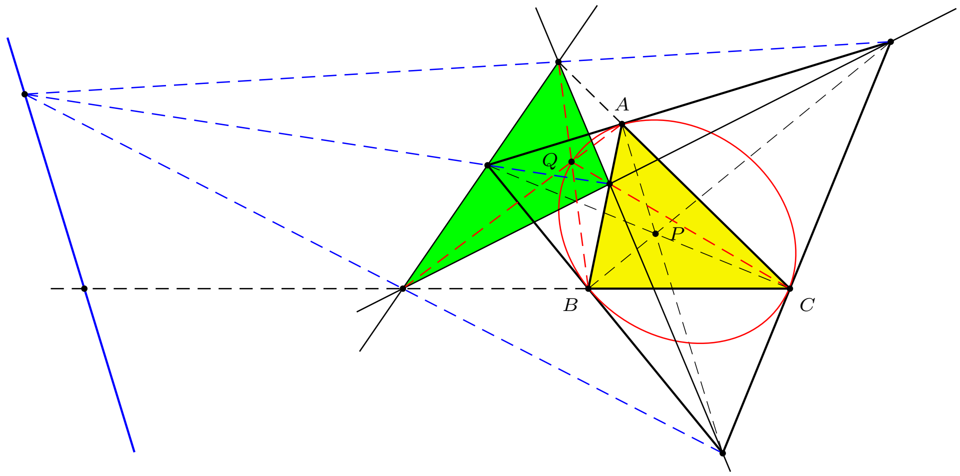

Figure 4.

Proof. Let $P = (u : v : w)$ so that

$$X' = (-u : v : w), \qquad Y' = (u : -v : w), \qquad Z' = (u : v : -w).$$

Since XYZ is an inscribed triangle of ABC,

$$X = (0 : t_1 : 1), \quad Y = (1 : 0 : t_2), \quad Z = (t_3 : 1 : 0),$$

for real numbers t_1, t_2, t_3. Here we assume that X, Y, Z do not coincide with the vertices of ABC. Since the lines YZ, ZX, XY contain the points respectively, we have

$$\begin{aligned} t_2u + t_2t_3v + w &= 0, \\ u + t_3v + t_3t_1w &= 0, \\ t_1t_2u + v + t_1w &= 0. \end{aligned}$$

From these,

$$0 = \begin{vmatrix} t_2 & t_2t_3 & 1 \\ 1 & t_3 & t_3t_1 \\ t_1t_2 & 1 & t_1 \end{vmatrix} = (t_1t_2t_3 - 1)^2.$$

It follows from the Ceva theorem that the lines AX, BY, CZ are concurrent. The inscribed triangle XYZ is the cevian triangle of a point $(p : q : r)$. The three collinearity conditions all reduce to

$$uqr + vrp + wpq = 0.$$

This means that $(p : q : r)$ lies on the circumconic with perspector $(u : v : w)$. $\square$

Proposition 6. *The locus of the perspector of the anticevian triangle of P and the cevian triangle of a point Q on the circumconic with perspector P is the trilinear polar of P.*

Proof. Let $Q = (u : v : w)$ be a point on the circumconic. The perspector is the cevian quotient

$$\left(p\left(-\frac{p}{u}+\frac{q}{v}+\frac{r}{w}\right) : \ q\left(\frac{p}{u}-\frac{q}{v}+\frac{r}{w}\right) : r\left(\frac{p}{u}+\frac{q}{v}-\frac{r}{w}\right)\right)$$
$$= (p(-pvw+qwu+ruv) : \ q(pvw-qwu+ruv) : \ r(pvw+qwu-ruv)) \, .$$

Since $pvw + qwu + ruv = 0$, this simplifies into $(p^2vw : q^2wu : r^2uv)$, which clearly lies on the line $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=0$, the trilinear polar of P. □

4. Brianchon-Poncelet theorem

For $P_0 = H$, the orthocenter, and $P_1 = X_7$, the Gergonne point, we have $T(P_0, P_1) = X_{650}$. The circumconic through P_0 and P_1 is

$$a(b-c)(b+c-a)yz + b(c-a)(c+a-b)zx + c(a-b)(a+b-c)xy = 0,$$

the Feuerbach conic, which is the isogonal conjugate of the line OI, and has center at the Feuerbach point

$$X_{11} = ((b-c)^2(b+c-a) : \ (c-a)^2(c+a-b) : \ (a-b)^2(a+b-c)).$$

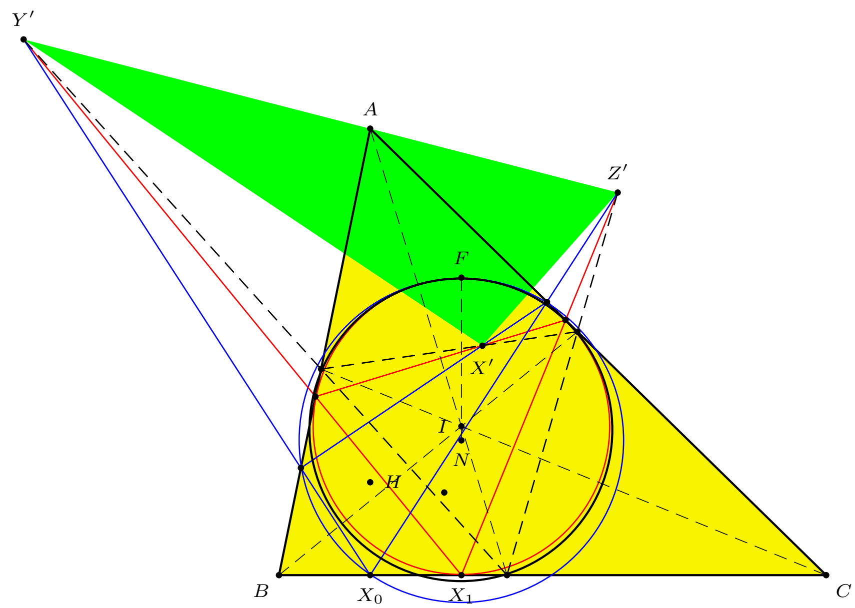

Figure 5.

The theorem of Emelyanov and Emelyanova therefore can be generalized as follows: *the the cevian circumcircle of a point on the Feuerbach hyperbola contains the Feuerbach point*. This in turn is a special case of a celebrated theorem of Brianchon and Poncelet in 1821.

Theorem 7 (Brianchon-Poncelet [1])**.** *Given a point P, the cevian circumcircle of an arbitrary point on the rectangular circum-hyperbola through P (and the orthocenter H) contains the center of the hyperbola which is on the nine-point circle of the reference triangle.*

At the end of their paper Brianchon and Poncelet made a remarkable conjecture about the locus of the centers of conics through four given points. This was subsequently proved by J. D. Gergonne [6].

Theorem 8 (Brianchon-Poncelet-Gergonne)**.** *The locus of the centers of conics through four given points in general positions in a plane is a conic through*
(i) the midpoints of the six segments joining them and
(ii) the intersections of the three pairs of lines joining them two by two.

Proposition 9. *The cevian circumcircle of a point on a nondegenerate circumconic contains the center of the conic if and only if the conic is a rectangular hyperbola.*

Proof. (a) The sufficiency part follows from Theorem 7.

(b) For the converse, consider a nondegenerate conic through A, B, C, P whose center W lies on the cevian circumcircle of P. The locus of centers of conics through A, B, C, P is, by Theorem 8, a conic $\mathcal{C}$ through the traces of P on the sidelines of triangle ABC. The four common points of $\mathcal{C}$ and the cevian circumcircle of P are the traces of P and W. By (a), the cevian circumcircle of P contains the center of the rectangular circum-hyperbola through P, which must coincide with W. Therefore the conic in question in rectangular. □

Since the Feuerbach hyperbola contain the incenter I, we have the following result. See Figure 5.

Corollary 10. *The cevian circumcircle of the incenter contains the Feuerbach point.*

Applying Brianchon-Poncelet theorem to the Kiepert perspectors, we obtain the following interesting result.

Corollary 11. *Given triangle ABC, construct on the sides similar isosceles triangles BCX', CAY', and ABZ'. Let AX', BY', CZ' intersect BC, CA, AB at X, Y, Z respectively. The circle through X, Y, Z also contains the center X_{115} of the Kiepert hyperbola, which is also the midpoint between the two Fermat points.*

5. Second tangents to an inscribed conic from the traces of a point

Consider an inscribed conic $\mathcal{C}_0$ with Brianchon point $P_0 = (u_0 : v_0 : w_0)$, so that its equation is

$$\left(\frac{x}{u_0}\right)^2 + \left(\frac{y}{v_0}\right)^2 + \left(\frac{z}{w_0}\right)^2 - \frac{2yz}{v_0w_0} - \frac{2zx}{w_0u_0} - \frac{2xy}{u_0v_0} = 0.$$

Let $P_1 = (u_1 : v_1 : w_1)$ be a given point with cevian triangle $X_1Y_1Z_1$. The sidelines of triangle ABC are tangents from X_1, Y_1, Z_1 to the conic $\mathcal{C}_0$. From each of these points there is a second tangent to the conic. J.-P. Ehrmann [5] has computed the second points of tangency X_2, Y_2, Z_2, and concluded that the triangle $X_2Y_2Z_2$ is perspective with ABC at the point

$$\left(\frac{u_1^2}{u_0} : \frac{v_1^2}{v_0} : \frac{w_1^2}{w_0}\right).$$

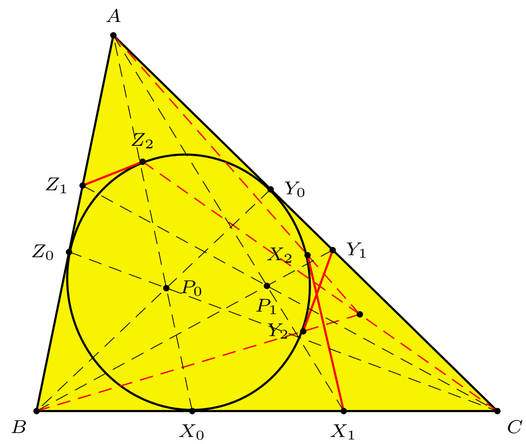

Figure 6.

More precisely, the coordinates of X_2, Y_2, Z_2 are as follows.

$$X_2 = \left(u_0\left(\frac{v_1}{v_0} - \frac{w_1}{w_0}\right)^2 : \frac{v_1^2}{v_0} : \frac{{w_1}^2}{w_0}\right),$$

$$Y_2 = \left(\frac{u_1^2}{u_0} : v_0\left(\frac{w_1}{w_0} - \frac{v_1}{v_0}\right)^2 : \frac{{w_1}^2}{w_0}\right),$$

$$Z_2 = \left(\frac{u_1^2}{u_0} : \frac{v_1^2}{v_0} : w_0\left(\frac{u_1}{u_0} - \frac{v_1}{v_0}\right)^2\right).$$

Proposition 12. *The lines Y_0Z_0, Y_1Z_1, Y_2Z_2 are concurrent; similarly for the triples Z_0X_0, Z_1X_1, Z_2X_2 and X_0Y_0, X_1Y_1, X_2Y_2.*

Proof. The line Y_2Z_2 has equation

$$u_1\left(\frac{x}{u_0}\left(-\frac{u_1}{u_0} + \frac{v_1}{v_0} + \frac{w_1}{w_0}\right) + \frac{y}{v_0}\left(\frac{u_1}{u_0} - \frac{v_1}{v_0} + \frac{w_1}{w_0}\right) + \frac{z}{w_0}\left(\frac{u_1}{u_0} + \frac{v_1}{v_0} - \frac{w_1}{w_0}\right)\right)$$
$$- \frac{2v_1w_1x}{v_0w_0} = 0.$$

With

$$(x : y : z) = \left(-u_1\left(\frac{v_1}{v_0} - \frac{w_1}{w_0}\right) : v_1\left(\frac{w_1}{w_0} - \frac{u_1}{u_0}\right) : w_1\left(\frac{u_1}{u_0} - \frac{v_1}{v_0}\right)\right),$$

we have, apart from a factor u_1,

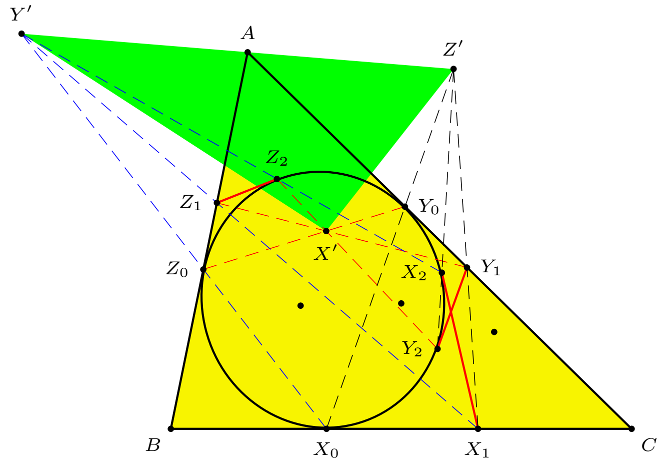

Figure 7.

$$
\begin{aligned}
&-\frac{u_1}{u_0}\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)\left(-\frac{u_1}{u_0}+\frac{v_1}{v_0}+\frac{w_1}{w_0}\right)+\frac{v_1}{v_0}\left(\frac{w_1}{w_0}-\frac{u_1}{u_0}\right)\left(\frac{u_1}{u_0}-\frac{v_1}{v_0}+\frac{w_1}{w_0}\right)\\
&+\frac{w_1}{w_0}\left(\frac{u_1}{u_0}-\frac{v_1}{v_0}\right)\left(\frac{u_1}{u_0}+\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)+\frac{2v_1w_1}{v_0w_0}\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)\\
=&-\frac{2v_1w_1}{v_0w_0}\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)+\frac{2v_1w_1}{v_0w_0}\left(\frac{v_1}{v_0}-\frac{w_1}{w_0}\right)\\
=&\,0.
\end{aligned}
$$

This shows that the line Y_2Z_2 contains the point $X' = Y_0Z_0 \cap Y_1Z_1$. $\square$

We conclude with some examples of the triangle centers from the inscribed conics with given perspectors P_0 and P_1. In the table below,

$$Q_{0,1} = \left(\frac{u_0^2}{u_1} : \frac{v_0^2}{v_1} : \frac{w_0^2}{w_1}\right) \quad \text{and} \quad Q_{1,0} = \left(\frac{u_1^2}{u_0} : \frac{v_1^2}{v_0} : \frac{w_1^2}{w_0}\right).$$

P_0	P_1	$T(P_0,\ P_1)$	$P_0/(T(P_0,\ P_1))$	$P_1/(T(P_0,\ P_1))$	$Q_{0,1}$	$Q_{1,0}$
G	H	X_{523}	X_{125}	X_{115}	X_{69}	X_{393}
G	K	X_{512}	Q_1	X_{1084}	X_{76}	X_{32}
G	G_e	X_{514}	X_{11}	X_{1086}	X_8	X_{279}
G	N_a	X_{522}	X_{11}	X_{1146}	X_7	X_{346}
H	K	X_{647}	Q_2	Q_3	X_{2052}	X_{184}
H	G_e	X_{650}	X_{3022}	Q_4	X_{1857}	Q_5
H	N_a	X_{650}	Q_6	Q_4	X_{1118}	X_{1265}
K	G_e	X_{665}	Q_7	Q_8	X_{2175}	Q_9
G_e	N_a	X_{650}	Q_6	X_{3022}	X_{479}	Q_{10}

The new triangle centers Q_i have simple coordinates given below.

Q_1	$a^2(b^2-c^2)^2$
Q_2	$a^2(b^2-c^2)^2(b^2+c^2-a^2)^2$
Q_3	$a^4(b^2-c^2)^2(b^2+c^2-a^2)^3$
Q_4	$a^2(b-c)^2(b+c-a)^2(b^2+c^2-a^2)$
Q_5	$\frac{b^2+c^2-a^2}{(b+c-a)^2}$
Q_6	$a^2(b-c)^2(b+c-a)$
Q_7	$a^4(b-c)^2(b+c-a)(a(b+c)-(b^2+c^2))^2$
Q_8	$a^2(b-c)^2(a(b+c)-(b^2+c^2))^2$
Q_9	$\frac{1}{a^2(b+c-a)^2}$
Q_{10}	$(b+c-a)^3$

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Forum Geometricorum
Volume 7 (2007) 181–185.

FORUM GEOM
ISSN 1534-1178

Steinhaus' Problem on Partition of a Triangle

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Abstract. H. Steinhaus has asked whether inside each acute triangle there is a point from which perpendiculars to the sides divide the triangle into three parts of equal areas. We present two solutions of Steinhaus' problem.

The n-dimensional case of Theorem 1 below was proved in [6], see also [2] and [4, Theorem 2.1, p. 152]. For an earlier mass-partition version of Theorem 1, for bounded convex masses in $\mathbb{R}^n$ and $r_1 = r_2 = ... = r_{n+1}$, see [7].

Theorem 1 (Kuratowski-Steinhaus)**.** *Let $\boldsymbol{T} \subseteq \mathbb{R}^2$ be a bounded measurable set, and let $|\boldsymbol{T}|$ be the measure of $\boldsymbol{T}$. Let α_1, α_2, α_3 be the angles determined by three rays emanating from a point, and let $\alpha_1 < \pi$, $\alpha_2 < \pi$, $\alpha_3 < \pi$. Let r_1, r_2, r_3 be nonnegative numbers such that $r_1 + r_2 + r_3 = |\boldsymbol{T}|$. Then there exists a translation $\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ such that $|\lambda(\boldsymbol{T}) \cap \alpha_1| = r_1$, $|\lambda(\boldsymbol{T}) \cap \alpha_2| = r_2$, $|\lambda(\boldsymbol{T}) \cap \alpha_3| = r_3$.*

H. Steinhaus asked ([10], [11]) whether *inside* each acute triangle there is a point from which perpendiculars to the sides divide the triangle into three parts with equal areas. Long and elementary solutions of Steinhaus' problem appeared in [8, pp. 101–104], [9, pp. 103–105], [12, pp. 133–138] and [13]. For some acute triangles with rational coordinates of vertices, the point solving Steinhaus' problem is not constructible with ruler and compass alone, see [15]. Following article [14], we will present two solutions of Steinhaus' problem.

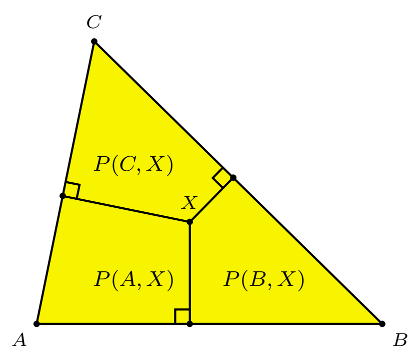

Figure 1

For $X \in \triangle ABC$, we denote by $P(A, X)$, $P(B, X)$, $P(C, X)$ the areas of the quadrangles containing vertices A, B, C respectively (see Figure 1). The areas

Publication Date: November 19, 2007. Communicating Editor: Paul Yiu.

$P(A, X)$, $P(B, X)$, $P(C, X)$ are continuous functions of X in the triangle ABC. The function

$$f(X) = \min\{P(A, X), P(B, X), P(C, X)\}$$

is also continuous. By Weierstrass' theorem f attains a maximum in triangle ABC, *i.e.*, there exists $X_0 \in \triangle ABC$ such that $f(X) \le f(X_0)$ for all $X \in \triangle ABC$.

Lemma 2. *For a point X lying on a side of an acute triangle, the area at the opposite vertex is greater than one of the remaining two areas.*

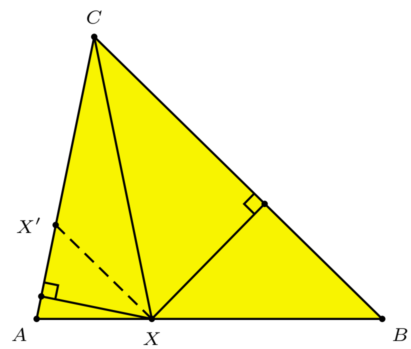

Figure 2

Proof. Without loss of generality, we may assume that $X \in \overline{AB}$ and $|AX| \le |BX|$, see Figure 2. Straight line XX' parallel to straight line BC cuts the triangle AXX' greater than $P(A, X)$ (as the angle ACB is acute), but not greater than the triangle CXX' because $|AX'| < \frac{|AC|}{2} < |X'C|$. Hence $P(A, X) < |\triangle AXX'| \le |\triangle CXX'| < P(C, X)$. $\square$

Theorem 3. *If a triangle ABC is acute and f attains a maximum at X_0, then $P(A, X_0) = P(B, X_0) = P(C, X_0) = \frac{|\triangle ABC|}{3}$.*

Proof. $f(A) = f(B) = f(C) = 0$, and 0 is not a maximum of f. Therefore X_0 is not a vertex of the triangle ABC. Let us assume that $f(X_0) = P(A, X_0)$. By Lemma 2, $X_0 \notin \overline{BC}$. Suppose, on the contrary, that some of the other areas, let's say $P(C, X_0)$, is greater than $P(A, X_0)$.

Case 1: $X_0 \notin \overline{AC}$. When shifting X_0 from the segment $\overline{AB}$ by appropriately small ε and perpendicularly to the segment $\overline{AB}$ (see Figure 3), we receive $P(C, X)$ further greater than $f(X_0)$ and at the same time $P(A, X) > P(A, X_0)$ and $P(B, X) > P(B, X_0)$. Hence $f(X) > f(X_0)$, a contradiction.

Case 2: $X_0 \in \overline{AC} \setminus \{A, C\}$. By Lemma 2,

$$\begin{aligned} P(B, X_0) &> \min\{P(A, X_0), P(C, X_0)\} \\ &\ge \min\{P(A, X_0), P(B, X_0), P(C, X_0)\} \\ &= f(X_0). \end{aligned}$$

When shifting X_0 from the segment $\overline{AC}$ by appropriately small ε and perpendicularly to the segment $\overline{AC}$ (see Figure 4), we receive $P(B, X)$ further greater than

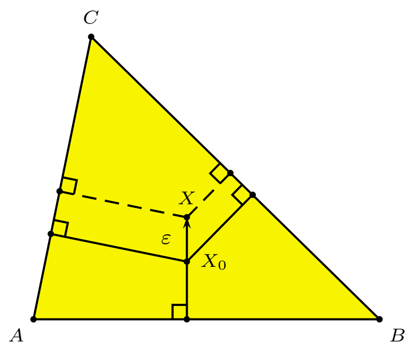

Figure 3

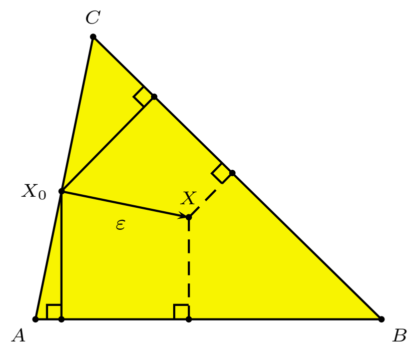

Figure 4

$f(X_0)$ and at the same time $P(A, X) > P(A, X_0)$ and $P(C, X) > P(C, X_0)$. Hence $f(X) > f(X_0)$, a contradiction. $\square$

For each acute triangle ABC there is a unique $X_0 \in \triangle ABC$ such that $P(A, X_0) = P(B, X_0) = P(C, X_0) = \frac{|\triangle ABC|}{3}$. Indeed, if $X \neq X_0$ then X lies in some of the quadrangles determined by X_0. Let us say that X lies in the quadrangle with vertex A (see Figure 5). Then $P(A, X) < P(A, X_0) = \frac{|\triangle ABC|}{3}$.

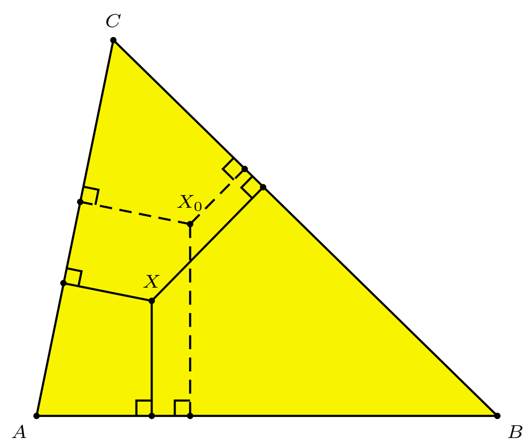

Figure 5.

The sets $R_A = \{X \in \triangle ABC : P(A, X) = f(X)\}$, $R_B = \{X \in \triangle ABC : P(B, X) = f(X)\}$ and $R_C = \{X \in \triangle ABC : P(C, X) = f(X)\}$ are closed and cover the triangle ABC. Assume that the triangle ABC is acute. By Lemma 2, $R_A \cap \overline{BC} = \emptyset$, $R_B \cap \overline{AC} = \emptyset$, and $R_C \cap \overline{AB} = \emptyset$. The theorem proved in [5] guarantees that $R_A \cap R_B \cap R_C \neq \emptyset$, see also [4, item D4, p. 101] and [1, item 2.23, p. 162]. Any point belonging to $R_A \cap R_B \cap R_C$ lies inside the triangle ABC and determines the partition of the triangle ABC into three parts with equal areas.

The above proof remains valid for all right triangles, because the hypothesis of Lemma 2 holds for all right triangles. For each triangle the following statements are true.

(1) There is a unique point in the plane which determines the partition of the triangle into three equal areas.
(2) The point of partition into three equal areas lies inside the triangle if and only if the hypothesis of Lemma 2 holds for the triangle.
(3) The point of partition into three equal areas lies inside the triangle if and only if the maximum of f on the boundary of the triangle is smaller than the maximum of f on the whole triangle. For each acute or right triangle ABC, the maximum of f on the boundary does not exceed $\frac{|\triangle ABC|}{4}$.
(4) The point of partition into three equal areas lies inside the triangle, if the triangle has two angles in the interval $\left(\arctan\frac{1}{\sqrt{2}}, \frac{\pi}{2}\right]$. This condition holds for each acute or right triangle.
(5) If the point of partition into three equal areas lies inside the triangle, then it is a partition into quadrangles.

Assume now $C > \frac{\pi}{2}$. The point of partition into three equal areas lies inside the triangle if and only if

$$\sqrt{(1+\tan^2 A)\tan B} + \sqrt{(1+\tan^2 B)\tan A} > \sqrt{3(\tan A + \tan B)}.$$

If, on the other hand,

$$\sqrt{(1+\tan^2 A)\tan B} + \sqrt{(1+\tan^2 B)\tan A} = \sqrt{3(\tan A + \tan B)},$$

then the unique $X_0 \in \overline{AB}$ such that

$$|AX_0| = \sqrt{\frac{(1+\tan^2 A)\tan B}{3(\tan A + \tan B)}}|AB|, \qquad |BX_0| = \sqrt{\frac{(1+\tan^2 B)\tan A}{3(\tan A + \tan B)}}|AB|$$

determines the partition of the triangle ABC into three equal areas. It is a partition into a triangle with vertex A, and a triangle with vertex B, and a quadrangle. Finally, when

$$\sqrt{(1+\tan^2 A)\tan B} + \sqrt{(1+\tan^2 B)\tan A} < \sqrt{3(\tan A + \tan B)}, \qquad (*)$$

there is a straight line a perpendicular to the segment $\overline{AC}$ which cuts from the triangle ABC a figure with the area $\frac{|\triangle ABC|}{3}$ (see Figure 6). There is a straight line b perpendicular to the segment $\overline{BC}$ which cuts from the triangle ABC a figure with the area $\frac{|\triangle ABC|}{3}$. By $(*)$, the intersection point of the straight lines a and b lies outside the triangle ABC. This point determines the partition of the triangle ABC into three equal areas.

J.-P. Ehrmann [3] has subsequently found a constructive solution of a generalization of Steinhaus' problem of partitioning a given triangle into three quadrangles with prescribed proportions.

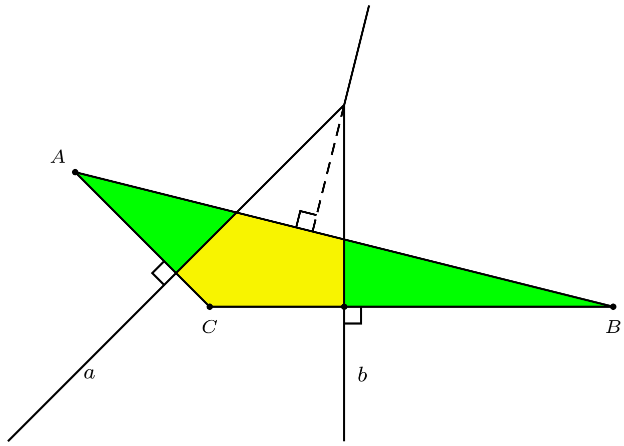

Figure 6.

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Forum Geometricorum
Volume 7 (2007) 187–190.

FORUM GEOM
ISSN 1534-1178

Constructive Solution of a Generalization of Steinhaus' Problem on Partition of a Triangle

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Abstract. We present a constructive solution to a generalization of Hugo Steinhaus' problem of partitioning a given triangle, by dropping perpendiculars from an interior point, into three quadrilaterals whose areas are in prescribed proportions.

1. Generalized Steinhaus problem

Given an acute angled triangle ABC, Steinhaus' problem asks a point P in its interior with pedals P_a, P_b, P_c on BC, CA, AB such that the quadrilaterals AP_bPP_c, BP_cP_a, and CP_aPP_b have equal areas. See [3] and the bibliographic information therein. A. Tyszka [2] has also shown that Steinhaus' problem is in general not soluble by ruler-and-compass. We present a simple constructive solution (using conics) of a generalization of Steinhaus' problem. In this note, the area of a polygon $\mathscr{P}$ will be denoted by $\Delta(\mathscr{P})$. In particular, $\Delta = \Delta(ABC)$. Thus, given three positive real numbers u, v, w, we look for the point(s) P such that
(1) P is inside ABC and P_a, P_b, P_c lie respectively in the segments BC, CA, AB,
(2) $\Delta(AP_bPP_c) : \Delta(BP_cPP_a) : \Delta(CP_aPP_b) = u : v : w$.

We do not require the triangle to be acute-angled.

Lemma 1. *Consider a point P inside the angular sector bounded by the half-lines AB and AC, with projections P_b and P_c on AC and AB respectively. For a positive real number k, $\Delta(AP_bPP_c) = k \cdot \Delta(ABC)$ if and only if P lies on the rectangular hyperbola with center A, focal axis the internal bisector AI, and semi-major axis $\sqrt{kbc}$.*

Proof. We take A for pole and the bisector AI for polar axis; let $(\rho,\ \theta)$ be the polar coordinates of P. As $AP_b = \rho\cos\left(\frac{A}{2} - \theta\right)$ and $PP_b = \rho\sin\left(\frac{A}{2} - \theta\right)$, we have $\Delta(APP_b) = \frac{1}{2}\rho^2\sin(A - 2\theta)$. Similarly, $\Delta(AP_cP) = \frac{1}{2}\rho^2\sin(A + 2\theta)$. Hence the quadrilateral AP_bPP_c has area $\frac{1}{2}\rho^2\sin A\cos 2\theta$. Therefore,

$$\Delta(AP_bPP_c) = k \cdot \Delta(ABC) \Longleftrightarrow \rho^2\cos 2\theta = \frac{2k \cdot \Delta(ABC)}{\sin A} = kbc.$$

□

Publication Date: November 21, 2007. Communicating Editor: Paul Yiu.

Theorem 2. *Let U be the point with barycentric coordinates $(u : v : w)$ and M_1, M_2, M_3 be the antipodes on the circumcircle Γ of ABC of the points whose Simson lines pass through U and P the incenter of the triangle $M_1M_2M_3$. If P verifies* (1), *then P is the unique solution of our problem. Otherwise, the generalized Steinhaus problem has no solution.*

Remarks. (a) Of course, if ABC is acute angled, and P inside ABC, then (1) will be verified.

(b) As U lies inside the Steiner deltoid, there exist three real Simson lines through U; so M_1, M_2, M_3 are real and distinct.

(c) Let h_A be the rectangular hyperbola with center A, focal axis AI, and semi-major axis $\sqrt{\dfrac{u}{u+v+w}} \cdot bc$, and define rectangular hyperbolas h_B and h_C analogously.

If P verifies (1), it will verify (2) if and only if $P \in h_A \cap h_B$. In this case, $P \in h_C$, and the solutions of our problem are the common points of h_A, h_B, h_C verifying (1).

(d) The four common points P_1, P_2, P_3, P_4 (real or imaginary) of the rectangular hyperbolae h_A, h_B, h_C form an orthocentric system. As h_A, h_B, h_C are centered respectively at A, B, C, any conic through P_1, P_2, P_3, P_4 is a rectangular hyperbola with center on Γ. As the vertices of the diagonal triangle of this orthocentric system are the centers of the degenerate conics through P_1, P_2, P_3, P_4 , they lie on Γ.

(e) We will see later that P_1, P_2, P_3, P_4 are always real.

2. Proof of Theorem 2

If P has homogeneous barycentric coordinates $(x : y : z)$ with reference to triangle ABC, then

$$(x+y+z)^2\Delta(APP_b) = y\left(z + \frac{b^2+c^2-a^2}{2b^2}y\right)\Delta,$$
$$(x+y+z)^2\Delta(AP_cP) = z\left(y + \frac{b^2+c^2-a^2}{2c^2}z\right)\Delta,$$

where $\Delta = \Delta(ABC)$. Hence the barycentric equation of h_A is

$$h_A(x,y,z) := \frac{b^2+c^2-a^2}{2}\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) + 2\,y\,z - \frac{u}{u+v+w}\,(x+y+z)^2 = 0.$$

We get h_B and h_C by cyclically permuting a, b, c; u, v, w; x, y, z.

If $M = (x : y : z)$ is a vertex of the diagonal triangle of $P_1P_2P_3P_4$, it has the same polar line (the opposite side) with respect to the three conics h_A, h_B, h_C. Hence,

$$\frac{\partial h_B}{\partial y}\frac{\partial h_C}{\partial z} - \frac{\partial h_B}{\partial z}\frac{\partial h_C}{\partial y} = \frac{\partial h_C}{\partial z}\frac{\partial h_A}{\partial x} - \frac{\partial h_C}{\partial x}\frac{\partial h_A}{\partial z} = \frac{\partial h_A}{\partial x}\frac{\partial h_B}{\partial y} - \frac{\partial h_A}{\partial y}\frac{\partial h_B}{\partial x} = 0.$$

Let N be the reflection of M in the circumcenter O; $N_aN_bN_c$ the pedal triangle of N. Clearly, N_a, N_b, N_c are the reflections of the vertices of the pedal triangle of M in the midpoints of the corresponding sides of ABC. Now, N_b and N_c have coordinates

$$(b^2 + c^2 - a^2)y + 2b^2z : \ 0 : \ (a^2 + b^2 - c^2)y + 2b^2x$$

and

$$(b^2 + c^2 - a^2)z + 2c^2y : \ (c^2 + a^2 - b^2)z + 2c^2x : \ 0$$

respectively. A straightforward computation shows that

$$\det[N_b, \ N_c, \ U] = b^2c^2\,(u + v + w)\left(\frac{\partial h_B}{\partial y}\frac{\partial h_C}{\partial z} - \frac{\partial h_B}{\partial z}\frac{\partial h_C}{\partial y}\right) = 0.$$

Similarly, $\det[N_c, \ N_a, \ U] = \det[N_a, \ N_b, \ U] = 0$. It follows that N lies on the circumcircle (we knew that already by Remark (d)), and the Simson line of N passes through U.

Hence, $M_1M_2M_3$ is the diagonal triangle of the orthocentric system $P_1P_2P_3P_4$, which means that $P_1P_2P_3P_4$ are real and are the incenter and the three excenters of $M_1M_2M_3$.

As the three excenters of a triangle lie outside his circumcircle, the incenter of $M_1M_2M_3$ is the only common point of h_A, h_B, h_C inside Γ. This completes the proof of Theorem 2.

3. Constructions

In [1], the author has given a construction of the points on the circumcircle whose Simson line pass through a given point. Let U^- and U^+ be the complement and the anticomplement of U, *i.e.*, the images of U under the homotheties $\mathsf{h}\left(G, -\frac{1}{2}\right)$ and $\mathsf{h}(G, -2)$ respectively. Since

$$(\text{Reflection in } O) \ \circ (\text{Translation by } \overrightarrow{HU}) = \text{Reflection in } U^-,$$

if h_0 is the reflection in U^- of the rectangular circumhyperbola through U, and M_4 the antipode of U^+ on h_0, then M_1, M_2, M_3, M_4 are the four common points of h_0 and the circumcircle.

In the case $u = v = w = 1$, h_0 is the reflection in the centroid G of the Kiepert hyperbola of ABC. It intersects the circumcircle Γ at M_1, M_2, M_3 and the Steiner point of ABC. See Figure 1.

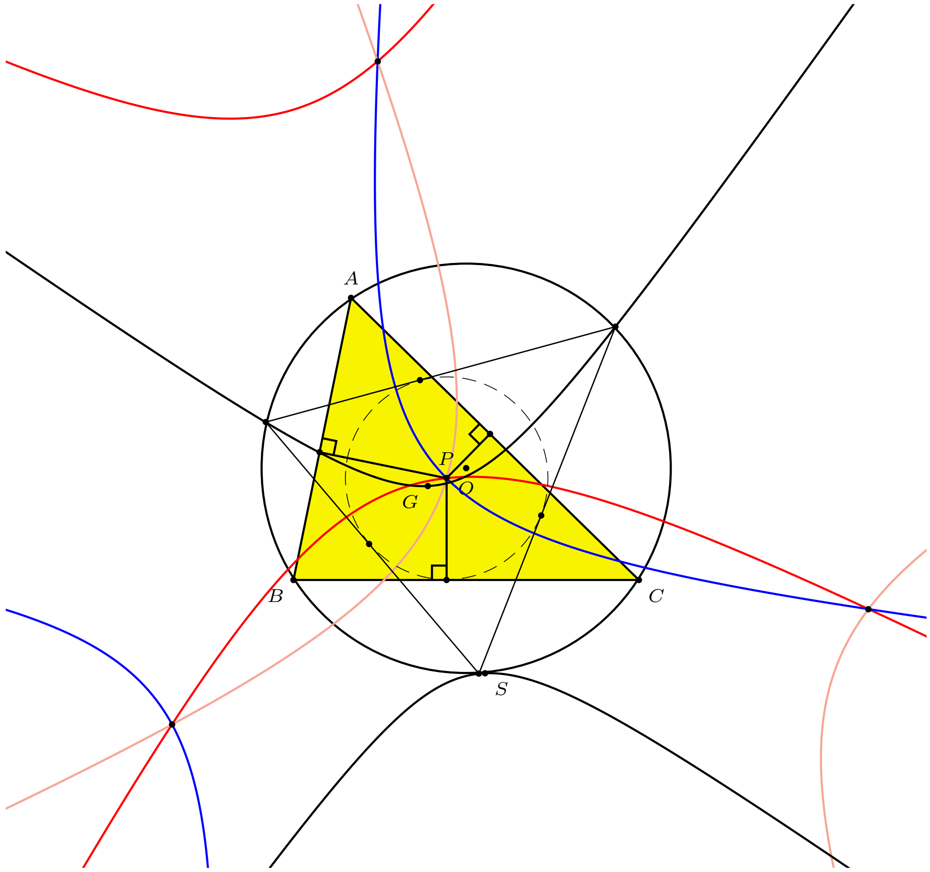

Figure 1.

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Forum Geometricorum
Volume 7 (2007) 191–197.

FORUM GEOM
ISSN 1534-1178

The Soddy Circles

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Abstract. Given three circles externally tangent to each other, we investigate the construction of the two so called Soddy circles, that are tangent to the given three circles. From this construction we get easily the formulas of the radii and the barycentric coordinates of Soddy centers relative to the triangle ABC that has vertices the centers of the three given circles.

1. Construction of Soddy circles

In the general Apollonius problem it is known that, given three arbitrary circles with noncollinear centers, there are at most 8 circles tangent to each of them. In the special case when three given circles are tangent externally to each other, there are only two such circles. These are called the inner and outer Soddy circles respectively of the given circles. Let the mutually externally tangent circles be $\mathscr{C}_a(A, r_1)$, $\mathscr{C}_b(B, r_2)$, $\mathscr{C}_c(C, r_3)$, and A_1, B_1, C_1 be their tangency points (see Figure 1).

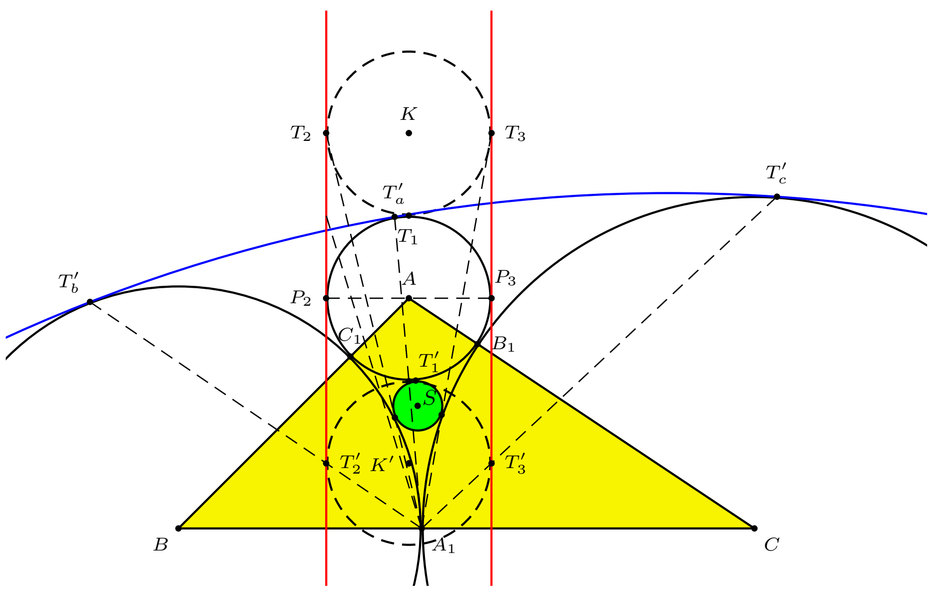

Figure 1.

Consider the inversion τ with pole A_1 that maps $\mathscr{C}_a$ to $\mathscr{C}_a$. This also maps the circles $\mathscr{C}_b$, $\mathscr{C}_c$ to the two lines perpendicular to BC and tangent to $\mathscr{C}_a$ at the points P_2, P_3 where P_2P_3 is parallel from A to BC. The only circles tangent to $\mathscr{C}_a$ and to the above lines are the circles $K(T_1)$, $K'(T_1')$ where T_1, T_1' are lying on $\mathscr{C}_a$ and

Publication Date: November 26, 2007. Communicating Editor: Paul Yiu.

the A-altitude of ABC. These circles are the images, in the above inversion, of the Soddy circles we are trying to construct. Since the circle $K(T_1)$ must be the inverse of the inner Soddy circle, the lines A_1T_1, A_1T_2, A_1T_3, $(P_2T_2 = P_3T_3 = P_2P_3)$ meet $\mathscr{C}_a, \mathscr{C}_b, \mathscr{C}_c$ at the points T_a, T_b, T_c respectively, that are the tangency points of the inner Soddy circle. Hence the lines BT_b and CT_c give the center S of the inner Soddy circle. Similarly the lines A_1T_1', A_1T_2', A_1T_3', $(P_2T_2' = P_3T_3' = P_2P_3)$, meet $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ at the points T_a', T_b', T_c' respectively, that are the tangency points of the outer Soddy circle. Triangles $T_aT_bT_c$, $T_a'T_b'T_c'$ are the inner and outer Soddy triangles. A construction by the so called Soddy hyperbolas can be found in [5, §12.4.2].

2. The radii of Soddy circles

If the sidelengths of ABC are a, b, c, and $s = \frac{1}{2}(a+b+c)$, then

$$\begin{array}{lll} a = r_2 + r_3, & b = r_3 + r_1, & c = r_1 + r_2; \\ r_1 = s - a, & r_2 = s - b, & r_3 = s - c. \end{array}$$

If $\triangle$ is the area of ABC, then $\triangle = \sqrt{r_1r_2r_3(r_1+r_2+r_3)}$. The A-altitude of ABC is $AD = h_a = \frac{2\triangle}{a}$, and the inradius is $r = \frac{\triangle}{r_1+r_2+r_3}$.

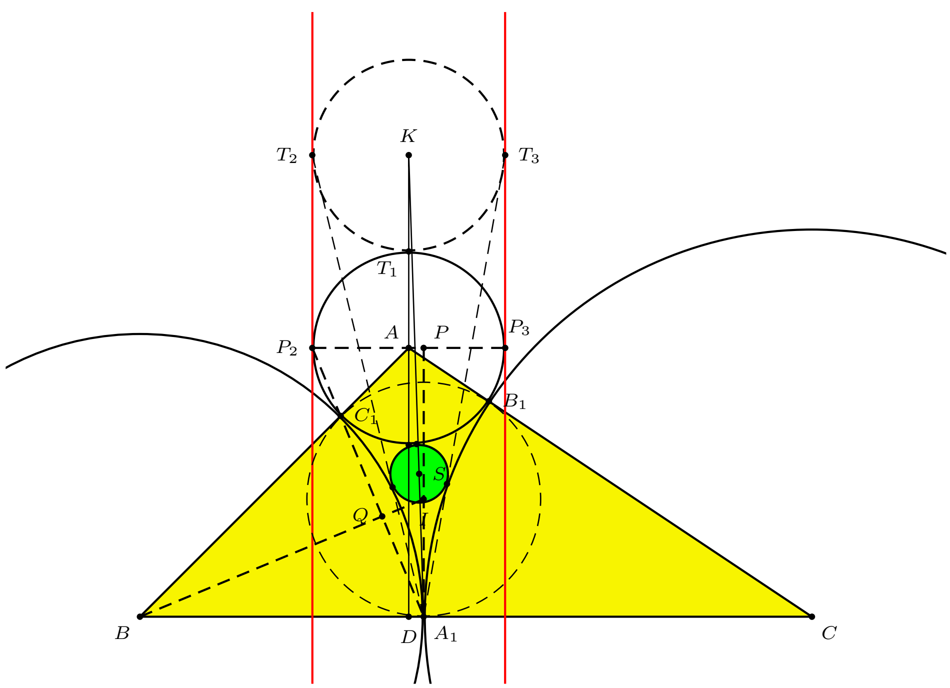

Figure 2.

The points A_1, B_1, C_1 are the points of tangency of the incircle $I(r)$ of ABC with the sidelines. If A_1P is perpendicular to P_2P_3 and IB meets A_1C_1 at Q, then the inversion τ maps C_1 to P_2, and the quadrilateral IQP_2P is cyclic (see Figure 2). The power of the inversion is

$$d^2 = A_1C_1 \cdot A_1P_2 = 2A_1Q \cdot A_1P_2 = 2A_1I \cdot A_1P = 2rh_a = \frac{4r_1r_2r_3}{r_2 + r_3}. \qquad (1)$$

2.1. *Inner Soddy circle.* Since the inner Soddy circle is the inverse of the circle $K(r_1)$, its radius is given by

$$x = \frac{d^2}{A_1K^2 - r_1^2} \cdot r_1. \qquad (2)$$

In triangle A_1AK, $A_1K^2 - A_1A^2 = 2AK \cdot T_1D = 4r_1(r_1 + h_a)$. Hence,

$$A_1K^2 - r_1^2 = A_1A^2 - r_1^2 + 4r_1(r_1 + h_a) = d^2 + 4r_1(r_1 + h_a),$$

and from (1), (2),

$$x = \frac{r_1r_2r_3}{r_2r_3 + r_3r_1 + r_1r_2 + 2\triangle}. \qquad (3)$$

Here is an alternative expression for x. If r_a, r_b, r_c are the exradii of triangle ABC, and R its circumradius, it is well known that

$$r_a + r_b + r_c = 4R + r.$$

See, for example, [4, §2.4.1]. Now also that $r_1r_a = r_2r_b = r_3r_c = \triangle$. Therefore,

$$\begin{aligned} x =& \frac{r_1r_2r_3}{r_2r_3 + r_3r_1 + r_1r_2 + 2\triangle} \\ =& \frac{\triangle}{\frac{\triangle}{r_1} + \frac{\triangle}{r_2} + \frac{\triangle}{r_3} + 2 \cdot \frac{\triangle^2}{r_1r_2r_3}} \\ =& \frac{\triangle}{r_a + r_b + r_c + 2(r_1 + r_2 + r_3)} \\ =& \frac{\triangle}{4R + r + 2s}. \end{aligned} \qquad (4)$$

As a special case, if $r_1 \to \infty$, then the circle $\mathscr{C}_a$ tends to a common tangent of $\mathscr{C}_b$, $\mathscr{C}_c$, and

$$\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{r_2}} + \frac{1}{\sqrt{r_3}}. \qquad (5)$$

In this case the outer Soddy circle degenerates into the common tangent of $\mathscr{C}_b$ and $\mathscr{C}_c$.

2.2. *Outer Soddy circle.* If $\mathscr{C}_a$ is the smallest of the three circles $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ and is greater than the circle of (5), *i.e.*, $\frac{1}{\sqrt{r_1}} < \frac{1}{\sqrt{r_2}} + \frac{1}{\sqrt{r_3}}$, then the outer Soddy circle is internally tangent to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$. Otherwise, the outer Soddy circle is externally tangent to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$.

Since the outer Soddy circle is the inverse of the circle $K'(r_1)$, its radius is given by

$$x' = \frac{d^2}{A_1K'^2 - r_1^2} \cdot r_1. \qquad (6)$$

This is a signed radius and is negative when A_1 is inside the circle $K'(r_1)$ or when the outer Soddy circle is tangent internally to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$. In triangle A_1AK', $A_1A^2 - A_1K'^2 = 2AK' \cdot T_1'D = 4r_1(h_a - r_1)$, and from (6),

$$x' = \frac{r_1r_2r_3}{r_1r_2 + r_2r_3 + r_3r_1 - 2\triangle}. \tag{7}$$

Analogous to (4) we also have

$$x' = \frac{\triangle}{4R + r - 2s}. \tag{8}$$

Hence this radius is negative, equivalently, the outer Soddy circle is tangent internally to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$, when $4R + r < 2s$. From (4) and (8), we have

$$\frac{1}{x} - \frac{1}{x'} = \frac{2s}{\triangle} = \frac{4}{r}.$$

If $4R + r = 2s$, then $x = \frac{r}{4}$.

3. The barycentric coordinates of Soddy centers

3.1. *The Inner Soddy center.* If d_1 is the distance of the inner Soddy circle center S from BC, then since A_1 is the center of similitute of the inner Soddy circle and the circle $K(r_1)$ we have $\frac{d_1}{KD} = \frac{x}{r_1}$, or

$$d_1 = \frac{x(2r_1 + h_a)}{r_1} = 2x\left(1 + \frac{h_a}{2r_1}\right) = 2x\left(1 + \frac{\triangle}{a(s-a)}\right).$$

Similarly we obtain the distances d_2, d_3 from S to the sides CA and AB respectively. Hence the homogeneous barycentric coordinates of S are

$$(ad_1 : bd_2 : cd_3) = \left(a + \frac{\triangle}{s-a} : \ b + \frac{\triangle}{s-b} : \ c + \frac{\triangle}{s-c}\right).$$

The inner Soddy center S appears in [3] as the triangle center X_{176}, also called the equal detour point. It is obvious that for the Inner Soddy center S, the "detour" of triangle SBC is

$$SB + SC - BC = (x + r_2) + (x + r_3) - (r_2 + r_3) = 2x.$$

Similarly the triangles SCA and SAB also have detours $2x$. Hence the three incircles of triangles SBC, SCA, SAB are tangent to each other and their three tangency point A_2, B_2, C_2 are the points T_a, T_b, T_c on the inner Soddy circle [1] since $SA_2 = SB_2 = SC_2 = x$. See Figure 3.

Working with absolute barycentric coordinates, we have

$$\begin{aligned} S =& \frac{\left(a + \frac{\triangle}{s-a}\right)A + \left(b + \frac{\triangle}{s-b}\right)B + \left(c + \frac{\triangle}{s-c}\right)C}{a + \frac{\triangle}{s-a} + b + \frac{\triangle}{s-b} + c + \frac{\triangle}{s-c}} \\ =& \frac{(a+b+c)I + \triangle\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right)G_{\mathrm{e}}}{\frac{\triangle}{x}}, \end{aligned} \tag{9}$$

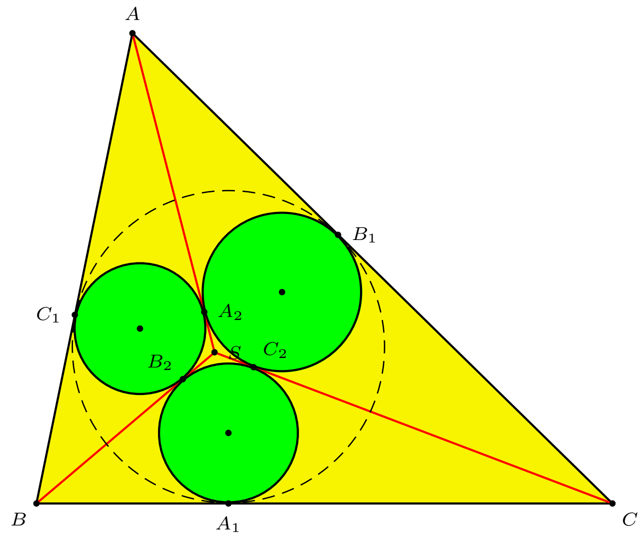

Figure 3.

where $G_{\rm e} = \left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c}\right)$ is the Gergonne point. Hence, the inner Soddy center S lies on the line connecting the incenter I and $G_{\rm e}$. This explains whey $IG_{\rm e}$ is called the Soddy line. Indeed, S divides $IG_{\rm e}$ in the ratio

$$IS : SG_{\rm e} = r_a + r_b + r_c : a + b + c = 4R + r : 2s.$$

3.2. *The outer Soddy center.* If d_1' is the distance of the outer Soddy circle center S' from BC, then since A_1 is the center of similitute of the outer Soddy circle and the circle $K'(r_1)$, a similar calculation referring to Figure 1 shows that

$$d_1' = -2x\left(1 - \frac{\triangle}{a(s-a)}\right).$$

Similarly, we have the distances d_2' and d_3' from S' to CA and AB respectively. The homogeneous barycentric coordinates of S' are

$$(ad_1' : bd_2' : cd_3') = \left(a - \frac{\triangle}{s-a} : b - \frac{\triangle}{s-b} : c - \frac{\triangle}{s-c}\right).$$

This is the triangle center X_{175} of [3], called the isoperimetric point. It is obvious that if the outer Soddy circle is tangent internally to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ or $4R+r < 2s$, then the perimeter of triangle $S'BC$ is

$$S'B + S'C + BC = (x' - r_2) + (x' - r_3) + (r_2 + r_3) = 2x'.$$

Similarly the perimeters of triangles $S'CA$ and $S'AB$ are also $2x'$. Therefore the S'-excircles of triangles $S'BC$, $S'CA$, $S'AB$ are tangent to each other at the tangency points T_a', T_b', T_c' of the outer Soddy circle with $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$.

If the outer Soddy circle is tangent externally to $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$, equivalently, $4R + r > 2s$, then the triangles $S'BC$, $S'CA$, $S'AB$ have equal detours $2x'$ because for triangle $S'BC$,

$$S'B + S'C - BC = (x' + r_2) + (x' + r_3) - (r_2 + r_3) = 2x',$$

and similarly for the other two triangles. In this case, S' is second equal detour point. Analogous to (9), we have

$$S' = \frac{(a+b+c)I - \triangle\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right) G_{\mathrm{e}}}{\frac{\triangle}{x'}}. \tag{10}$$

A comparison of (9) and (10) shows that S and S' are harmonic conjugates with respect to IG_{e}.

4. The barycentric equations of Soddy circles

We find the barycentric equation of the inner Soddy circle in the form

$$a^2yz + b^2zx + c^2xy - (x+y+z)(p_1x + p_2y + p_3z) = 0,$$

where p_1, p_2, p_3 are the powers of A, B, C with respect to the circle. See [5, Proposition 7.2.3]. It is easy to see that

$$\begin{aligned} p_1 =& r_1(r_1 + 2x) = (s-a)(s-a+2x), \\ p_2 =& r_2(r_2 + 2x) = (s-b)(s-b+2x), \\ p_3 =& r_3(r_3 + 2x) = (s-c)(s-c+2x). \end{aligned}$$

Similarly, the barycentric equation of the outer Soddy circle is

$$a^2yz + b^2zx + c^2xy - (x+y+z)(q_1x + q_2y + q_3z) = 0,$$

where

$$\begin{aligned} q_1 =& (s-a)(s-a+2x'), \\ q_2 =& (s-b)(s-b+2x'), \\ q_3 =& (s-c)(s-c+2x'), \end{aligned}$$

where x' is the *signed* radius of the circle given by (8), treated as negative when $2s > 4R + r$.

5. The Soddy triangles and the Eppstein points

The incenter I of ABC is the radical center of the circles $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$. The inversion with respect to the incircle leaves each of $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ invariant and swaps the inner and outer Soddy circles. In particular, it interchanges the points of tangency T_a and T'_a; similarly, T_b and T'_b, T_c and T'_c. The Soddy triangles $T_aT_bT_c$ and $T'_aT'_bT'_c$ are clearly perspective at the incenter I. They are also perspective with ABC, at S and S' respectively. Since $AT_a : T_aS = r_1 : x$, we have, $T_a = \frac{xA + r_1S}{x + r_1}$. In homogeneous barycentric coordinates,

$$T_a = \left(a + \frac{2\triangle}{r_1} : b + \frac{\triangle}{r_2} : c + \frac{\triangle}{r_3}\right).$$

Since the intouch point A_1 has coordinates $\left(0 : \frac{1}{r_2} : \frac{1}{r_3}\right)$, the line T_aA_1 clearly contains the point

$$E = \left(a + \frac{2\triangle}{r_1} : b + \frac{2\triangle}{r_2} : c + \frac{2\triangle}{r_3}\right).$$

Similarly, the lines T_bB_1 and T_cC_1 also contain the same point E, which is therefore the perspector of the triangles $T_aT_bT_c$ and the intouch triangle. This is the Eppstein pont X_{481} in [3]. See also [2]. It is clear that E also lies on the Soddy line. See Figure 4.

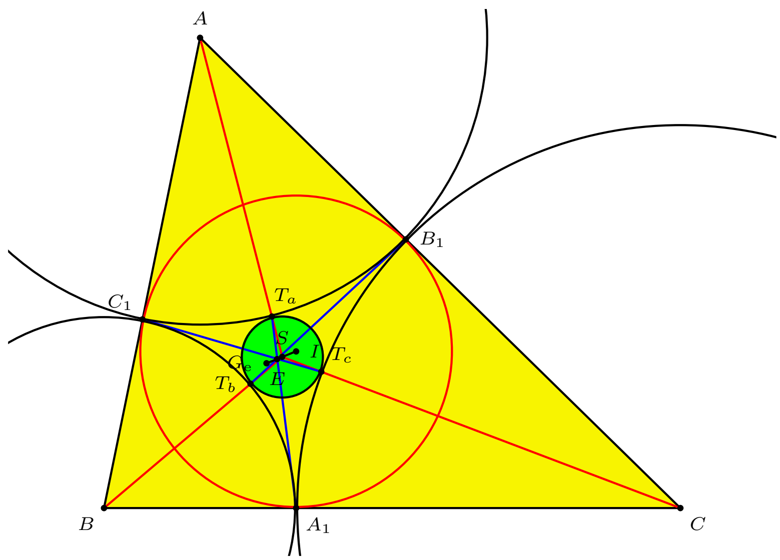

Figure 4.

The triangle $T'_aT'_bT'_c$ is also perspective with the intouch triangle, at a point

$$E' = \left(a - \frac{2\triangle}{r_1} : b - \frac{2\triangle}{r_2} : c - \frac{2\triangle}{r_3}\right),$$

on the Soddy line, dividing with E the segment IG_e harmonically. This is the second Eppstein point X_{482} of [3].

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Forum Geometricorum
Volume 7 (2007) 199–206.

FORUM GEOM
ISSN 1534-1178

Cyclic Quadrilaterals with Prescribed Varignon Parallelogram

Michel Bataille

Abstract. We prove that the vertices of a given parallelogram $\mathcal{P}$ are the midpoints of the sides of infinitely many *cyclic* quadrilaterals and show how to construct such quadrilaterals. Then we discuss some of their properties and identify related loci. Lastly, the cases when $\mathcal{P}$ is a rectangle or a rhombus are examined.

1. Introduction

The following well-known theorem of elementary geometry, attributed to the French mathematician Pierre Varignon (1654-1722), was published in 1731: if A, B, C, D are four points in the plane, the respective midpoints P, Q, R, S of AB, BC, CD, DA are the vertices of a parallelogram. We will say that $PQRS$ is the Varignon parallelogram of $ABCD$, in short $PQRS = \mathcal{V}(ABCD)$. In a converse way, given a parallelogram $\mathcal{P}$, there exist infinitely many quadrilaterals $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. In §2, we offer a quick review of this general result, introducing the diagonal midpoints of $ABCD$ which are of constant use afterwards. The primary result of this paper, namely that infinitely many of these quadrilaterals $ABCD$ are cyclic, is proved in §3 and the proof leads naturally to a construction of such quadrilaterals. Further results, including a simpler construction, are established in §4, all centering on a rectangular hyperbola determined by $\mathcal{P}$. Finally, §5 is devoted to particular results that hold if $\mathcal{P}$ is either a rectangle or a rhombus.

In what follows, $\mathcal{P} = PQRS$ denotes a parallelogram whose vertices are not collinear. The whole work takes place in the plane of $\mathcal{P}$.

2. Quadrilaterals $ABCD$ with $\mathcal{P} = \mathcal{V}(ABCD)$

The construction of a quadrilateral $ABCD$ satisfying $\mathcal{P} = \mathcal{V}(ABCD)$ is usually presented as follows: start with an arbitrary point A and construct successively the symmetric B of A about P, the symmetric C of B about Q and the symmetric D of C about R (see Figure 1). Because $\mathcal{P}$ is a parallelogram, A is automatically the symmetric of D about S and $ABCD$ is a solution (see [1, 2]).

Let M, M' be the midpoints of the diagonals of $ABCD$ (in brief, the diagonal midpoints of $ABCD$) and let O be the center of $\mathcal{P}$. Since $4O = 2P + 2R = A + B + C + D = 2M + 2M'$, the midpoint of MM' is O. This simple property allows another construction of $ABCD$ from $\mathcal{P}$ that will be preferred in the next sections: start with two points M, M' symmetric about O; then obtain A, C such that $\overrightarrow{AM} = \overrightarrow{PQ} = \overrightarrow{MC}$ and B, D such that $\overrightarrow{BM'} = \overrightarrow{QR} = \overrightarrow{M'D}$. Exchanging the roles of M, M' provides another solution $A'B'C'D'$ with the same set $\{M, M'\}$ of diagonal

Publication Date: December 3, 2007. Communicating Editor: Paul Yiu.

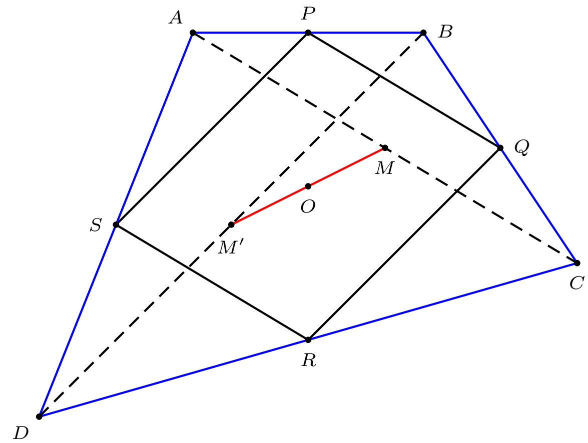

Figure 1

midpoints (see Figure 2). Clearly, $ABCD$ and $A'B'C'D'$ are symmetrical about O.

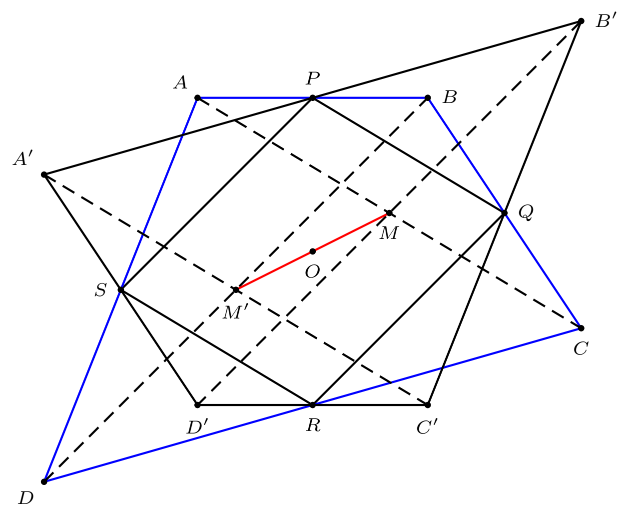

Figure 2

3. Cyclic quadrilaterals $ABCD$ with $\mathcal{P} = \mathcal{V}(ABCD)$

The previous section has brought out the role of diagonal midpoints when looking for quadrilaterals $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. We characterize the diagonal midpoints of *cyclic* solutions and show how to construct them from $\mathcal{P}$, obtaining the following theorem.

Theorem 1. *Given $\mathcal{P}$, there exist infinitely many cyclic quadrilaterals $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. Such quadrilaterals can be constructed from $\mathcal{P}$ by ruler and compass.*

Proof. Consider a Cartesian system with origin at O and x-axis parallel to PQ (see Figure 3). The affix of a point Z is denoted by z. For example, $q-p$ is a real number.

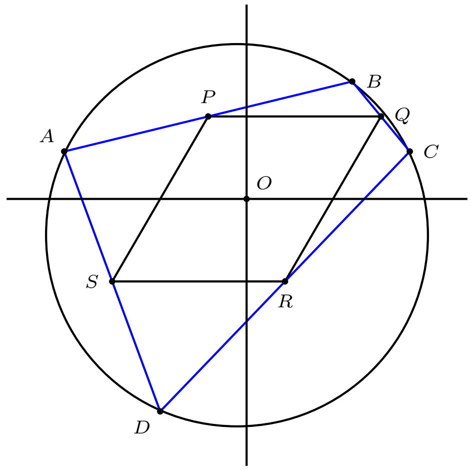

Figure 3

Let $ABCD$ be such that $\mathcal{P} = \mathcal{V}(ABCD)$. Then $A \neq C,\ B \neq D$ and the quadrilateral $ABCD$ is cyclic if and only if the cross-ratio $\rho = \frac{d-a}{d-b} \cdot \frac{c-b}{c-a}$ is a real number. With $b = 2p - a,\ c = 2q - 2p + a,\ d = -2q - a$ and allowing for $q - p \in \mathbb{R}$, the calculation of ρ yields the condition:

$$(q - p + a)^2 = p^2 + \lambda(p + q)$$

for some real number λ. Thus, $ABCD$ is cyclic if and only if the affixes $m,\ m' = -m$ of its diagonal midpoints M, M' are the square roots of a complex number of the form $p^2 + \lambda(p+q)$, where $\lambda \in \mathbb{R}$. Clearly, distinct values λ_1, λ_2 for λ lead to corresponding disjoint sets $\{M_1, M_1'\}, \{M_2, M_2'\}$ of diagonal midpoints, hence to distinct solutions for cyclic quadrilaterals. It follows that our problem has infinitely many solutions.

Consider P_2 with affix p^2 and choose a point K on the line through P_2 parallel to QR. The affix k of K is of the form $p^2 + \lambda(p+q)$ with $\lambda \in \mathbb{R}$. The construction of the corresponding pair M, M' is straightforward and achieved in Figure 4 where for the sake of simplification we take OP as the unit of length: M, M' are on the angle bisector of $\angle xOK$ and $OM = OM' = \sqrt{OK}$ (we skip the classical construction of the square root of a given length). $\square$

Exchanging the roles of M and M' (as in §2) evidently gives a solution inscribed in the symmetric of the circle $(ABCD)$ about O. In §4, we will indicate a different construction of suitable diagonal midpoints $M,\ M'$.

4. The rectangular hyperbola $\mathcal{H}(\mathcal{P})$

With the aim of obtaining the diagonal midpoints M, M' more directly, it seems interesting to identify their locus as the real number λ varies. This brings to light

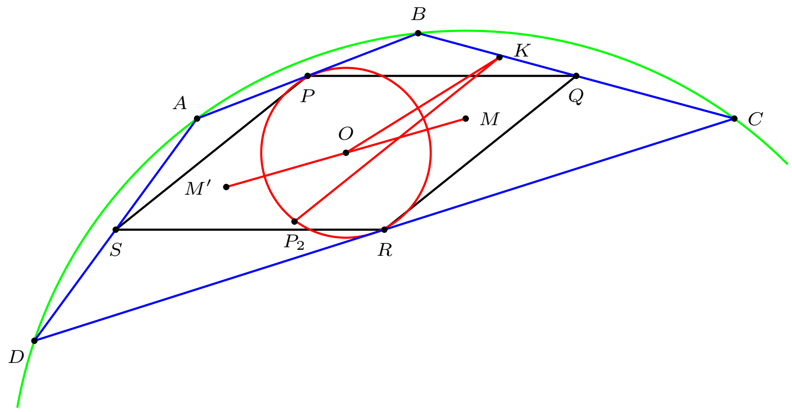

Figure 4

an unexpected hyperbola which will also provide more results about our quadrilaterals.

Theorem 2. *Consider the cyclic quadrilaterals $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. If $\mathcal{P}$ is not a rhombus, the locus of their diagonal midpoints is the rectangular hyperbola $\mathcal{H}(\mathcal{P})$ with the same center O as $\mathcal{P}$, passing through the vertices P, Q, R, S of $\mathcal{P}$. If $\mathcal{P}$ is a rhombus, the locus is the pair of diagonals of $\mathcal{P}$.*

Proof. We use the same system of axes as in the preceding section and continue to suppose that $OP = 1$. We denote by θ the directed angle $\angle(\overrightarrow{SR}, \overrightarrow{SP})$ that is, $\theta = \arg(p+q)$. Note that $\sin\theta \neq 0$. Let $m = x + iy$ with $x, y \in \mathbb{R}$. From $m^2 = p^2 + \lambda(p+q)$, we obtain $(x+iy)^2 = e^{2it} + \lambda\mu e^{i\theta}$ where $t = \arg(p)$ and $\mu = |p+q|$ and we readily deduce:

$$x^2 - y^2 = \cos 2t + \lambda\mu\cos\theta, \qquad 2xy = \sin 2t + \lambda\mu\sin\theta.$$

The elimination of λ shows that the locus of M (and of M' as well) is the curve $\mathcal{C}$ with equation

$$x^2 - y^2 - 2(\cot\theta)xy + \nu = 0, \tag{1}$$

where $\nu = \cot\theta\sin 2t - \cos 2t = \frac{\sin(2t-\theta)}{\sin\theta}$. Thus, when $\nu \neq 0$, $\mathcal{C}$ is a rectangular hyperbola centered at O with asymptotes

$$(\ell) \qquad y = x\tan(\theta/2),$$

and

$$(\ell') \qquad y = -x\cot(\theta/2),$$

and $\mathcal{C}$ degenerates into these two lines if $\nu = 0$ (we shall soon see that the latter occurs if and only if $\mathcal{P}$ is a rhombus). Note that (ℓ) and (ℓ') are the axes of symmetry of the medians of $\mathcal{P}$. An easy calculation shows that the coordinates $x_P = \cos t$, $y_P = \sin t$ of P satisfy (1), meaning that $P \in \mathcal{C}$. As for Q, the coordinates are $x_Q = \mu\cos\theta - \cos t$, $y_Q = \mu\sin\theta - \sin t$, but observing that

$y_Q = y_P$, we find $x_Q = 2\sin t \cot\theta - \cos t,\ y_Q = \sin t$. Again, x_Q, y_Q satisfy (1) and Q is a point of $\mathcal{C}$ as well. Thus, the parallelogram $\mathcal{P}$ is inscribed in $\mathcal{C}$. It follows that $\nu = 0$ if and only if (ℓ) and (ℓ') are the diagonals of $\mathcal{P}$. Since (ℓ) and (ℓ') are perpendicular, the situation occurs if $\mathcal{P}$ is a rhombus and only in that case. Otherwise, $\mathcal{C}$ is the rectangular hyperbola $\mathcal{H}(\mathcal{P})$, as defined in the statement of the theorem (see Figure 5). $\square$

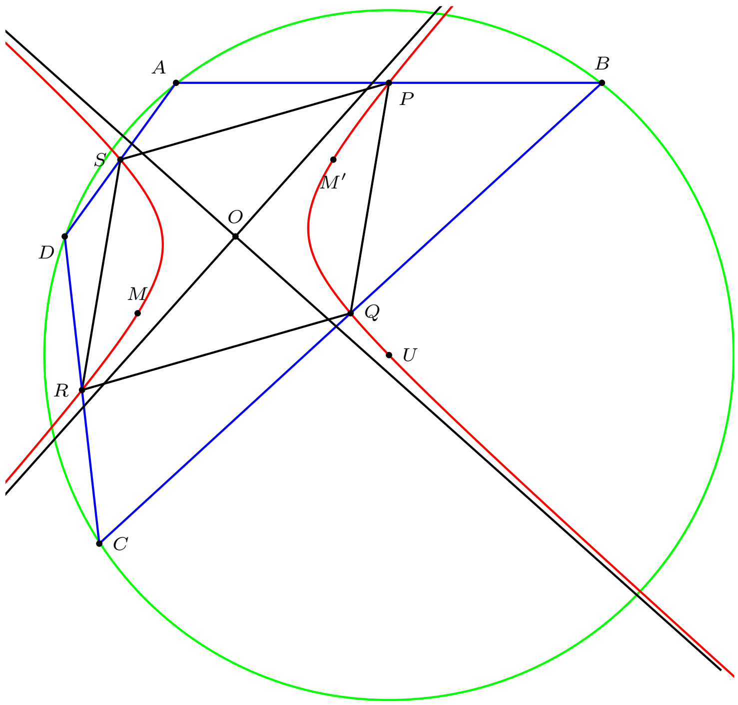

Figure 5

Figure 5 shows the center U of the circle through A, B, C, D as a point of $\mathcal{H}(\mathcal{P})$. This is no coincidence! Being the circumcenter of ΔABC, U is also the orthocenter of its median triangle MPQ. Since the latter is inscribed in $\mathcal{H}(\mathcal{P})$, a well-known property of the rectangular hyperbola ensures that its orthocenter is on $\mathcal{H}(\mathcal{P})$ as well. Conversely, any point U of $\mathcal{H}(\mathcal{P})$ can be obtained in this way by taking for M the orthocenter of ΔUPQ. We have proved:

Theorem 3. *If $\mathcal{P}$ is not a rhombus, $\mathcal{H}(\mathcal{P})$ is the locus of the circumcenter of a cyclic quadrilateral $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$.*

Of course, if $\mathcal{P}$ is a rhombus, the locus is the pair of diagonals of $\mathcal{P}$.

As another consequence of Theorem 2, we give a construction of a pair M, M' of diagonal midpoints simpler than the one in §3: through a vertex of $\mathcal{P}$, say Q, draw a line intersecting (ℓ) and (ℓ') at W and W'. As is well-known, the symmetric M of Q about the midpoint of WW' is on $\mathcal{H}(\mathcal{P})$. This point M and its symmetric M' about O provide a suitable pair. In addition, the orthocenter of ΔMPQ is the center U of the circumcircle of $ABCD$ (see Figure 6).

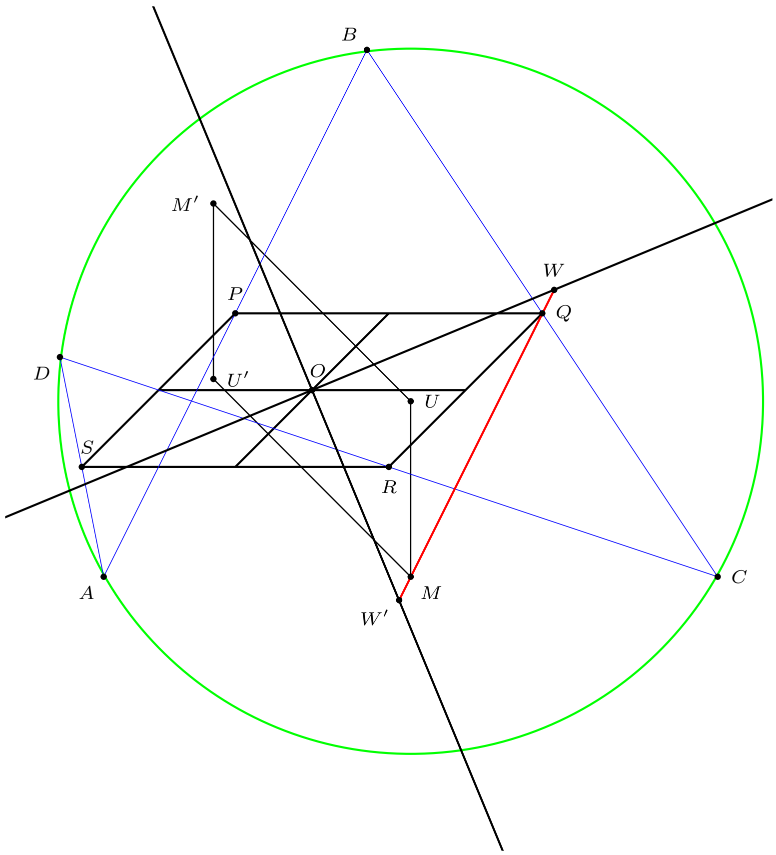

Figure 6

We shall end this section with a remark about the circumcenter U' of the quadrilateral $A'B'C'D'$ which shares the diagonal midpoints M, M' of $ABCD$ (as seen in §2). Clearly, $UMU'M'$ is a parallelogram with center O, inscribed in $\mathcal{H}(\mathcal{P})$ (Figure 6). Since UM and UM' are respectively perpendicular to PQ and PS, the directed angles of lines $\angle(UM, UM')$ and $\angle(PQ, PS)$ are equal (modulo π). Thus, $UMU'M'$ and $\mathcal{P}$ are equiangular.

5. Special cases

First, suppose that $\mathcal{P}$ is a rectangle and consider a cyclic quadrilateral $ABCD$ such that $\mathcal{P} = \mathcal{V}(ABCD)$. From the final remark of the previous section, $UMU'M'$ is a rectangle and since UM is perpendicular to PQ, the sides of $UMU'M'$ are parallel to those of $\mathcal{P}$. Recalling that M is on AC and M' on BD, we conclude that

U' is the point of intersection of the (perpendicular) diagonals of $ABCD$. Now, suppose that AC intersects PS at A_1, QR at C_1 and that BD intersects PQ at B_1, RS at D_1 (see Figure 7). Obviously, A_1, B_1, C_1 and D_1 are the midpoints of $U'A$, $U'B$, $U'C$ and $U'D$, so that A_1, B_1, C_1, D_1 are on the circle image of $(ABCD)$ under the homothety with center U' and ratio $\frac{1}{2}$. Since $\overrightarrow{U'O} = \frac{1}{2}\overrightarrow{U'U}$, the center of this circle $(A_1B_1C_1D_1)$ is just the center O of $\mathcal{P}$.

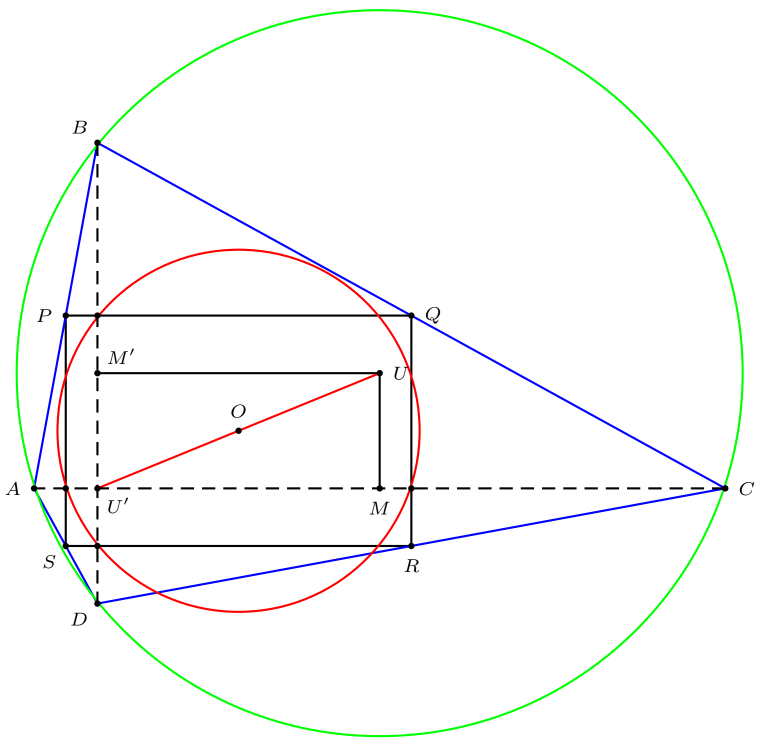

Figure 7

Conversely, draw any circle with center O intersecting the lines SP at A_1, A_1', PQ at B_1, B_1', QR at C_1, C_1' and RS at D_1, D_1', the notations being chosen so that A_1C_1, $A_1'C_1'$ are parallel to PQ and B_1D_1, $B_1'D_1'$ are parallel to QR. If $U' = A_1C_1 \cap B_1D_1$, then the image $ABCD$ of $A_1B_1C_1D_1$ under the homothety with center U' and ratio 2 is cyclic and satisfies $\mathcal{P} = \mathcal{V}(ABCD)$. For instance, because $U'A_1PB_1$ is a rectangle, P is the image of the midpoint of A_1B_1 and as such, is the midpoint of AB. The companion solution $A'B'C'D'$ is similarly obtained from $A_1'B_1'C_1'D_1'$.

Thus, in the case when $\mathcal{P}$ is a rectangle, a very quick construction provides suitable quadrilaterals $ABCD$. As a corollary of the analysis above, we have the following property that can also be proved directly:

Theorem 4. *If A, B, C, D are on a circle with center U and AC is perpendicular to BD at U', then the midpoint of UU' is the center of the rectangle $\mathcal{V}(ABCD)$.*

We conclude with a brief comment on the case when $\mathcal{P}$ is a rhombus. Remarking that if $\mathcal{P} = \mathcal{V}(ABCD)$, then $AC = 2PQ = 2QR = BD$, we see that any cyclic solution for $ABCD$ must be an isosceles trapezoid (possibly a self-crossing one). Conversely, if $ABCD$ is an isosceles trapezoid, then it is cyclic and $\mathcal{V}(ABCD)$ is a rhombus. The construction of a solution $ABCD$ from $\mathcal{P}$ simply follows from the choice of two points M, M' as diagonal midpoints of $ABCD$ on either diagonal of $\mathcal{P}$ (see Figure 8).

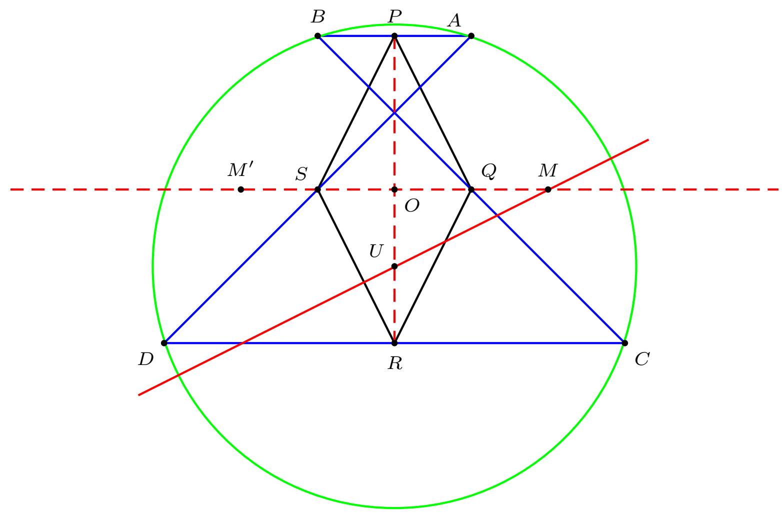

Figure 8

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Forum Geometricorum
Volume 7 (2007) 207–210.

FORUM GEOM
ISSN 1534-1178

Another Verification of Fagnano's Theorem

Finbarr Holland

Abstract. We present a trigonometrical proof of Fagnano's theorem which states that, among all inscribed triangles in a given acute-angled triangle, the feet of its altitudes are the vertices of the one with the least perimeter.

1. Introduction

At the outset, and to avoid ambiguity, we fix the following terminology. Let ABC be any triangle. The feet of its altitudes are the vertices of what we call its *orthic* triangle, and, if X, Y, and Z, respectively, are interior points of the sides AB, BC, and CA, respectively, we call the triangle XYZ an *inscribed triangle* of ABC.

In 1775, Fagnano proved the following theorem.

Theorem 1. *Suppose ABC is an acute-angled triangle. Of all inscribed triangles in ABC, its orthic triangle has the smallest perimeter.*

Not surprisingly, over the years this beautiful result has attracted the attentions of many mathematicians, and there are several proofs known of it [1]. Fagnano himself apparently used differential calculus to prove it, though, by modern standards, it seems to me that this is far from being a routine exercise. Perhaps the most appealing proofs of the theorem are those based on the Reflection Principle, and two of these, in particular, due independently to L. Fejér and H. A. Schwarz, have made their appearance in several books aimed at general audiences [2], [3], [4], [6]. A proof based on vector calculus appeared recently [5]. The purpose of this note is to offer one based on trigonometry.

Theorem 2. *Let ABC be any triangle, with $a = |BC|, b = |CA|, c = |AB|$, and area Δ. If XYZ is inscribed in ABC, then*

$$|XY| + |YZ| + |ZX| \geq \frac{8\Delta^2}{abc}. \tag{1}$$

Equality holds in (1) *if and only if ABC is acute-angled; and then only if XYZ is its orthic triangle. If ABC is right-angled* (*respectively, obtuse-angled*)*, and C*

Publication Date: December 5, 2007. Communicating Editor: Paul Yiu.
The author is grateful to the referee for his helpful remarks.

is the right-angle (respectively, the obtuse-angle), then an inequality stronger than (1) holds, viz.,

$$|XY| + |YZ| + |ZX| > 2h_c, \tag{2}$$

where h_c denotes the length of the altitude from C; and, in either case, this estimate is best possible.

2. Proof of Theorem 2

Let XYZ be a triangle inscribed in ABC. Let $x = |BX|, y = |CY|, z = |AZ|$. Then $0 < x < a$, $0 < y < b$, $0 < z < c$. By applying the Cosine Rule in the triangle ZBX we have

$$\begin{aligned}
|ZX|^2 &= (c-z)^2 + x^2 - 2x(c-z)\cos B \\
&= (c-z)^2 + x^2 + 2x(c-z)\cos(A+C) \\
&= (x\cos A + (c-z)\cos C)^2 + (x\sin A - (c-z)\sin C)^2.
\end{aligned}$$

Hence,

$$|ZX| \geq |x\cos A + (c-z)\cos C|\,,$$

with equality if and only if $x\sin A = (c-z)\sin C$, *i.e.*, if and only if

$$ax + cz = c^2, \tag{3}$$

by the Sine Rule. Similarly,

$$|XY| \geq |y\cos B + (a-x)\cos A|\,,$$

with equality if and only if

$$ax + by = a^2. \tag{4}$$

And

$$|YZ| \geq |z\cos C + (b-y)\cos B|\,,$$

with equality if and only if

$$by + cz = b^2. \tag{5}$$

Thus, by the triangle inequality for real numbers,

$$\begin{aligned}
&|XY| + |YZ| + |ZX| \\
&\geq |y\cos B + (a-x)\cos A| + |z\cos C + (b-y)\cos B| + |x\cos A + (c-z)\cos C| \\
&\geq |y\cos B + (a-x)\cos A + z\cos C + (b-y)\cos B + x\cos A + (c-z)\cos C| \\
&= |a\cos A + b\cos B + c\cos C| \\
&= \frac{|a^2(b^2+c^2-a^2) + b^2(c^2+a^2-b^2) + c^2(a^2+b^2-c^2)|}{2abc} \\
&= \frac{8\Delta^2}{abc}.
\end{aligned}$$

This proves (1). Moreover, there is equality here if and only if equations (3), (4), and (5) hold, and the expressions

$$\begin{aligned} u &= x\cos A + (c-z)\cos C, \\ v &= y\cos B + (a-x)\cos A, \\ w &= z\cos C + (b-y)\cos B, \end{aligned}$$

are either all non-negative or all non-positive. Now it is easy to verify that the system of equations (3), (4), and (5), has a unique solution given by

$$x = c\cos B,\ \ y = a\cos C,\ \ z = b\cos A,$$

in which case

$$u = b\cos B,\ v = c\cos C,\ w = a\cos A.$$

Thus, in this case, at most one of u, v, w can be non-positive. But, if one of u, v, w is zero, then one of x, y, z must be zero, which is not possible. It follows that

$$|XY| + |YZ| + |ZX| > \frac{8\Delta^2}{abc},$$

unless ABC is acute-angled, and XYZ is its orthic triangle. If ABC is acute-angled, then $\frac{8\Delta^2}{abc}$ is the perimeter of its orthic triangle, in which case we recover Fagnano's theorem, equality being attained in (1) when and only when XYZ is the orthic triangle.

Turning now to the case when ABC is not acute-angled, suppose first that C is a right-angle. Then

$$|XY| + |YZ| + |ZX| > \frac{8\Delta^2}{abc} = \frac{4\Delta}{c} = 2h_c,$$

and so (2) holds in this case. Next, if C is an obtuse-angle, denote by D and E, respectively, the points of intersection of the side AB and the lines through C that are perpendicular to the sides BC and CA, respectively. Then Z is an interior point of one of the line segments $[B, D]$ and $[E, A]$. Suppose, for definiteness, that Z is an interior point of $[B, D]$. If Y' is the point of intersection of $[X, Y]$ and $[C, D]$, then

$$\begin{aligned} |XY| + |YZ| + |ZX| &= |XY'| + |Y'Y| + |YZ| + |ZX| \\ &> |XY'| + |Y'Z| + |ZX| \\ &> 2h_c, \end{aligned}$$

since the triangle $XY'Z$ is inscribed in the right-angled triangle BCD. A similar argument works if Z is an interior point of $[E, A]$. Hence, (2) also holds if C is obtuse.

That (2) is stronger than (1), for a non acute-angled triangle, follows from the fact that, in any triangle ABC,

$$\frac{4\Delta^2}{abc} = \frac{2\Delta\sin C}{c} = a\sin B\sin C \le a\sin B = h_c.$$

It remains to prove that inequality (2) cannot be improved when the angle C is right or obtuse. To see this, let Z be the foot of the perpendicular from C to AB, and $0 < \varepsilon < 1$. Choose Y on CA so that $|CY| = \varepsilon b$, and X on BC so that XY is parallel to AB. Then, as $\varepsilon \to 0^+$, both X and Y converge to C, and so

$$\lim_{\varepsilon \to 0^+} (|XY| + |YZ| + |ZX|) = |CC| + |CZ| + |ZC| = 2|CZ| = 2h_c.$$

This finishes the proof.

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Forum Geometricorum
Volume 7 (2007) 211–229.

FORUM GEOM
ISSN 1534-1178

How Pivotal Isocubics Intersect the Circumcircle

Bernard Gibert

Abstract. Given the pivotal isocubic $\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$, we seek its common points with the circumcircle and we also study the tangents at these points.

1. Introduction

A pivotal cubic $\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$ with pole Ω, pivot P, is the locus of point M such that P, M and its Ω-isoconjugate M^* are collinear. It is also the locus of point M such that P^* (the isopivot or secondary pivot), M and the cevian quotient P/M are collinear. See [2] for more information. [1] The isocubic $\mathcal{K}$ meets the circumcircle $(\mathcal{O})$ of the reference triangle ABC at its vertices and three other points Q_1, Q_2, Q_3, one of them being always real. This paper is devoted to a study of these points and special emphasis on their tangents.

2. Isogonal pivotal cubics

We first consider the case where the pivotal isocubic $\mathcal{K} = \mathrm{p}\mathcal{K}(X_6, P)$ is isogonal with pole the Lemoine point K.

2.1. *Circular isogonal cubics.* When the pivot P lies at infinity, $\mathcal{K}$ contains the two circular points at infinity. Hence it is a circular cubic of the class **CL035** in [3], and has only one real intersection with $(\mathcal{O})$. This is the isogonal conjugate P^* of the pivot.

The tangent at P is the real asymptote PP^* of the cubic and the isotropic tangents meet at the singular focus F of the circular cubic. F is the antipode of P^* on $(\mathcal{O})$.

The pair P and P^* are the foci of an inscribed conic, which is a parabola with focal axis PP^*. When P traverses the line at infinity, this axis envelopes the deltoid $\mathcal{H}_3$ tritangent to $(\mathcal{O})$ at the vertices of the circumtangential triangle. The contact of the deltoid with this axis is the reflection in P^* of the second intersection of PP^* with the circumcircle. See Figure 1 with the Neuberg cubic **K001** and the Brocard cubic **K021**. For example, with the Neuberg cubic, $P^* = X_{74}$, the second point on the axis is X_{476}, the contact is the reflection of X_{476} in X_{74}.

Publication Date: December 10, 2007. Communicating Editor: Paul Yiu.

The author thanks Paul Yiu for his help in the preparation of this paper.

[1] Most of the cubics cited here are now available on the web-site http://perso.orange.fr/bernard.gibert/index.html, where they are referenced under a catalogue number of the form **Knnn**.

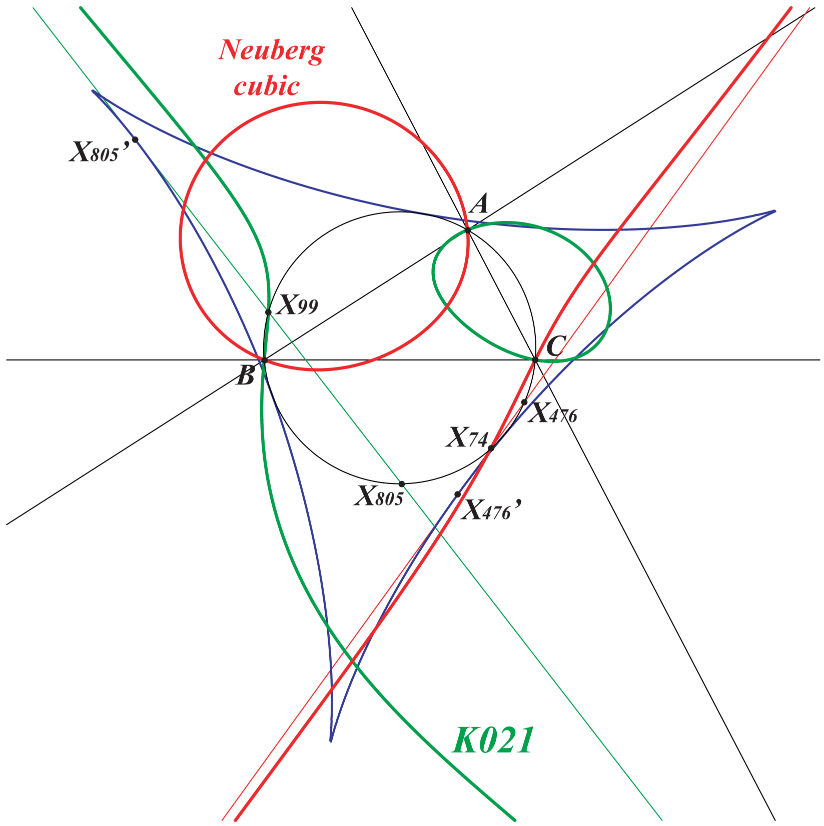

Figure 1. Isogonal circular cubic with pivot at infinity

2.2. *Isogonal cubics with pivot on the circumcircle.* When P lies on $(\mathcal{O})$, the remaining two intersections Q_1, Q_2 are antipodes on $(\mathcal{O})$. They lie on the perpendicular at O to the line PP^* or the parallel at O to the Simson line of P. The isocubic $\mathcal{K}$ has three real asymptotes:
(i) One is the parallel at P/P^* (cevian quotient) to the line PP^*.
(ii) The two others are perpendicular and can be obtained as follows. Reflect P in Q_1, Q_2 to get S_1, S_2 and draw the parallels at S_1^*, S_2^* to the lines PQ_1, PQ_2. These asymptotes meet at X on the line OP. Note that the tangent to the cubic at Q_1, Q_2 are the lines $Q_1S_1^*$, $Q_2S_2^*$. See Figure 2.

2.3. *The general case.* In both cases above, the orthocenter of the triangle formed by the points Q_1, Q_2, Q_3 is the pivot P of the cubic, although this triangle is not a proper triangle in the former case and a right triangle in the latter case. More generally, we have the following

Theorem 1. *For any point P, the isogonal cubic $\mathcal{K} = \mathrm{p}\mathcal{K}(X_6, P)$ meets the circumcircle at A, B, C and three other points Q_1, Q_2, Q_3 such that P is the orthocenter of the triangle $Q_1Q_2Q_3$.*

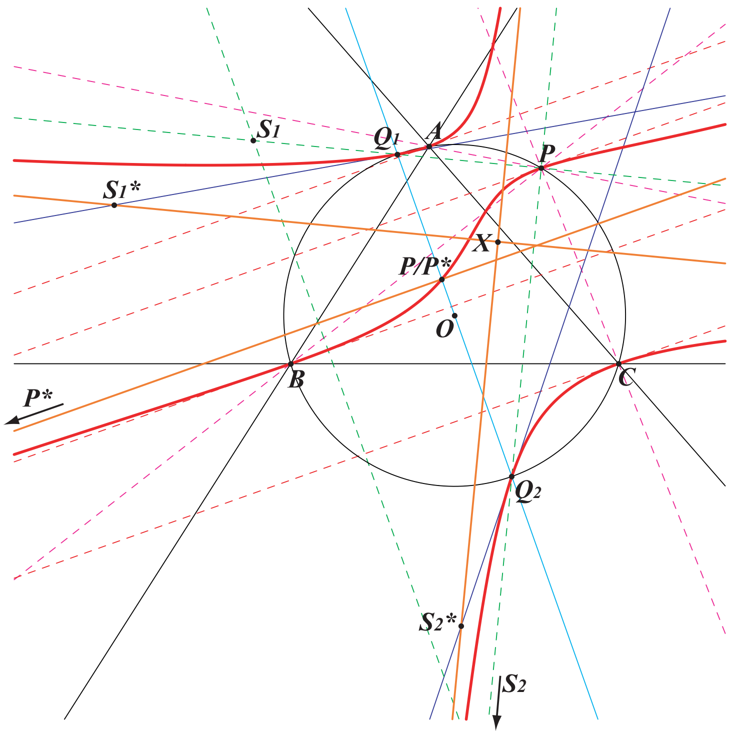

Figure 2. Isogonal cubic with pivot on the circumcircle

Proof. The lines $Q_1Q_1^*$, $Q_2Q_2^*$, $Q_3Q_3^*$ pass through the pivot P and are parallel to the asymptotes of the cubic. Since they are the axes of three inscribed parabolas, they must be tangent to the deltoid $\mathcal{H}_3$, the anticomplement of the Steiner deltoid. This deltoid is a bicircular quartic of class 3. Hence, for a given P, there are only three tangents (at least one of which is real) to the deltoid passing through P.

According to a known result, Q_1 must be the antipode on $(\mathcal{O})$ of Q_1', the isogonal conjugate of the infinite point of the line Q_2Q_3. The Simson lines of Q_1', Q_2, Q_3 are concurrent. Hence, the axes are also concurrent at P. But the Simson line of Q_1 is parallel to Q_2Q_3. Hence $Q_1Q_1^*$ is an altitude of $Q_1Q_2Q_3$. This completes the proof. See Figure 3. □

Remark. These points Q_1, Q_2, Q_3 are not necessarily all real nor distinct. In [1], H. M. Cundy and C. F. Parry have shown that this depends of the position of P with respect to $\mathcal{H}_3$. More precisely, these points are all real if and only if P lies strictly inside $\mathcal{H}_3$. One only is real when P lies outside $\mathcal{H}_3$. This leaves a special case when P lies on $\mathcal{H}_3$. See §2.4.

Recall that the contacts of the deltoid $\mathcal{H}_3$ with the line $PQ_1Q_1^*$ is the reflection in Q_1 of the second intersection of the circumcircle and the line $PQ_1Q_1^*$. Consequently, every conic passing through P, Q_1, Q_2, Q_3 is a rectangular hyperbola and all these hyperbolas form a pencil $\mathcal{F}$ of rectangular hyperbolas.

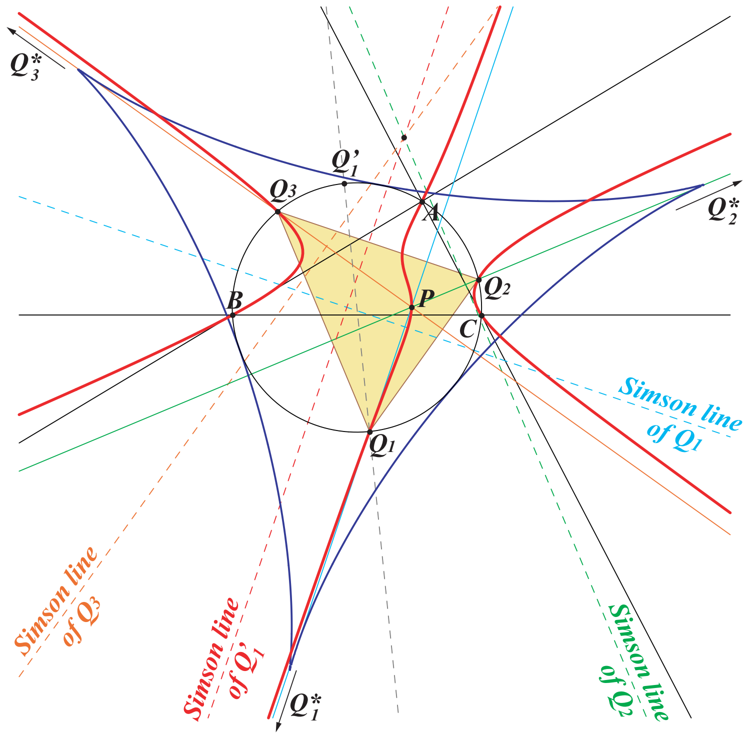

Figure 3. The deltoid $\mathcal{H}_3$ and the points Q_1, Q_2, Q_3

Let $\mathcal{D}$ be the diagonal rectangular hyperbola which contains the four in/excenters of ABC, P^*, and P/P^*. Its center is $\Omega_{\mathcal{D}}$. Note that the tangent at P^* to $\mathcal{D}$ contains P and the tangent at P/P^* to $\mathcal{D}$ contains P. In other words, the polar line of P in $\mathcal{D}$ is the line through P^* and P/P^*.

The pencil $\mathcal{F}$ contains the hyperbola $\mathcal{H}$ passing through P, P^*, P/P^* and $\Omega_{\mathcal{D}}$ having the same asymptotic directions as $\mathcal{D}$. The center of $\mathcal{H}$ is the midpoint of P and $\Omega_{\mathcal{D}}$. This gives an easy conic construction of the points Q_1, Q_2, Q_3 when P is given. See Figure 4. The pencil $\mathcal{F}$ contains another very simple rectangular hyperbola $\mathcal{H}'$, which is the homothetic of the polar conic of P in $\mathcal{K}$ under $\mathsf{h}\left(P, \frac{1}{2}\right)$. Since this polar conic is the diagonal conic passing through the in/excenters and P, $\mathcal{H}'$ contains P and the four midpoints of the segments joining P to the in/excenters.

Corollary 2. *The isocubic $\mathcal{K}$ contains the projections R_1, R_2, R_3 of P^* on the sidelines of $Q_1Q_2Q_3$. These three points lie on the bicevian conic $\mathcal{C}(G, P)$.* [2]

Proof. Let R_1 be the third point of $\mathcal{K}$ on the line Q_2Q_3. The following table shows the collinearity relations of nine points on $\mathcal{K}$ and proves that P^*, R_1 and Q_1^* are collinear.

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[2]This is the conic through the vertices of the cevian triangles of G and P. This is the P-Ceva conjugate of the line at infinity.

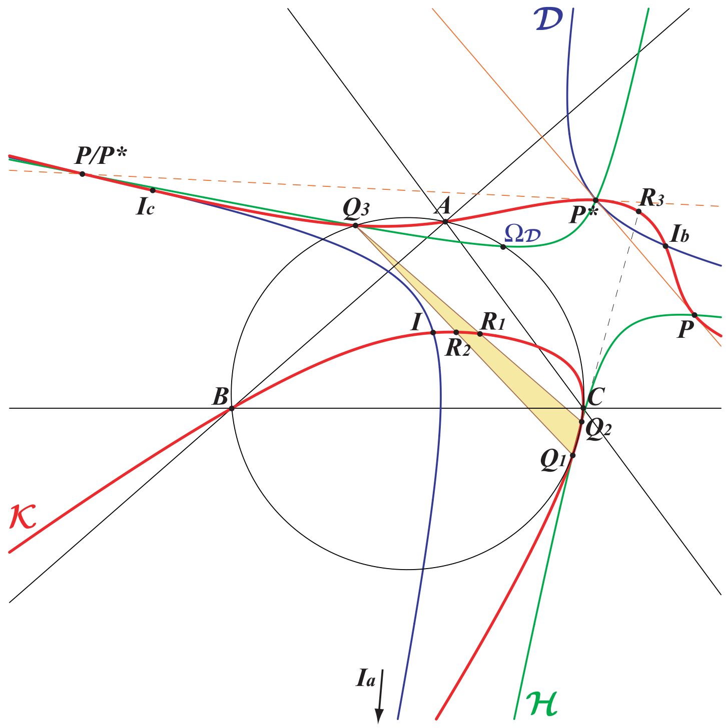

Figure 4. The hyperbolas $\mathcal{H}$ and $\mathcal{D}$

P	P	P^*	$\leftarrow P^*$ is the tangential of P
Q_2	Q_3	R_1	$\leftarrow$ definition of R_1
Q_2^*	Q_3^*	Q_1^*	$\leftarrow$ these three points lie at infinity

This shows that, for $i = 1, 2, 3$, the points P^*, R_i and Q_i^* are collinear and, since P, Q_i and Q_i^* are also collinear, the lines PQ_i and P^*R_i are parallel. It follows from Theorem 1 that R_i is the projection of P^* onto the line R_jR_k.

Recall that P^* is the secondary pivot of $\mathcal{K}$ hence, for any point M on $\mathcal{K}$, the points P^*, M and P/M (cevian quotient) are three collinear points on $\mathcal{K}$. Consequently, $R_i = P/Q_i^*$ and, since Q_i^* lies at infinity, R_i is a point on $\mathcal{C}(G, P)$. □

Corollary 3. *The lines* $Q_iR_i^*$, $i = 1, 2, 3$, *pass through the cevian quotient* P/P^*.

Proof. This is obvious from the following table.

P^*	P^*	P	$\leftarrow P/P^*$ is the tangential of P^*
P	Q_1^*	Q_1	$\leftarrow Q_1Q_1^*$ must contain the pivot P
P	R_1	R_1^*	$\leftarrow R_1R_1^*$ must contain the pivot P

Recall that P^* is the tangential of P (first column). The second column is the corollary above. □

Corollary 4. *Let S_1, S_2, S_3 be the reflections of P in Q_1, Q_2, Q_3 respectively. The asymptotes of $\mathcal{K}$ are the parallel at S_i^* to the lines PQ_i or P^*R_i.*

Proof. These points S_i lie on the polar conic of the pivot P since they are the harmonic conjugate of P with respect to Q_i and Q_i^*. The construction of the asymptotes derives from [2, §1.4.4]. □

Theorem 5. *The inconic $\mathcal{I}(P)$ concentric with $\mathcal{C}(G, P)$* [3] *is also inscribed in the triangle $Q_1Q_2Q_3$ and in the triangle formed by the Simson lines of Q_1, Q_2, Q_3.*

Proof. Since the triangles ABC and $Q_1Q_2Q_3$ are inscribed in the circumcircle, there must be a conic inscribed in both triangles. The rest is mere calculation. □

In [4, §29, p.88], A. Haarbleicher remarks that the triangle ABC and the reflection of $Q_1Q_2Q_3$ in O circumscribe the same parabola. These two parabolas are obviously symmetric about O. Their directrices are the line through H and the reflection P' of P in O in the former case, and its reflection in O in the latter case. The foci are the isogonal conjugates of the infinite points of these directrices and its reflection about O.

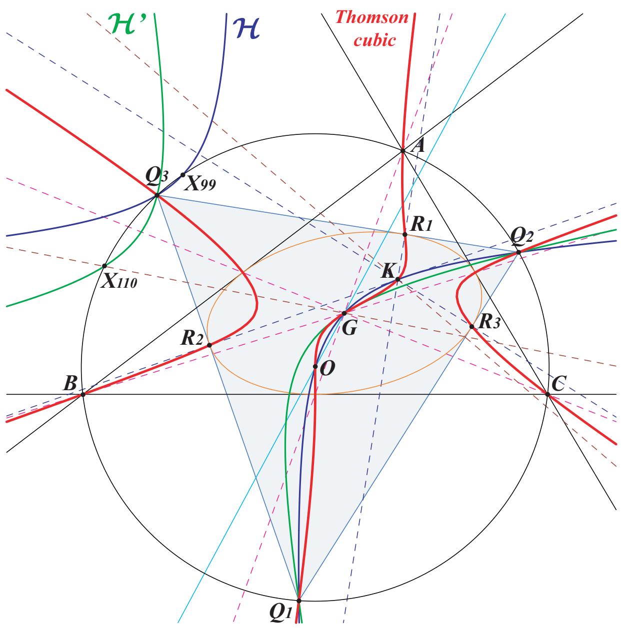

Figure 5. Thomson cubic

[3]This center is the complement of the complement of P, *i.e.*, the homothetic of P under $\mathsf{h}(G, \frac{1}{4})$. Note that these two conics $\mathcal{I}(P)$ and $\mathcal{C}(G, P)$ are bitangent at two points on the line GP. When $P = G$, they coincide since they both are the Steiner in-ellipse.

For example, Figures 5 and 6 show the case $P = G$. Note, in particular,
– $\mathcal{K}$ is the Thomson cubic,
– $\mathcal{D}$ is the Steiner (or Don Wallace) hyperbola,
– $\mathcal{H}$ contains X_2, X_3, X_6, X_{110}, X_{154}, X_{354}, X_{392}, X_{1201}, X_{2574}, X_{2575},
– $\mathcal{H}'$ contains X_2, X_{99}, X_{376}, X_{551},
– the inconic $\mathcal{I}(P)$ and the bicevian conic $\mathcal{C}(G, P)$ are the Steiner in-ellipse,
– the two parabolas are the Kiepert parabola and its reflection in O.

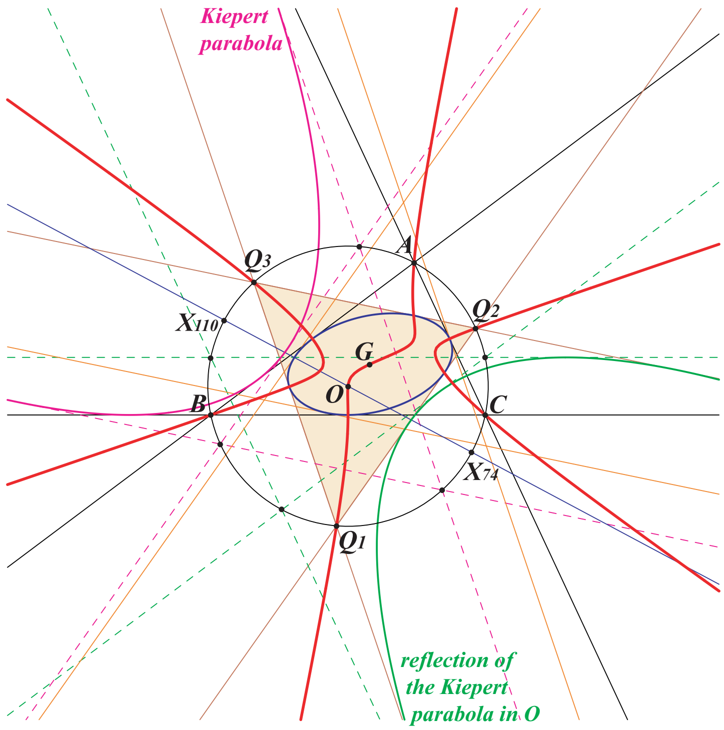

Figure 6. The Thomson cubic and the two parabolas

More generally, any $\mathrm{p}\mathcal{K}(X_6, P)$ with pivot P on the Euler line is obviously associated to the same two parabolas. In other words, any cubic of the Euler pencil meets the circumcircle at three (not always real) points Q_1, Q_2, Q_3 such that the reflection of the Kiepert parabola in O is inscribed in the triangle $Q_1Q_2Q_3$ and in the circumcevian triangle of O.

In particular, taking $P = O$, we obtain the McCay cubic and this shows that the reflection of the Kiepert parabola in O is inscribed in the circumnormal triangle.

Another interesting case is $\mathrm{p}\mathcal{K}(X_6, X_{145})$ in Figure 7 since the incircle is inscribed in the triangle $Q_1Q_2Q_3$.

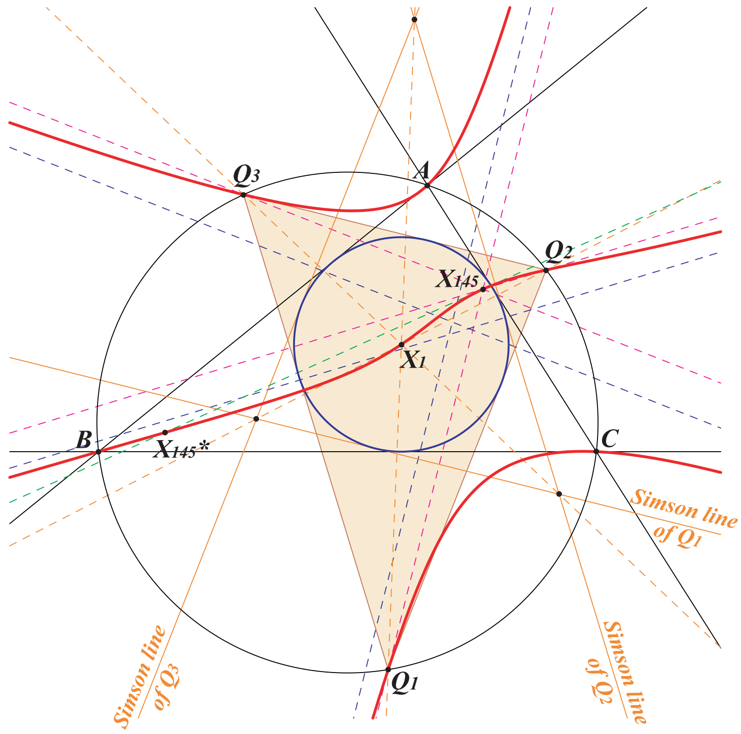

Figure 7. p$\mathcal{K}(X_6, X_{145})$

2.4. *Isogonal pivotal cubics tangent to the circumcircle.* In this section, we take P on $\mathcal{H}_3$ so that $\mathcal{K}$ has a multiple point at infinity.

Here is a special case. $\mathcal{H}_3$ is tangent to the six bisectors of ABC. If we take the bisector AI, the contact P is the reflection of A in the second intersection A_i of AI with the circumcircle. The corresponding cubic $\mathcal{K}$ is the union of the bisector AI and the conic passing through B, C, the excenters I_b and I_c, A_i, the antipode of A on the circumcircle.

Let us now take M on the circle $\mathcal{C}_H$ with center H, radius $2R$ and let us denote by $\mathcal{T}_M$ the tangent at M to $\mathcal{C}_H$. The orthopole P of $\mathcal{T}_M$ with respect to the antimedial triangle is a point on $\mathcal{H}_3$.

The corresponding cubic $\mathcal{K}$ meets $(\mathcal{O})$ at P_1 (double) and P_3. The common tangent at P_1 to $\mathcal{K}$ and $(\mathcal{O})$ is parallel to $\mathcal{T}_M$. Note that P_1 lies on the Simson line $\mathcal{S}_P$ of P with respect to the antimedial triangle.

The perpendicular at P_1 to $\mathcal{S}_P$ meets $(\mathcal{O})$ again at P_3 which is the antipode on $(\mathcal{O})$ of the second intersection Q_3 of $\mathcal{S}_P$ and $(\mathcal{O})$. The Simson line of P_3 is parallel to $\mathcal{T}_M$.

It follows that $\mathcal{K}$ has a triple common point with $(\mathcal{O})$ if and only if P_1 and Q_3 are antipodes on $(\mathcal{O})$ i.e. if and only if $\mathcal{S}_P$ passes through O. This gives the following theorem.

Theorem 6. *There are exactly three isogonal pivotal cubics which are tridents.*

Their pivots are the cusps of the deltoid $\mathcal{H}_3$. The triple contacts with $(\mathcal{O})$ are the vertices of the circumnormal triangle.[4] These points are obviously inflexion points and the inflexional tangent is parallel to a sideline of the Morley triangle. See Figure 8.

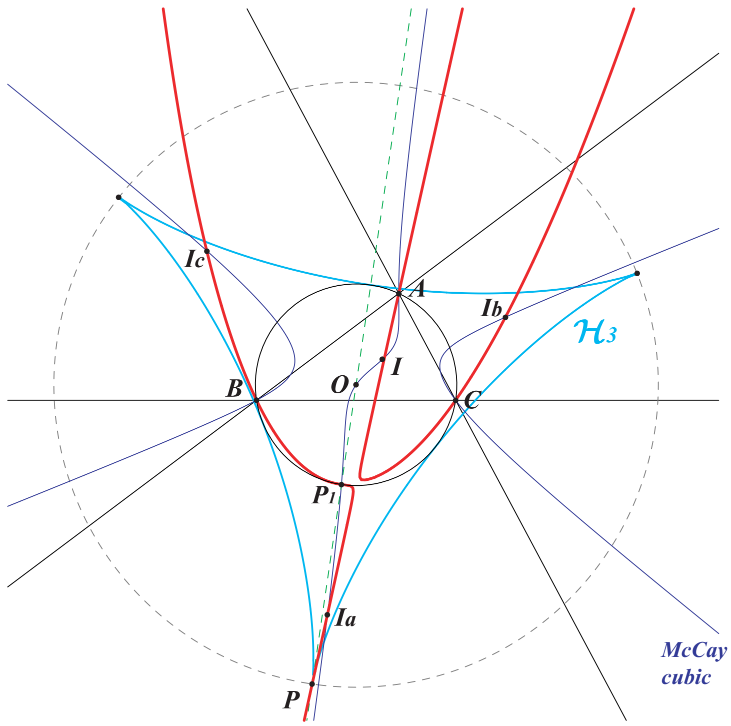

Figure 8. Isogonal pivotal trident

2.5. *Tangents at* Q_1, Q_2, Q_3. We know that the tangents at A, B, C to any pivotal cubic concur at P^*. This is not necessarily true for those at Q_1, Q_2, Q_3.

Theorem 7. *The tangents at* Q_1*,* Q_2*,* Q_3 *to the isogonal cubic* $\mathrm{p}\mathcal{K}(X_6, P)$ *concur if and only if* P *lies on the quintic* **Q063** *with equation*

$$\sum_{\text{cyclic}} a^2y^2z^2\left(S_C(x+y) - S_B(x+z)\right) = 0.$$

Q063 is a circular quintic with singular focus X_{376}, the reflection of G in O. It has three real asymptotes parallel to those of the Thomson cubic and concurrent at G.

[4]These three points are the common points of the circumcircle and the McCay cubic apart A, B, C.

A, B, C are nodes and the fifth points on the sidelines of ABC are the vertices A', B', C' of the pedal triangle of X_{20}, the de Longchamps point. The tangents at these points pass through X_{20} and meet the corresponding bisectors at six points on the curve. See Figure 9.

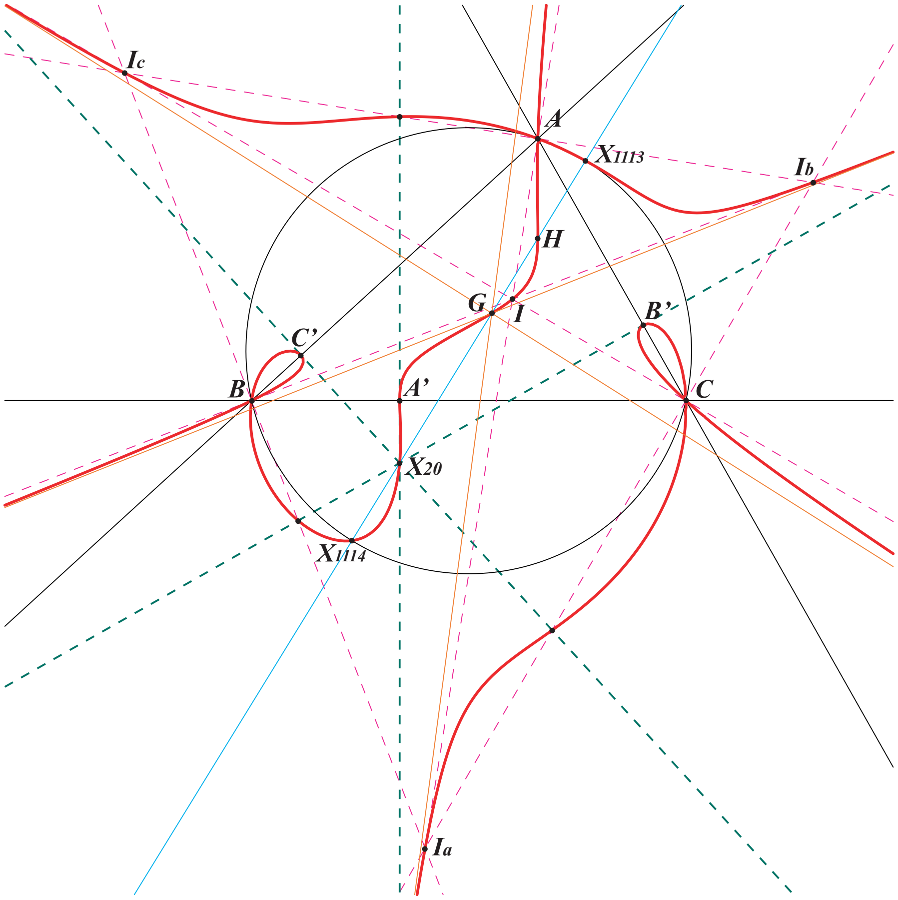

Figure 9. The quintic **Q063**

Q063 contains I, the excenters, G, H, X_{20}, X_{1113}, X_{1114}. Hence, for the Thomson cubic, the orthocubic, and the Darboux cubic, the tangents at Q_1, Q_2, Q_3 concur. The intersection of these tangents are X_{25} for the orthocubic, and X_{1498} for the Darboux cubic. For the Thomson cubic, this is an unknown point[5] in the current edition of ETC on the line GX_{1350}.

[5]This has first barycentric coordinate

$$a^2(3S_A^2 + 2a^2 S_A + 5b^2c^2).$$

3. Non-isogonal pivotal cubics

We now consider a non-isogonal pivotal cubic $\mathcal{K}$ with pole $\omega \neq K$ and pivot π.

We recall that π^* is the $\omega-$isoconjugate of π and that π/π^* is the cevian quotient of π and π^*, these three points lying on the cubic.

3.1. *Circular cubics.* In this special case, two of the points, say Q_2 and Q_3, are the circular points at infinity. This gives already five common points of the cubic on the circumcircle and the sixth point Q_1 must be real.

The isoconjugation with pole ω swaps the pivot π and the isopivot π^* which must be the inverse (in the circumcircle of ABC) of the isogonal conjugate of π. In this case, the cubic contains the point T, isogonal conjugate of the complement of π. This gives the following

Theorem 8. *A non isogonal circular pivotal cubic $\mathcal{K}$ meets the circumcircle at A, B, C, the circular points at infinity and another (real) point Q_1 which is the second intersection of the line through T and π/π^* with the circle passing through π, π^* and π/π^*.*

Example: The Droussent cubic **K008**. This is the only circular isotomic pivotal cubic. See Figure 10.

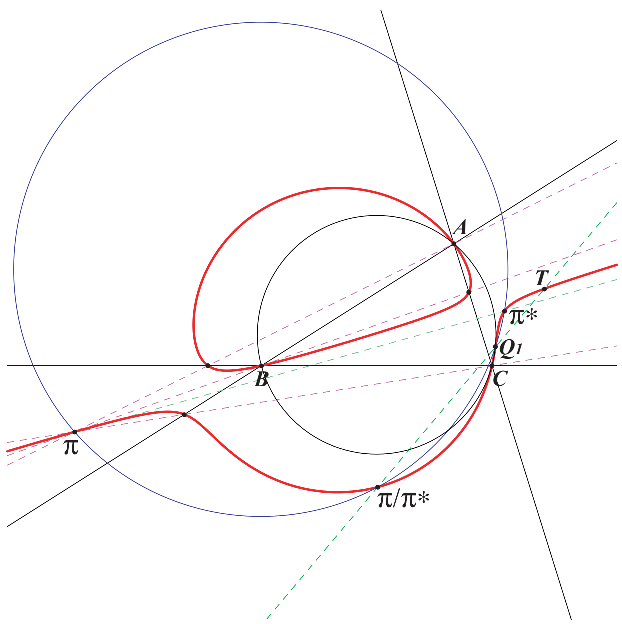

Figure 10. The Droussent cubic **K008**

The points π, π^*, T, Q_1 are X_{316}, X_{67}, X_{671}, X_{2373} respectively. The point π/π^* is not mentioned in the current edition of [6].

Note that when $\pi = H$, there are infinitely many circular pivotal cubics with pivot H, with isopivot π^* at infinity. These cubics are the isogonal circular pivotal cubics with respect to the orthic triangle. They have their singular focus F on the nine point circle and their pole ω on the orthic axis. The isoconjugate H^* of H is the point at infinity of the cubic. The intersection with their real asymptote is X, the antipode of F on the nine point circle and, in this case, $X = \pi/\pi^*$. This asymptote envelopes the Steiner deltoid $\mathcal{H}_3$. The sixth point Q_1 on the circumcircle is the orthoassociate of X, i.e. the inverse of X in the polar circle.

Example: The Neuberg orthic cubic **K050**. This is the Neuberg cubic of the orthic triangle. See [3].

3.2. *General theorems for non circular cubics.*

Theorem 9. *$\mathcal{K}$ meets the circumcircle at A, B, C and three other points Q_1, Q_2, Q_3 (one at least is real) lying on a same conic passing through π, π^* and π/π^*.*

Note that this conic meets the circumcircle again at the isogonal conjugate of the infinite point of the trilinear polar of the isoconjugate of ω under the isoconjugation with fixed point π.

With $\omega = p : q : r$ and $\pi = u : v : w$, this conic has equation

$$\sum_{\text{cyclic}} p^2v^2w^2(c^2y+b^2z)(wy-vz)+qru^2x(vw(c^2v-b^2w)x+u(b^2w^2y-c^2v^2z)) = 0,$$

and the point on the circumcircle is :

$$\frac{a^2}{u^2(rv^2-qw^2)} : \frac{b^2}{v^2(pw^2-ru^2)} : \frac{c^2}{w^2(qu^2-pv^2)}$$

Theorem 10. *The conic inscribed in triangles ABC and $Q_1Q_2Q_3$ is that with perspector the cevian product of π and $tg\omega$, the isotomic of the isogonal of ω.*

3.3. *Relation with isogonal pivotal cubics.*

Theorem 11. *$\mathcal{K}$ meets the circumcircle at the same points as the isogonal pivotal cubic with pivot $P = u : v : w$ if and only if its pole ω lie on the cubic $\mathcal{K}_{\text{pole}}$ with equation*

$$\sum_{\text{cyclic}} (v+w)(c^4y-b^4z)\,\frac{x^2}{a^2} - \left(\sum_{\text{cyclic}} (b^2-c^2)u\right) xyz = 0$$
$$\iff \sum_{\text{cyclic}} a^2u(c^2y-b^2z)(-a^4yz+b^4zx+c^4xy) = 0.$$

In other words, for any point ω on $\mathcal{K}_{\text{pole}}$, there is a pivotal cubic with pole ω meeting the circumcircle at the same points as the isogonal pivotal cubic with pivot $P = u : v : w$.

$\mathcal{K}_{\text{pole}}$ is a circum-cubic passing through K, the vertices of the cevian triangle of gc$P$, the isogonal conjugate of the complement of P. The tangents at A, B, C are the cevians of X_{32}.

The second equation above clearly shows that all these cubics belong to a same net of circum-cubics passing through K having the same tangents at A, B, C.

This net can be generated by three decomposed cubics, one of them being the union of the symmedian AK and the circum-conic with perspector the A-harmonic associate of X_{32}.

For example, with $P = H$, $\mathcal{K}_{\rm pole}$ is a nodal cubic with node K and nodal tangents parallel to the asymptotes of the Jerabek hyperbola. It contains X_6, X_{66}, X_{193}, X_{393}, X_{571}, X_{608}, X_{1974}, X_{2911} which are the poles of cubics meeting the circumcircle at the same points as the orthocubic **K006**.

Theorem 12. *$\mathcal{K}$ meets the circumcircle at the same points as the isogonal pivotal cubic with pivot $P = u : v : w$ if and only if its pivot π lie on the cubic $\mathcal{K}_{\rm pivot}$ with equation*

$$\sum_{\text{cyclic}} (v+w)(c^4y - b^4z)\; x^2 + \left(\sum_{\text{cyclic}} (b^2 - c^2)u\right) xyz = 0.$$

In other words, for any point π on $\mathcal{K}_{\rm pivot}$, there is a pivotal cubic with pivot π meeting the circumcircle at the same points as the isogonal pivotal cubic with pivot $P = u : v : w$.

$\mathcal{K}_{\rm pivot}$ is a circum-cubic tangent at A, B, C to the symmedians. It passes through P, the points on the circumcircle and on the isogonal pivotal cubic with pivot P, the infinite points of the isogonal pivotal cubic with pivot the complement of P, the vertices of the cevian triangle of tc$P$, the isotomic conjugate of the complement of P.

Following the example above, with $P = H$, $\mathcal{K}_{\rm pivot}$ is also a nodal cubic with node H and nodal tangents parallel to the asymptotes of the Jerabek hyperbola. It contains X_3, X_4, X_8, X_{76}, X_{847} which are the pivots of cubics meeting the circumcircle at the same points as the orthocubic, three of them being $\mathrm{p}\mathcal{K}(X_{193}, X_{76})$, $\mathrm{p}\mathcal{K}(X_{571}, X_3)$ and $\mathrm{p}\mathcal{K}(X_{2911}, X_8)$.

Remark. Adding up the equations of $\mathcal{K}_{\rm pole}$ and $\mathcal{K}_{\rm pivot}$ shows that these two cubics generate a pencil containing the $\mathrm{p}\mathcal{K}$ with pole the X_{32}-isoconjugate of cP, pivot the X_{39}-isoconjugate of cP and isopivot X_{251}.

For example, with $P = X_{69}$, this cubic is $\mathrm{p}\mathcal{K}(X_6, X_{141})$. The nine common points of all the cubics of the pencil are A, B, C, K, X_{1169} and the four foci of the inscribed ellipse with center X_{141}, perspector X_{76}.

3.4. *Pivotal $\mathcal{K}_{\rm pole}$ and $\mathcal{K}_{\rm pivot}$.* The equations of $\mathcal{K}_{\rm pole}$ and $\mathcal{K}_{\rm pivot}$ clearly show that these two cubics are pivotal cubics if and only if P lies on the line GK. This gives the two following corollaries.

Corollary 13. *When P lies on the line GK, $\mathcal{K}_{\rm pole}$ is a pivotal cubic and contains K, X_{25}, X_{32}. Its pivot is gc$P$ (on the circum-conic through G and K) and its isopivot is X_{32}. Its pole is the barycentric product of X_{32} and gc$P$. It lies on the circum-conic through X_{32} and X_{251}.*

All these cubics belong to a same pencil of pivotal cubics. Furthermore, $\mathcal{K}_{\rm pole}$ contains the cevian quotients of the pivot gc$P$ and K, X_{25}, X_{32}. Each of these points is the third point of the cubic on the corresponding sideline of the triangle with vertices K, X_{25}, X_{32}. In particular, X_{25} gives the point gt$P$.

Table 1 shows a selection of these cubics.

P	$\mathcal{K}_{\text{pole}}$ contains K, X_{25}, X_{32} and	cubic
X_2	X_{31}, X_{41}, X_{184}, X_{604}, X_{2199}	**K346**
X_{69}	X_2, X_3, X_{66}, X_{206}, X_{1676}, X_{1677}	**K177**
X_{81}	X_{1169}, X_{1333}, X_{2194}, X_{2206}	
X_{86}	X_{58}, X_{1171}	
X_{193}	X_{1974}, X_{3053}	
X_{298}	X_{15}, X_{2981}	
X_{323}	X_{50}, X_{1495}	
X_{325}	X_{511}, X_{2987}	
X_{385}	X_{1691}, X_{1976}	
X_{394}	X_{154}, X_{577}	
X_{491}	X_{372}, X_{589}	
X_{492}	X_{371}, X_{588}	
X_{524}	X_{111}, X_{187}	
X_{1270}	X_{493}, X_{1151}	
X_{1271}	X_{494}, X_{1152}	
X_{1654}	X_{42}, X_{1918}, X_{2200}	
X_{1992}	X_{1383}, X_{1384}	
X_{1994}	X_{51}, X_{2965}	
X_{2895}	X_{37}, X_{213}, X_{228}, X_{1030}	
atX_{1916}	X_{237}, X_{384}, X_{385}, X_{694}, X_{733}, X_{904}, X_{1911}, X_{2076}, X_{3051}	

Table 1. $\mathcal{K}_{\text{pole}}$ with P on the line GK.

Remark. atX_{1916} is the anticomplement of the isotomic conjugate of X_{1916}.

Corollary 14. *When P lies on the line GK, $\mathcal{K}_{\text{pivot}}$ contains P, G, H, K. Its pole is gc$P$ (on the cubic) and its pivot is tc$P$ on the Kiepert hyperbola.*

All these cubics also belong to a same pencil of pivotal cubics.

Table 2 shows a selection of these cubics.

We remark that $\mathcal{K}_{\text{pole}}$ is the isogonal of the isotomic transform of $\mathcal{K}_{\text{pivot}}$ but this correspondence is not generally true for the pivot π and the pole ω. To be more precise, for π on $\mathcal{K}_{\text{pivot}}$, the pole ω on $\mathcal{K}_{\text{pole}}$ is the Ceva-conjugate of gc$P$ and gtπ.

From the two corollaries above, we see that, given an isogonal pivotal cubic $\mathcal{K}$ with pivot on the line GK, we can always find two cubics with poles X_{25}, X_{32} and three cubics with pivots G, H, K sharing the same points on the circumcircle as $\mathcal{K}$. Obviously, there are other such cubics but their pole and pivot both depend of P. In particular, we have $\text{p}\mathcal{K}(\text{gc}P, \text{tc}P)$ and $\text{p}\mathcal{K}(O\times \text{gc}P, \text{gc}P)$.

We illustrate this with $P = G$ (and gc$P = K$) in which case $\mathcal{K}_{\text{pivot}}$ is the Thomson cubic **K002** and $\mathcal{K}_{\text{pole}}$ is **K346**. For π and ω chosen accordingly on these

P	$\mathcal{K}_{\rm pivot}$ contains X_2, X_4, X_6 and	cubic
X_2	X_1, X_3, X_9, X_{57}, X_{223}, X_{282}, X_{1073}, X_{1249}	**K002**
X_6	X_{83}, X_{251}, X_{1176}	
X_{69}	X_{22}, X_{69}, X_{76}, X_{1670}, X_{1671}	**K141**
X_{81}	X_{21}, X_{58}, X_{81}, X_{572}, X_{961}, X_{1169}, X_{1220}, X_{1798}, X_{2298}	**K379**
X_{86}	X_{86}, X_{1126}, X_{1171}	
X_{193}	X_{25}, X_{193}, X_{371}, X_{372}, X_{2362}	**K233**
X_{298}	X_{298}, X_{2981}	
X_{323}	X_{30}, X_{323}, X_{2986}	
X_{325}	X_{325}, X_{2065}, X_{2987}	
X_{385}	X_{98}, X_{237}, X_{248}, X_{385}, X_{1687}, X_{1688}, X_{1976}	**K380**
X_{394}	X_{20}, X_{394}, X_{801}	
X_{491}	X_{491}, X_{589}	
X_{492}	X_{492}, X_{588}	
X_{524}	X_{23}, X_{111}, X_{524}, X_{671}, X_{895}	**K273**
X_{1270}	X_{493}, X_{1270}	
X_{1271}	X_{494}, X_{1271}	
X_{1611}	X_{439}, X_{1611}	
X_{1654}	X_{10}, X_{42}, X_{71}, X_{199}, X_{1654}	
X_{1992}	X_{598}, X_{1383}, X_{1992}, X_{1995}	**K283**
X_{1993}	X_{54}, X_{275}, X_{1993}	
X_{1994}	X_5, X_{1166}, X_{1994}	
X_{2287}	X_{1817}, X_{2287}	
X_{2895}	X_{37}, X_{72}, X_{321}, X_{2895}, X_{2915}	
X_{3051}	X_{384}, X_{3051}	
atX_{1916}	X_{39}, X_{256}, X_{291}, X_{511}, X_{694}, X_{1432}, X_{1916}	**K354**

Table 2. $\mathcal{K}_{\rm pivot}$ with P on the line GK

cubics, we obtain a family of pivotal cubics meeting the circumcircle at the same points as the Thomson cubic. See Table 3 and Figure 11.

With $P = X_{69}$ (isotomic conjugate of H), we obtain several interesting cubics related to the centroid G=gc$P$, the circumcenter O=gt$P$. $\mathcal{K}_{\rm pole}$ is **K177**, $\mathcal{K}_{\rm pivot}$ is **K141** and the cubics p$\mathcal{K}(X_2, X_{76})$ = **K141**, p$\mathcal{K}(X_3, X_2)$ = **K168**, p$\mathcal{K}(X_6, X_{69})$ = **K169**, p$\mathcal{K}(X_{32}, X_{22})$ = **K174**, p$\mathcal{K}(X_{206}, X_6)$ have the same common points on the circumcircle.

π	ω (X_i or SEARCH)	cubic or X_i on the cubic
X_1	X_{41}	X_1, X_6, X_9, X_{55}, X_{259}
X_2	X_6	**K002**
X_3	X_{32}	**K172**
X_4	0.1732184721703	X_4, X_6, X_{20}, X_{25}, X_{154}, X_{1249}
X_6	X_{184}	**K167**
X_9	X_{31}	X_1, X_6, X_9, X_{56}, X_{84}, X_{165}, X_{198}, X_{365}
X_{57}	X_{2199}	X_6, X_{40}, X_{56}, X_{57}, X_{198}, X_{223}
X_{223}	X_{604}	X_6, X_{57}, X_{223}, X_{266}, X_{1035}, X_{1436}
X_{282}	0.3666241407629	X_6, X_{282}, X_{1035}, X_{1436}, X_{1490}
X_{1073}	0.6990940852287	X_6, X_{64}, X_{1033}, X_{1073}, X_{1498}
X_{1249}	X_{25}	X_4, X_6, X_{64}, X_{1033}, X_{1249}

Table 3. Thomson cubic **K002** and some related cubics

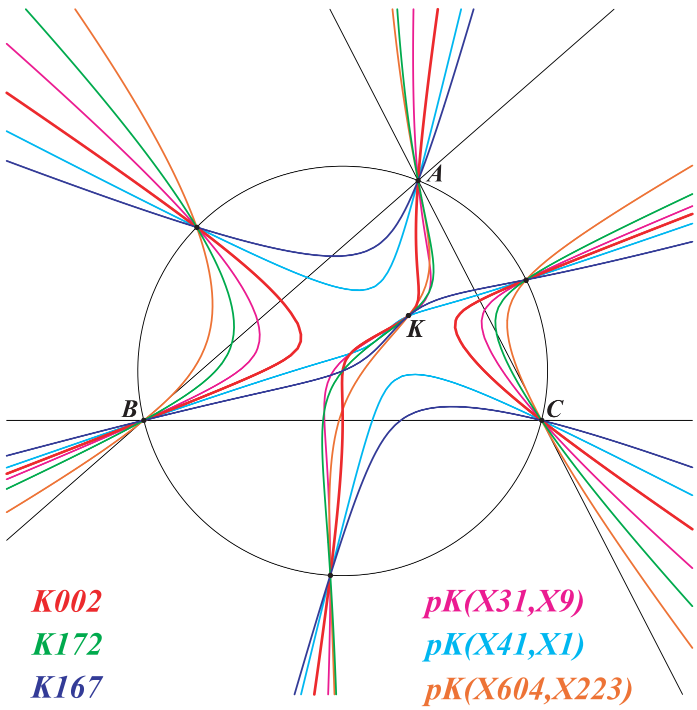

Figure 11. Thomson cubic **K002** and some related cubics

4. Non isogonal pivotal cubics and concurrent tangents

We now generalize Theorem 7 for any pivotal cubic with pole $\Omega = p : q : r$ and pivot $P = u : v : w$, meeting the circumcircle at A, B, C and three other points Q_1, Q_2, Q_3. We obtain the two following theorems.

Theorem 15. *For a given pole Ω, the tangents at Q_1, Q_2, Q_3 to the pivotal cubic with pole Ω are concurrent if and only if its pivot P lies on the quintic $\mathcal{Q}(\Omega)$.*

Remark. $\mathcal{Q}(\Omega)$ contains the following points:
– A, B, C which are nodes,
– the square roots of Ω,
– tgΩ, the Ω-isoconjugate of K,
– the vertices of the cevian triangle of $Z = \left(\frac{(c^4pq+b^4rp-a^4qr)p}{a^2} : \cdots : \cdots\right)$, the isoconjugate of the crossconjugate of K and tgΩ in the isoconjugation with fixed point tgΩ,
– the common points of the circumcircle and the trilinear polar Δ_1 of tgΩ,
– the common points of the circumcircle and the line Δ_2 passing through tgΩ and the cross-conjugate of K and tgΩ.

Theorem 16. *For a given pivot P, the tangents at Q_1, Q_2, Q_3 to the pivotal cubic with pivot P are concurrent if and only if its pole Ω lies on the quintic $\mathcal{Q}'(P)$.*

Remark. $\mathcal{Q}'(P)$ contains the following points:
– the barycentric product $P \times K$,
– A, B, C which are nodes, the tangents being the cevian lines of X_{32} and the sidelines of the anticevian triangle of $P \times K$,
– the barycentric square P^2 of P and the vertices of its cevian triangle, the tangent at P^2 passing through $P \times K$.

5. Equilateral triangles

The McCay cubic meets the circumcircle at A, B, C and three other points N_a, N_b, N_c which are the vertices of an equilateral triangle. In this section, we characterize all the pivotal cubics $\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$ having the same property.

We know that the isogonal conjugates of three such points N_a, N_b, N_c are the infinite points of an equilateral cubic (a $\mathcal{K}_{60}$, see [2]) and that the isogonal transform of $\mathcal{K}$ is another pivotal cubic $\mathcal{K}' = \mathrm{p}\mathcal{K}(\Omega', P')$ with pole Ω' the X_{32}-isocongate of Ω, with pivot P' the barycentric product of P and the isogonal conjugate of Ω. Hence $\mathcal{K}$ meets the circumcircle at the vertices of an equilateral triangle if and only if $\mathcal{K}'$ is a $\mathrm{p}\mathcal{K}_{60}$.

Following [2, §6.2], we obtain the following theorem.

Theorem 17. *For a given pole Ω or a given pivot P, there is one and only one pivotal cubic $\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$ meeting the circumcircle at the vertices of an equilateral triangle.*

With $\Omega = K$ (or $P = O$) we obviously obtain the McCay cubic and the equilateral triangle is the circumnormal triangle. More generally, a $\mathrm{p}\mathcal{K}$ meets the circumcircle at the vertices of circumnormal triangle if and only if its pole Ω lies on the circum-cubic **K378** passing through K, the vertices of the cevian triangle of the Kosnita point X_{54}, the isogonal conjugates of X_{324}, X_{343}. The tangents at A, B, C are the cevians of X_{32}. The cubic is tangent at K to the Brocard axis and K is a flex on the cubic. See [3] and Figure 12.

The locus of pivots of these same cubics is **K361**. See [3] and Figure 13.

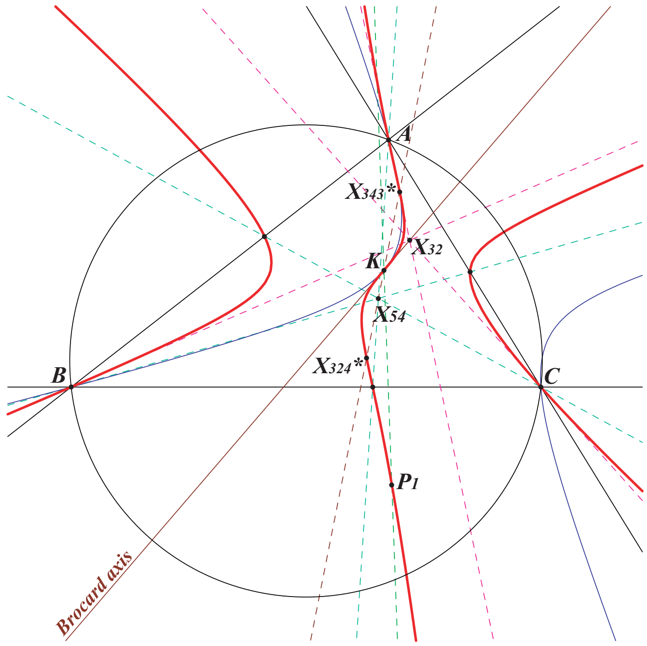

Figure 12. **K378**, the locus of poles of circumnormal p$\mathcal{K}$s

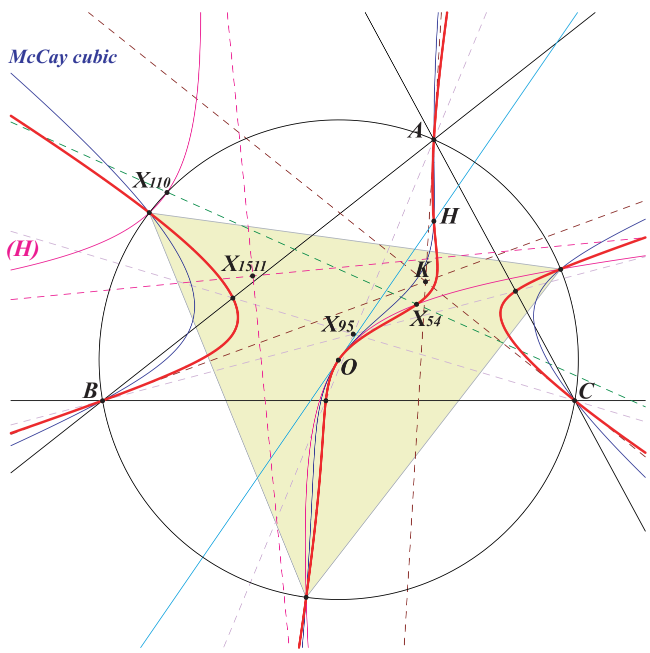

Figure 13. **K361**, the locus of pivots of circumnormal p$\mathcal{K}$s

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Forum Geometricorum
Volume 7 (2007) 231–247.

FORUM GEOM
ISSN 1534-1178

On a Construction of Hagge

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Abstract. In 1907 Hagge constructed a circle associated with each cevian point P of triangle ABC. If P is on the circumcircle this circle degenerates to a straight line through the orthocenter which is parallel to the Wallace-Simson line of P. We give a new proof of Hagge's result by a method based on reflections. We introduce an axis associated with the construction, and (via an areal analysis) a conic which generalizes the nine-point circle. The precise locus of the orthocenter in a Brocard porism is identified by using Hagge's theorem as a tool. Other natural loci associated with Hagge's construction are discussed.

1. Introduction

One hundred years ago, Karl Hagge wrote an article in *Zeitschrift für Mathematischen und Naturwissenschaftliche Unterricht* entitled (in loose translation) "The Fuhrmann and Brocard circles as special cases of a general circle construction" [5]. In this paper he managed to find an elegant extension of the Wallace-Simson theorem when the generating point is not on the circumcircle. Instead of creating a line, one makes a circle through seven important points. In §2 we give a new proof of the correctness of Hagge's construction, extend and apply the idea in various ways. As a tribute to Hagge's beautiful insight, we present this work as a centenary celebration. Note that the name Hagge is also associated with other circles [6], but here we refer only to the construction just described. Here we present new synthetic arguments to justify Hagge's construction, but the first author has also performed detailed areal calculations which provide an algebraic alternative in [2].

The triangle ABC has circumcircle Γ, circumcenter O and orthocenter H. See Figure 1. Choose P a point in the plane of ABC. The cevian lines AP, BP, CP meet Γ again at D, E and F respectively. Reflect D in BC to a point U, E in CA to a point V and F in AB to a point W. Let UP meet AH at X, VP meet BH at Y and WP meet CH at Z. Hagge proved that there is a circle passing through X, Y, Z, U, V, W and H [5, 7]. See Figure 1. Our purpose is to amplify this observation.

Hagge explicitly notes [5] the similarities between ABC and XYZ, between DEF and UVW, and the fact that both pairs of triangles ABC, DEF and XYZ, UVW are in perspective through P. There is an indirect similarity which carries the points $ABCDEFP$ to $XYZUVWP$.

Peiser [8] later proved that the center $h(P)$ of this Hagge circle is the rotation through π about the nine-point center of ABC of the isogonal conjugate P^* of P. His proof was by complex numbers, but we have found a direct proof by classical

Publication Date: December 18, 2007. Communicating Editor: Paul Yiu.

We thank the editor Paul Yiu for very helpful suggestions which improved the development of Section 5.

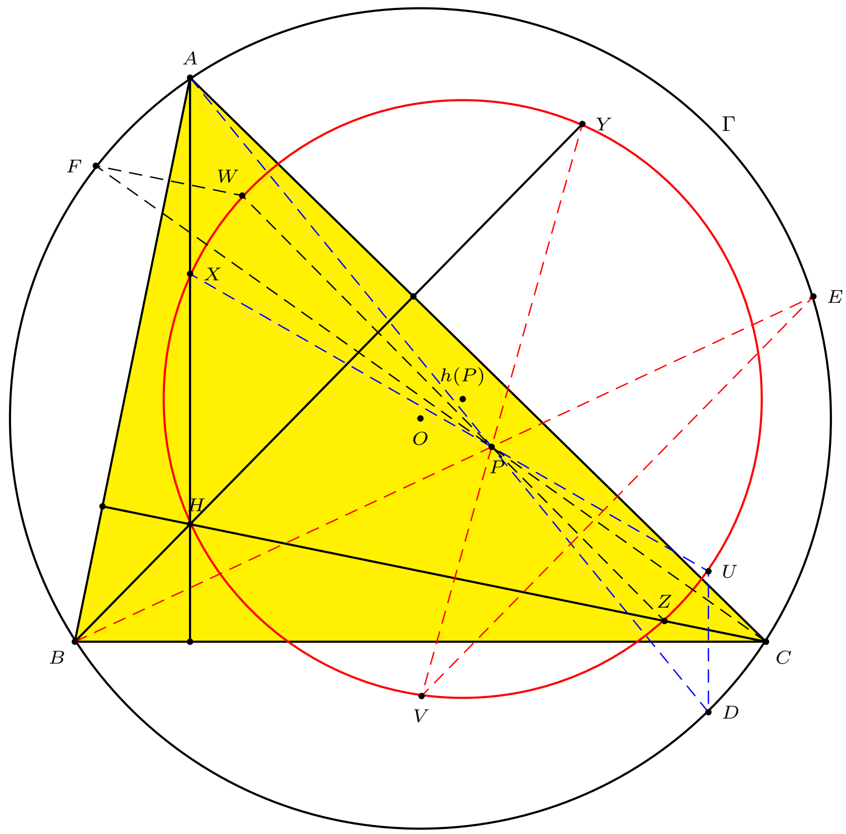

Figure 1. The Hagge construction

means [4]. In our proof of the validity of Hagge's construction we work directly with the center of the circle, whereas Hagge worked with the point at the far end of the diameter through H. This gives us the advantage of being able to study the distribution of points on a Hagge circle by means of reflections in lines through its center, a device which was not available with the original approach.

The point P^* is collinear with G and T, the far end of the diameter from H. The vector argument which justifies this is given at the start of §5.1. Indeed, we show that $P^*G : GT = 1 : 2$.

There are many important special cases. Here are some examples, but Hagge [5] listed even more.

(i) When $P = K$, the symmedian point, the Hagge circle is the orthocentroidal circle. [1]

(ii) When $P = I$, the incenter, the Hagge circle is the Fuhrmann circle.

(iii) When $P = O$, the circumcenter, the Hagge circle and the circumcircle are concentric.

---

[1]In [5] Hagge associates the name Böklen with the study of this circle (there were two geometers with this name active at around that time), and refers the reader to a work of Prof Dr Lieber, possibly H. Lieber who wrote extensively on advanced elementary mathematics in the *fin de siècle*.

(iv) When $P = H$, the orthocenter, the Hagge circle degenerates to the point H.

(v) The circumcenter is the orthocenter of the medial triangle, and the Brocard circle on diameter OK arises as a Hagge circle of the medial triangle with respect to the centroid G of ABC.

Note that UH is the doubled Wallace-Simson line of D, by which we mean the enlargement of the Wallace-Simson line with scale factor 2 from center D. Similarly VH and WH are the doubled Wallace-Simson lines of E and F. Now it is well known that the angle between two Wallace-Simson lines is half the angle subtended at O by the generating points. This applies equally well to doubled Wallace-Simson lines. A careful analysis (taking care to distinguish between angles and their supplements) will yield the angles between UH, VH and WH, from which it can be deduced that UVW is indirectly similar to DEF. We will not explain the details but rather we present a robust argument for Proposition 2 which does not rely on scrupulous bookkeeping.

Incidentally, if P is on Γ, then the Hagge circle degenerates to the doubled Wallace-Simson line of P. For the rest of this paper, we make the explicit assumption that P is not on Γ. The work described in the rest of this introduction is not foreshadowed in [5]. Since $ABCDEFP$ is similar to $XYZUVWP$, it follows that ABC is indirectly similar to XYZ and the similarity sends DEF to UVW. The point P turns out to be the unique fixed point of this similarity. This similarity must carry a distinguished point H^+ on Γ to H. We will give a geometric recipe for locating H^+ in Proposition 3.

This process admits of extension both inwards and outwards. One may construct the Hagge circle of XYZ with respect to P, or find the triangle RST so that the Hagge circle of RST with respect to P is Γ (with ABC playing the former role of XYZ). The composition of two of these indirect similarities is an enlargement with positive scale factor from P.

Proposition 2 sheds light on some of our earlier work [3]. Let G be the centroid, K the symmedian point, and ω the Brocard angle of triangle ABC. Also, let J be the center of the orthocentroidal circle (the circle on diameter GH). We have long been intrigued by the fact that $\dfrac{OK^2}{R^2} = \dfrac{JK^2}{JG^2}$ since areal algebra can be used to show that each quantity is $1-3\tan^2\omega$. In §3.3 we will explain how the similarity is a geometric explanation of this suggestive algebraic coincidence. In [3] we showed how to construct the sides of (non-equilateral) triangle ABC given only the data O, G, K. The method was based on finding a cubic which had a^2, b^2, c^2 as roots. We will present an improved algebraic explanation in §3.2.

We show in Proposition 4 that there is a point F which when used as a cevian point, generates the same Hagge circle for every triangle in a Brocard porism. Thus the locus of the orthocenter in a Brocard porism must be confined to a circle. We describe its center and radius. We also exhibit a point which gives rise to a fixed Hagge circle with respect to the medial triangles, as the reference triangle ranges over a Brocard porism.

We make more observations about Hagge's configuration. Given the large number of points lying on conics (circles), it is not surprising that Pascal's hexagon theorem comes into play. Let VW meet AH at L, WU meet BH at M, and UV meets CH at N. In §4 we will show that $LMNP$ are collinear, and we introduce the term Hagge axis for this line.

In §5 we will exhibit a *midpoint conic* which passes through six points associated with the Hagge construction. In special case (iv), when $P = H$, this conic is the nine-point circle of ABC. Drawings lead us to conjecture that the center of the midpoint conic is N.

In §6 we study some natural loci associated with Hagge's construction.

2. The Hagge Similarity

We first locate the center of the Hagge circle, but not, as Peiser [8] did, by using complex numbers. A more leisurely exposition of the next result appears in [4].

Proposition 1. *Given a point P in the plane of triangle ABC, the center $h(P)$ of the Hagge circle associated with P is the point such the nine-point center N is the midpoint of $h(P)P^*$ where P^* denotes the isogonal conjugate of P.*

Proof. Let AP meet the circumcircle at D, and reflect D in BC to the point U. The line UH is the doubled Simson line of D, and the reflections of D in the other two sides are also on this line. The isogonal conjugate of D is well known to be the point at infinity in the direction parallel to AP^* . (This is the degenerate case of the result that if D' is not on the circumcircle, then the isogonal conjugate of D' is the center of the circumcircle of the triangle with vertices the reflections of D' in the sides of ABC).

Thus $UH \perp AP^*$. To finish the proof it suffices to show that if OU' is the rotation through π of UH about N, then AP^* is the perpendicular bisector of OU'. However, $AO = R$ so it is enough to show that $AU' = R$. Let A' denote the rotation through π of A about N. From the theory of the nine-point circle it follows that A' is also the reflection of O in BC. Therefore $OUDA'$ is an isosceles trapezium with $OA'//UD$. Therefore $AU' = A'U = OD = R$. □

We are now in a position to prove what we call the Hagge similarity which is the essence of the construction [5].

Proposition 2. *The triangle ABC has circumcircle Γ, circumcenter O and orthocenter H. Choose a point P in the plane of ABC other than A, B, C. The cevian lines AP, BP, CP meet Γ again at D, E, F respectively. Reflect D in BC to a point U, E in CA to a point V and F in AB to a point W. Let UP meet AH at X, VP meet BH at Y and WP meet CH at Z. The points $XYZUVWH$ are concyclic, and there is an indirect similarity carrying $ABCDEFP$ to $XYZUVWP$.*

Discussion. The strategy of the proof is as follows. We consider six lines meeting at a point. Any point of the plane will have reflections in the six lines which are concyclic. The angles between the lines will be arranged so that there is an indirect similarity carrying $ABCDEF$ to the reflections of H in the six lines. The location of the point of concurrency of the six lines will be chosen so that the relevant six reflections of H are $UVWX_1Y_1Z_1$ where X_1, Y_1 and Z_1 are to be determined, but are placed on the appropriate altitudes so that they are candidates to become X, Y and Z respectively. The similarity then ensures that UVW and $X_1Y_1Z_1$ are in perspective from a point P'. Finally we show that $P = P'$, and it follows immediately that $X = X_1, Y = Y_1$ and $Z = Z_1$. We rely on the fact that we know where to make the six lines cross, thanks to Proposition 1. This is not the proof given in [5].

Proof of Proposition 2. Let $\angle DAC = a_1$ and $\angle BAD = a_2$. Similarly we define b_1, b_2, c_1 and c_2. We deduce that the angles subtended by A, F, B, D, D and E at O as shown in Figure 2.

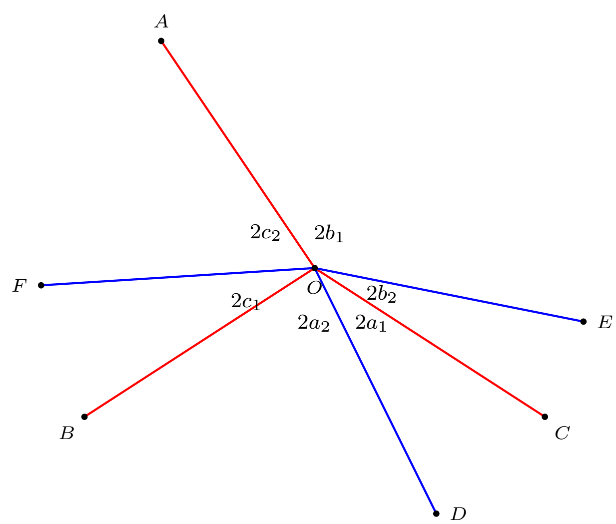

Figure 2. Angles subtended at the circumcenter of ABC

By Proposition 1, $h(P)$ is on the perpendicular bisector of UH which is parallel to AP^* (and similar results by cyclic change).

Draw three lines through $h(P)$ which are parallel to the sides of ABC and three more lines which are parallel to AP^* , BP^* and CP^* . See Figure 3.

Let X_1, Y_1 and Z_1 be the reflections of H in the lines parallel to BC, CA and AB respectively. Also U, V and W are the reflections of H in the lines parallel to AP^*, BP^* and CP^*. Thus $X_1Y_1Z_1UVW$ are all points on the Hagge circle. The angles between the lines are as shown, and the consequences for the six reflections of H are that $X_1Y_1Z_1UVW$ is a collection of points which are indirectly similar to $ABCDEF$. It is not necessary to know the location of H in Figure 3 to deduce this result. Just compare Figures 2 and 4. The point is that $\angle X_1h(P)V = \angle EOA$.

A similar argument works for each adjacent pair of vertices in the cyclic list $X_1VZ_1UY_1W$ and an indirect similarity is established. Let this similarity carrying

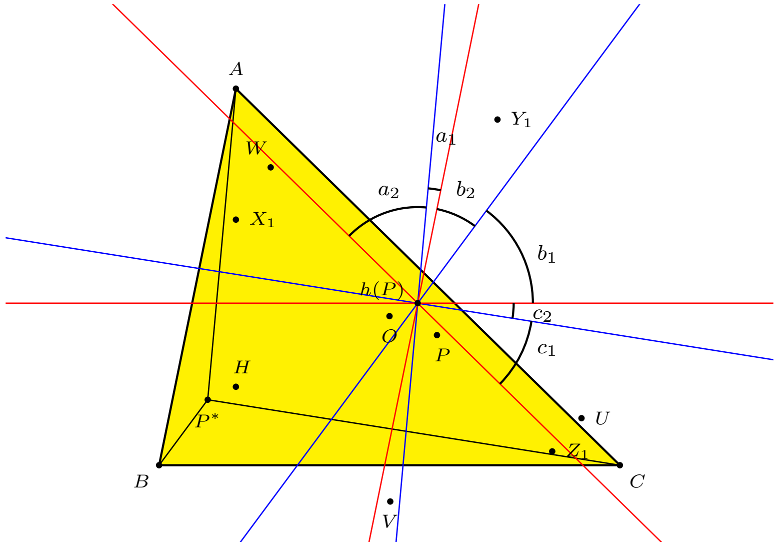

Figure 3. Reflections of the orthocenter

$ABCDEF$ to $X_1Y_1Z_1UVW$ be κ. It remains to show that $\kappa(P) = P$ (for then it will follow immediately that $X_1 = X$, $Y_1 = Y$ and $Z_1 = Z$).

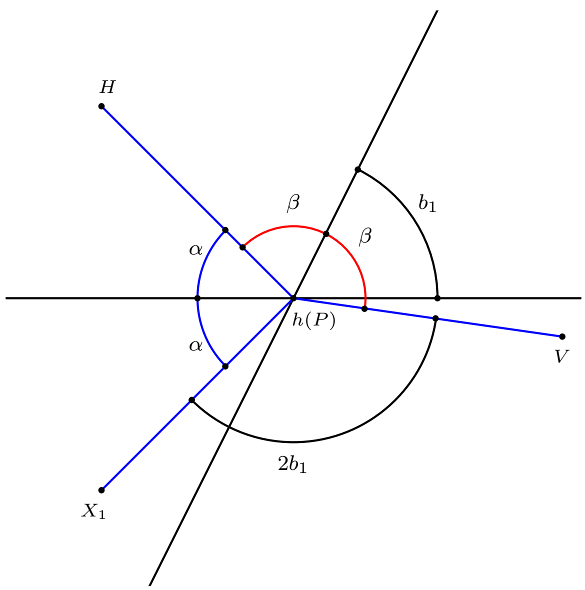

Figure 4. Two reflections of H

Now $X_1Y_1Z_1$ is similar to ABC, and the vertices of $X_1Y_1Z_1$ are on the altitudes of ABC. Also UVW is similar to DEF, and the lines X_1U, Y_1V and Z_1W are concurrent at a point P_1. Consider the directed line segments AD and X_1U which meet at Q. The lines AX_1 and UD are parallel so AX_1Q and DUQ are similar triangles, so in terms of lengths, $AQ : QD = X_1Q : QU$. Since κ carries AD to X_1U, it follows that Q is a fixed point of κ. Now if κ had at least two fixed points, then it would have a line of fixed points, and would be a reflection in that line. However κ takes DEF to UVW, to this line would have to be BC, CA and AB. This is absurd, so Q is the unique fixed point of κ. By cyclic change Q is on AD, BE and CF so $Q = P$. Also Q is on X_1U, Y_1V and Z_1W so $Q = P_1$. Thus X_1U, Y_1V and Z_1W concur at P. Therefore $X_1 = X$, $Y_1 = Y$ and $Z_1 = Z$. □

Proposition 3. *The similarity of Proposition 2 applied to ABC, P carries a point H^+ on Γ to H. The same result applied to XYZ, P carries H to the orthocenter H^- of XYZ. We may construct H^+ by drawing the ray PH^- to meet Γ at H^+.*

Proof. The similarity associated with ABC and P is expressible as: reflect in PA, scale by a factor of λ from P, and rotate about P through a certain angle. Note that if we repeat the process, constructing a similarity using the XYZ as the reference triangle, but still with cevian point P, the resulting similarity will be expressible as: reflect in XP, scale by a factor of λ from P, and rotate about P through a certain angle. Since $XYZP$ is indirectly similar to $ABCP$, the angles through which the rotation takes place are equal and opposite. The effect of composing the two similarities will be an enlargement with center P and (positive) scale factor λ^2. □

Thus in a natural example one would expect the point H^+ to be a natural point. Drawings indicate that when we consider the Brocard circle, H^+ is the Tarry point.

3. Implications for the Symmedian Point and Brocard geometry

3.1. *Standard formulas.* We first give a summary of useful formulas which can be found or derived from many sources, including Wolfram Mathworld [11]. The variables have their usual meanings.

$$abc = 4R\triangle, \tag{1}$$
$$a^2 + b^2 + c^2 = 4\triangle \cot\omega, \tag{2}$$
$$a^2b^2 + b^2c^2 + c^2a^2 = 4\triangle^2 \csc^2\omega, \tag{3}$$
$$a^4 + b^4 + c^4 = 8\triangle^2(\csc^2\omega - 2), \tag{4}$$

where (3) can be derived from the formula

$$\begin{aligned} R_B &= \frac{abc\sqrt{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}}{4(a^2+b^2+c^2)\triangle} \\ &= \frac{R\sqrt{1-4\sin^2\omega}}{2\cos\omega} \end{aligned}$$

for the radius R_B of the Brocard circle given in [11]. The square of the distance between the Brocard points was determined by Shail [9]:

$$\Omega\Omega'^2 = 4R^2\sin^2\omega(1-4\sin^2\omega) \tag{5}$$

which in turn is an economical way of expressing

$$\frac{a^2b^2c^2(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)}{(a^2b^2+b^2c^2+c^2a^2)^2}.$$

We will use these formulas in impending algebraic manipulations.

3.2. *The symmedian point.* Let G be the centroid, K the symmedian point, and ω be the Brocard angle of triangle ABC. Also let J be the center of the orthocentroidal circle (the circle on diameter GH). It is an intriguing fact that

$$\frac{OK^2}{R^2}=\frac{JK^2}{JG^2} \tag{6}$$

since one can calculate that each quantity is $1-3\tan^2\omega$. The similarity of Proposition 2 explains this suggestive algebraic coincidence via the following paragraph.

We first elaborate on Remark (v) of §1. Let h_{med} denote the function which assigns to a point P the center $h_{\mathrm{med}}(P)$ of the Hagge circle associated with P when the triangle of reference is the medial triangle. The medial triangle is the enlargement of ABC from G with scale factor $-\frac{1}{2}$. Let K_{med} be the symmedian point of the medial triangle. Now K_{med}, G, K are collinear and $K_{\mathrm{med}}G : GK = 1:2 = QG : GN$, where Q is the midpoint of ON. Thus, triangle GNK and GQK_{med} are similar and Q is the nine-point center of the medial triangle. By [8], $h\mathrm{med}(G)$ is the reflection in Q of K_{med}. But the line $Qh_{\mathrm{med}}(G)$ is parallel to NK and Q is the midpoint of ON. Therefore, $h_{\mathrm{med}}(G)$ is the midpoint of OK, and so is the center of the Brocard circle of ABC. The similarity of Proposition 2 and the one between the reference and medial triangle, serve to explain (6).

3.3. *The Brocard porism.* A Brocard porism is obtained in the following way. Take a triangle ABC and its circumcircle. Draw cevian lines through the symmedian point. There is a unique conic (the Brocard ellipse) which is tangent to the sides where the cevians cuts the sides. The Brocard points are the foci of the ellipse. There are infinitely many triangle with this circumcircle and this inconic. Indeed, every point of the circumcircle arises as a vertex of a unique such triangle.

These poristic triangles have the same circumcenter, symmedian point, Brocard points and Brocard angle. For each of them, the inconic is their Brocard ellipse. Any geometrical feature of the triangle which can be expressed exclusively in terms of R, ω and the locations of O and K will give rise to a conserved quantity among the poristic triangles.

This point of view also allows an improved version of the algebraic proof that a, b and c are determined by O, G and K [3]. Because of the ratios on the Euler line, the orthocenter H and the orthocentroidal center are determined. Now Equation (6) determines R and angle ω. However, $9R^2-(a^2+b^2+c^2)=OH^2$ so $a^2+b^2+c^2$ is determined. Also the area $\triangle$ of ABC is determined by (2). Now (1) means abc and so $a^2b^2c^2$ is determined. Also, (3) determines $a^2b^2+b^2c^2+c^2a^2$. Thus the polynomial $(X-a^2)(X-b^2)(X-c^2)$ is determined and so the sides of the triangle can be deduced.

As we move through triangles in a Brocard porism using a fixed cevian point P, the Hagge circles of the triangles vary in general, but if P is chosen appropriately, the Hagge circle if each triangle in the porism is the same.

Proposition 4. *Let F be the fourth power point* [2] *of a triangle in a Brocard porism, so that it has areal coordinates (a^4, b^4, c^4). The fourth power point F is the same point for all triangles in the porism. Moreover, when $P = F$, the Hagge circle of each triangle is the same.*

Proof. Our plan is to show that the point $h(F)$ is the same for all triangles in the porism, and then to show that the distance $h(F)H$ is also constant (though the orthocenters H vary). Recall that the nine-point center is the midpoint of O and H, and of F^* and $h(P)$. Thus there is a (variable) parallelogram $Oh(F)HF^*$ which will prove very useful.

The fourth power point F is well known to lie on the Brocard axis where the tangents to the Brocard circle at Ω and Ω' meet. Thus F is the same point for all triangles in the Brocard porism. The isogonal conjugate of F (incidentally the isotomic conjugate of the symmedian point) is $F^* = K_t = \left(\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}\right)$.

In any triangle OK is parallel to F^*H. To see this, note that OK has equation

$$b^2c^2(b^2 - c^2)x + c^2a^2(c^2 - a^2)y + a^2b^2(a^2 - b^2)z = 0.$$

Also F^*H has equation

$$\sum_{\text{cyclic}} b^2c^2(b^2 - c^2)(b^2 + c^2 - a^2)x = 0.$$

These equations are linearly dependent with $x + y + z = 0$ and hence the lines are parallel. (DERIVE confirms that the 3×3 determinant vanishes). In a Hagge circle with $P = F$, $P^* = F^*$ and $F^*Hh(F)O$ is a parallelogram. Thus OK is parallel to F^*H and because of the parallelogram, $h(F)$ is a (possibly variable) point on the Brocard axis OK.

Next we show that the point $h(F)$ us a common point for the poristic triangles. The first component of the normalized coordinates of F^* and H are

$$F^*_x = \frac{b^2c^2}{a^2b^2 + b^2c^2 + c^2a^2}$$

and

$$H_x = \frac{(a^2 + b^2 - c^2)(c^2 + a^2 - b^2)}{16\triangle^2}$$

where $\triangle$ is the area of the triangle in question. The components of the displacement F^*H are therefore

$$\frac{a^2 + b^2 + c^2}{16\triangle^2}(a^2b^2 + b^2c^2 + c^2a^2)(x, y, z)$$

[2] Geometers who speak trilinear rather than areal are apt to call F the third power point for obvious reasons.

where $x = a^2(a^2b^2 + a^2c^2 - b^4 - c^4)$, with y and z found by cyclic change of a, b, c. Using the areal distance formula this provides

$$F^*H^2 = \frac{a^2b^2c^2(a^2+b^2+c^2)^2(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)}{16\triangle^2(a^2b^2+b^2c^2+c^2a^2)^2}.$$

Using the formulas of §3.1 we see that

$$Oh(F) = F^*H = 2R\cos\omega\sqrt{1-4\sin^2\omega}$$

is constant for the poristic triangles. The point O is fixed so there are just two candidates for the location of $h(F)$ on the common Brocard axis. By continuity $h(F)$ cannot move between these places and so $h(F)$ is a fixed point.

To finish this analysis we must show that the distance $h(F)H$ is constant for the poristic triangles. This distance is the same as F^*O by the parallelogram. If a point X has good areal coordinates, it is often easy to find a formula for OX^2 using the generalized parallel axis theorem [10] because $OX^2 = R^2 - \sigma_X^2$ and σ_X^2 denotes the mean square distance of the triangle vertices from themselves, given that they carry weights which are the corresponding areal coordinates of X.

In our case $F^* = (a^{-2},\ b^{-2},\ c^{-2})$, so

$$\begin{aligned}\sigma_{F^*}^2 &= \frac{1}{(a^{-2}+b^{-2}+c^{-2})^2}(a^2b^{-2}c^{-2}+a^{-2}b^2c^{-2}+a^{-2}b^{-2}c^2)\\ &= \frac{a^2b^2c^2}{a^2b^2+b^2c^2+c^2a^2}(a^4+b^4+c^4).\end{aligned}$$

This can be tidied using the standard formulas to show that $F^*O = R(1-4\sin^2\omega)$. The distance $Hh(F) = F^*O$ is constant for the poristic triangles and $h(F)$ is a fixed point, so the Hagge circle associated with F is the same for all the poristic triangles. □

Corollary 5. *In a Brocard porism, as the poristic triangles vary, the locus of their orthocenters is contained in a circle with their common center $h(F)$ on the Brocard axis, where F is the* (areal) *fourth power point of the triangles. The radius of this circle is $R(1-4\sin^2\omega)$.*

In fact there is a direct method to show that the locus of H in the Brocard porism is a subset of a circle, but this approach reveals neither center nor radius. We have already observed that $\dfrac{JK^2}{JG^2} = 1 - 3\tan^2\omega$ so for triangles in a Brocard porism (with common O and K) we have $\dfrac{JK^2}{JO^2} = \dfrac{1-3\tan^2\omega}{4}$ is constant. So as you consider the various triangle in the porism, J is constrained to move on a circle of Apollonius with center some point on the fixed line OK. Now the vector $\mathbf{OH}$ is $\frac{2}{3}\mathbf{OJ}$, so H is constrained to move on a circle with its center M on the line OK. In fact H can occupy any position on this circle but we do not need this result (which follows from K ranging over a circle center J for triangles in a Brocard porism [3]).

There is a point which, when used as P for the Hagge construction using medial triangles, gives rise to a common Hagge circle as we range over reference triangles in a Brocard porism. We use dashes to indicate the names of points with respect to the medial triangle $A'B'C'$ of a poristic triangle ABC. We now know that F is a common point for the porism, so the distance OF is fixed. Since O is fixed in the Brocard porism and the locus of H is a circle, it follows that the locus of N is a circle with center half way between O and the center of the locus of H.

Proposition 6. *Let P be the center of the Brocard ellipse (the midpoint of the segment joining the Brocard points of ABC). When the Hagge construction is made for the medial triangle $A'B'C'$ using this point P, then for each ABC in the porism, the Hagge circle is the same.*

Proof. If the areal coordinates of a point are (l, m, n) with respect to ABC, then the areal coordinates of this point with respect to the medial triangle are $(m+n-l, n+l-m, l+n-m)$. The reference areals of P are $(a^2(b^2+c^2), b^2(c^2+a^2), c^2(a^2+b^2))$ so the medial areals are (b^2c^2, c^2a^2, a^2b^2). The medial areals of the medial isogonal conjugate $P^\dagger$ of P are (a^4, b^4, c^4). Now the similarity carrying ABC to $A'B'C'$ takes O to N and F to $P^\dagger$. Thus in terms of distance $OF = 2P^\dagger N$ and moreover OF is parallel to $P^\dagger N$. Now, $OP^\dagger Nh'(P)$ is a parallelogram with center the nine-point center of the medial triangle and $h'(P)$ is the center of the medial Hagge circle. It follows that $h'(P)$ lies on OK at the midpoint of OF. Therefore all triangles in the Brocard porism give rise to a Hagge circle of P (with respect to the medial triangle) which is the circle diameter OF. □

Incidentally, P is the center of the locus of N in the Brocard porism. To see this, note that N is the midpoint of OH, so it suffices to show that $OP = PX$ where X is the center of the locus of H in the Brocard cycle (given that P is on the Brocard axis of ABC). However, it is well known that $OP = R\sqrt{1-4\sin^2\omega}$ and in Proposition 4 we showed that $OX = 2R\cos\omega\sqrt{1-4\sin^2\omega}$. We must eliminate the possibility that X and P are on different sides of O. If this happened, there would be at least one triangle for which $\angle HOK = \pi$. However, K is confined to the orthocentroidal disk [3] so this is impossible.

4. The Hagge axis

Proposition 7. *In the Hagge configuration, let VW meet AH at L, WU meet BH at M and UV meet CH and N. Then the points L, M, N and P are collinear.*

We prove the following more general result. In order to apply it, the letters should be interpreted in the usual manner for the Hagge configuration, and Σ should be taken as the Hagge circle.

Proposition 8. *Let three points X, Y and Z lie on a conic Σ and let l_1, l_2, l_3 be three chords XH, YH, ZH all passing through a point H on Σ. Suppose further that P is any point in the plane of Σ, and let XP, YP, ZP meet Σ again at U, V and W respectively. Now, let VW meet l_1 at L, WU meet l_2 at M, UV meet l_3 at N. Then LMN is a straight line passing through P.*

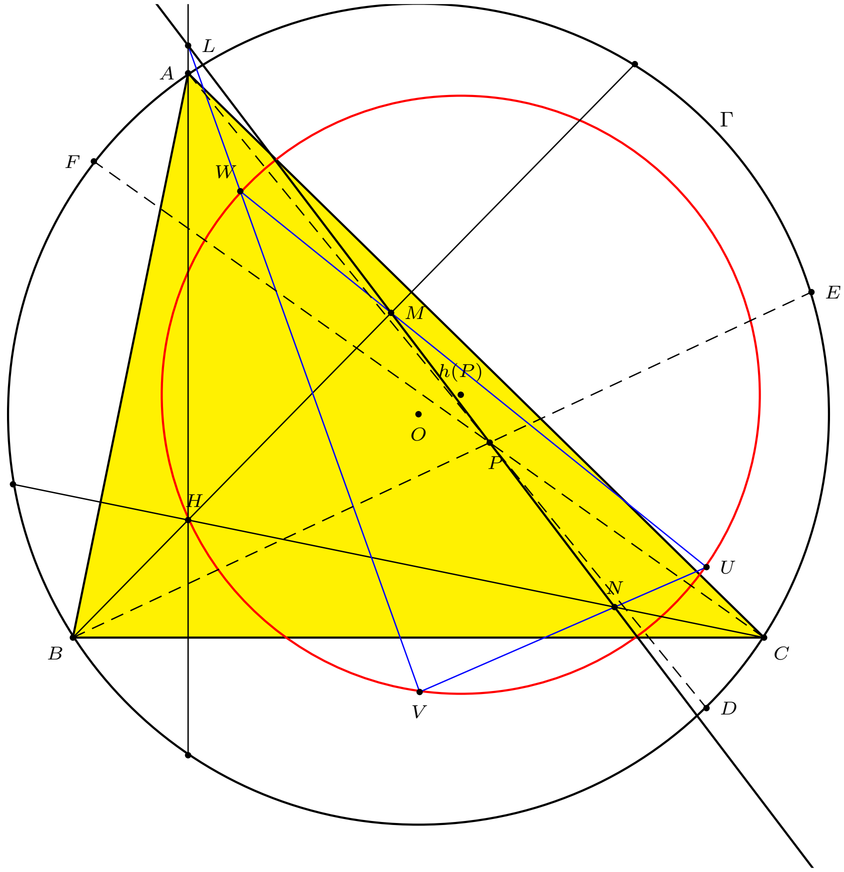

Figure 5. The Hagge axis LMN

Proof. Consider the hexagon $HYVUWZ$ inscribed in Σ. Apply Pascal's hexagon theorem. It follows that M, P, N are collinear. By taking another hexagon N, P, L are collinear. □

5. The Hagge configuration and associated Conics

In this section we give an analysis of the Hagge configuration using barycentric (areal) coordinates. This is both an enterprise in its own right, serving to confirm the earlier synthetic work, but also reveals the existence of an interesting sequence of conics. In what follows ABC is the reference triangle and we take P to have homogeneous barycentric coordinates (u, v, w). The algebra computer package DERIVE is used throughout the calculations.

5.1. *The Hagge circle and the Hagge axis .* The equation of AP is $wy = vz$. This meets the circumcircle, with equation $a^2yz + b^2zx + c^2xy = 0$, at the point D with coordinates $(-a^2vw, v(b^2w + c^2v), w(b^2w + c^2v))$. Note that the sum of these coordinates is $-a^2vw + v(b^2w + c^2v) + w(b^2w + c^2v))$. We now want to find the coordinates of $U(l,m,n)$, the reflection of D in the side BC. It is convenient to take the normalization of D to be the same as that of U so that

$$l+m+n=-a^2vw+v(b^2w+c^2v)+w(b^2w+c^2v)). \tag{7}$$

In order that the midpoint of UD lies on BC the requirement is that $l=a^2vw$. There is also the condition that the displacements $BC(0,-1,1)$ and $UD(-a^2vw-l, v(b^2w+c^2v)-m, w(b^2w+c^2v)-n)$ should be at right angles. The condition for perpendicular displacements may be found in [1, p.180]. When these conditions are taken into account we find the coordinates of U are

$$(l,m,n)=(a^2vw, v(c^2(v+w)-a^2w), w(b^2(v+w)-a^2v)). \tag{8}$$

The coordinates of E, F, V, W can be obtained from those of D, U by cyclic permutations of a, b, c and u, v, w.

The Hagge circle is the circle through U,V,W and its equation, which may be obtained by standard means, is

$$\begin{aligned}&(a^2vw+b^2wu+c^2uv)(a^2yz+b^2zx+c^2xy)\\ -&(x+y+z)(a^2(b^2+c^2-a^2)vwx+b^2(c^2+a^2-b^2)wuy+c^2(a^2+b^2-c^2)uvz)\\ =&0.\end{aligned} \tag{9}$$

It may now be checked that this circle has the characteristic property of a Hagge circle that it passes through H, whose coordinates are

$$\left(\frac{1}{b^2+c^2-a^2}, \frac{1}{c^2+a^2-b^2}, \frac{1}{a^2+b^2-c^2}\right).$$

Now the equation of AH is $(c^2+a^2-b^2)y=(a^2+b^2-c^2)z$ and this meets the Hagge circle with Equation (9) again at the point X with coordinates $(-a^2vw+b^2wu+c^2uv, (a^2+b^2-c^2)vw, (c^2+a^2-b^2)vw)$. The coordinates of Y,Z can be obtained from those of X by cyclic permutations of a,b,c and u,v,w.

Proposition 9. *XU, YV, ZW are concurrent at P.*

This has already been proved in Proposition 2, but may be verified by checking that when the coordinates of X,U,P are placed as entries in the rows of a 3×3 determinant, then this determinant vanishes. This shows that X, U, P are collinear as are Y, V, P and Z, W, P.

If the equation of a conic is $lx^2+my^2+nz^2+2fyz+2gzx+2hxy=0$, then the first coordinate of its center is $(mn-gm-hn-f^2+fg+hf)$ and other coordinates are obtained by cyclic change of letters. This is because it is the pole of the line at infinity. The x-coordinate of the center $h(P)$ of the Hagge circle is therefore $-a^4(b^2+c^2-a^2)vw+(a^2(b^2+c^2)-(b^2-c^2)^2)(b^2wu+c^2uv)$ with y- and z-coordinates following by cyclic permutations of a, b, c and u, v, w.

In §4 we introduced the Hagge axis and we now deduce its equation. The lines VW and AH meet at the point L with coordinates

$$\begin{aligned}&(u(a^2(b^2w(u+v)(w+u-v)+c^2v(w+u)(u+v-w))+b^4w(u+v)(v+w-u)\\&\quad -b^2c^2(u^2(v+w)+u(v^2+w^2)+2vw(v+w))+c^4v(w+u)(v+w-u)),\\&vw(a^2+b^2-c^2)(a^2(u+v)(w+u)-u(b^2(u+v)+c^2(w+u))),\\&vw(c^2+a^2-b^2)(a^2(u+v)(w+u)-u(b^2(u+v)+c^2(w+u)))).\end{aligned}$$

The coordinates of M and N follow by cyclic permutations of a, b, c and u, v, w. From these we obtain the equation of the Hagge axis LMN as

$$\sum_{\text{cyclic}} vw(a^2(u+v)(w+u)-u(b^2(u+v)+c^2(w+u)))(a^2(v-w)-(b^2-c^2)(v+w))x = 0. \tag{10}$$

It may now be verified that this line passes through P.

5.2. *The midpoint Hagge conic* . We now obtain a dividend from the areal analysis in §5.1. The midpoints in question are those of AX, BY, CZ, DU, EV, FW and in Figure 6 these points are labeled X_1, Y_1, Z_1, U_1, V_1, W_1. This notation is not to be confused with the now discarded notation X_1, Y_1 and Z_1 of Proposition 2. We now show these six points lie on a conic.

Proposition 10. *The points X_1, Y_1, Z_1, U_1, V_1, W_1 lie on a conic (the Hagge midpoint conic).*

Their coordinates are easily obtained and are

$$\begin{aligned}&X_1\ (2u(b^2w+c^2v), vw(a^2+b^2-c^2), vw(c^2+a^2-b^2)),\\&U_1\ (0, v(2c^2v+w(b^2+c^2-a^2)), w(2b^2w+v(b^2+c^2-a^2))),\end{aligned}$$

with coordinates of Y_1, Z_1, V_1, W_1 following by cyclic change of letters. It may now be checked that these six points lie on the conic with equation

$$\begin{aligned}&4(a^2vw+b^2wu+c^2uv)\left(\sum_{\text{cyclic}} u^2(-a^2vw+b^2(v+w)w+c^2v(v+w))yz\right)\\&-(x+y+z)\left(\sum_{\text{cyclic}} v^2w^2((a^2+b^2-c^2)u+2a^2v)((c^2+a^2-b^2)u+2a^2w)x\right)=0.\end{aligned} \tag{11}$$

Following the same method as before for the center, we find that its coordinates are $(u(b^2w+c^2v), v(c^2u+a^2w), w(a^2v+b^2u))$.

Proposition 11. *U_1, X_1, P are collinear.*

This is proved by checking that when the coordinates of X_1, U_1, P are placed as entries in the rows of a 3×3 determinant, then this determinant vanishes. This shows that X_1, U_1, P are collinear as are Y_1, V_1, P and Z_1, W_1, P.

Proposition 12. *The center of the Hagge midpoint conic is the midpoint of $Oh(P)$. It divides P^*G in the ratio $3:-1$.*

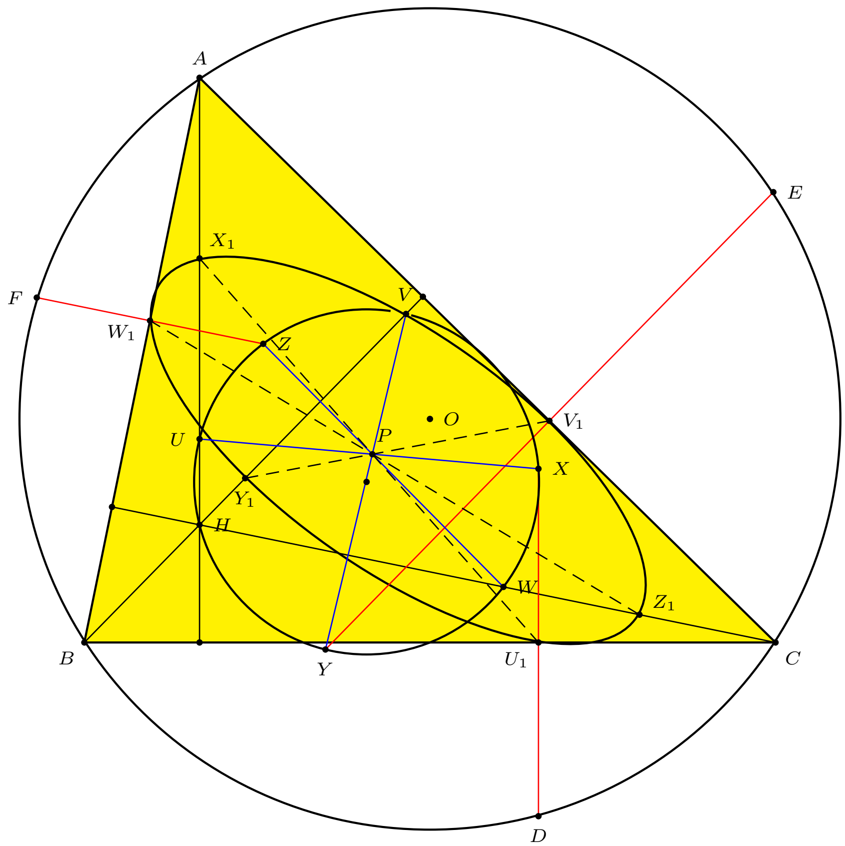

Figure 6.

The proof is straightforward and is left to the reader.

In similar fashion to above we define the six points X_k, Y_k, Z_k, U_k, V_k, W_k that divide the six lines AX, BY, CZ, DU, EV, FW respectively in the ratio $k:1$ (k real and $\neq 1$).

Proposition 13. *The six points X_k, Y_k, Z_k, U_k, V_k, W_k lie on a conic and the centers of these conics, for all values of k, lie on the line $Oh(P)$ and divide it in the ratio $k:1$.*

This proposition was originally conjectured by us on the basis of drawings by the geometry software package CABRI and we are grateful to the Editor for confirming the conjecture to be correct. We have rechecked his calculation and for the record the coordinates of X_k and U_k are

$$((1-k)a^2vw+(1+k)u(b^2w+c^2v), k(a^2+b^2-c^2)vw, k(c^2+a^2-b^2)vw),$$

and

$$(-a^2(1-k)vw, v((1+k)c^2v+(b^2+kc^2-ka^2)w), w((1+k)b^2w+(c^2+kb^2-ka^2)v)),$$

respectively. The conic involved has center with coordinates

$$((a^2(b^2+c^2-a^2)(a^2vw+b^2wu+c^2uv)$$
$$+k(-a^4(b^2+c^2-a^2)vw+(a^2(b^2+c^2)-(b^2-c^2)^2)(u(b^2w+c^2v)),$$
$$\cdots,\cdots).$$

Proposition 14. *U_k, X_k, P are collinear.*

The proof is by the same method as for Proposition 11.

6. Loci of Haggi circle centers

The Macbeath conic of ABC is the inconic with foci at the circumcenter O and the orthocenter H. The center of this conic is N, the nine-point center.

Proposition 15. *The locus of centers of those Hagge circles which are tangent to the circumcircle is the Macbeath conic.*

Proof. We address the elliptical case (see Figure 7) when ABC is acute and H is inside the circumcircle of radius R. The major axis of the Macbeath ellipse Σ is well known to have length R. Suppose that P is a point of the plane. Now $h(P)$ is on Σ if and only if $Oh(P)+h(P)H=R$, but $h(P)H$ is the radius of the Hagge circle, so this condition holds if and only if the Hagge circle is internally tangent to the circumcircle. Note that $h(P)$ is on Σ if and only if P^* is on Σ, and as P^* moves continuously round Σ, the Hagge circle moves around the inside of the circumcircle. The point P moved around the 'deltoid' shape as shown in Figure 7.

The case where ABC is obtuse and the Macbeath conic is a hyperbola is very similar. The associated Hagge circles are externally tangent to the circumcircle. □

Proposition 16. *The locus of centers of those Hagge circles which cut the circumcircle at diametrically opposite points is a straight line perpendicular to the Euler line.*

Proof. Let ABC have circumcenter O and orthocenter H. Choose H' on HO produced so that $HO\cdot OH'=R^2$ where R is the circumradius of ABC. Now if X, Y are diametrically opposite points on S (but not on the Euler line), then the circumcircle S' of XYH is of interest. By the converse of the power of a point theorem, H' lies on each S'. These circles S' form an intersecting coaxal system through H and H' and their centers lie on the perpendicular bisector of HH'. □

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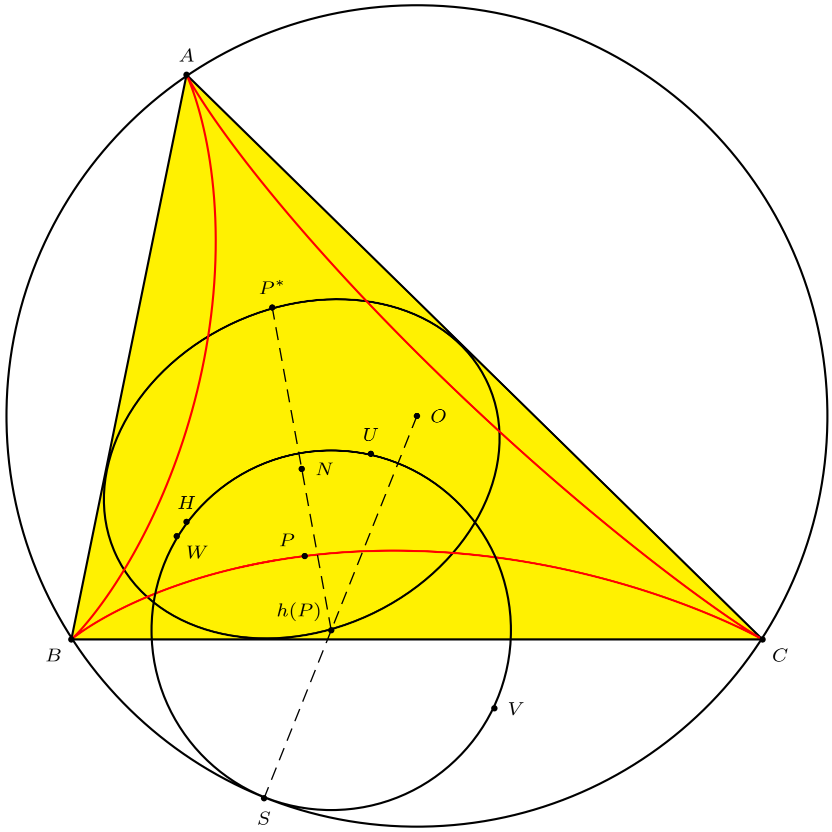

Figure 7.

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Forum Geometricorum
Volume 7 (2007) 249–250.

FORUM GEOM
ISSN 1534-1178

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