On an Affine Variant of a Steinhaus Problem

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Abstract. Given a triangle $ABC$ and three positive real numbers $u, v, w$, we prove that there exists a unique point $P$ in the interior of the triangle, with cevian triangle $P_aP_bP_c$, such that the areas of the three quadrilaterals $PP_aAP_c$, $PP_bBP_a$, $PP_cCP_b$ are in the ratio $u : v : w$. We locate $P$ as an intersection of three hyperbolas.

In this note we study a variation of the theme of [2], a generalization of a problem initiated by H. Steinhaus on partition of a triangle (see [1]). Given a triangle $ABC$ with interior $T$, and a point $P \in T$ with cevian triangle $P_aP_bP_c$, we denote by $\Delta_A(P), \Delta_B(P), \Delta_C(P)$ the areas of the oriented quadrilaterals $PP_bAP_c$, $PP_cBP_a$, $PP_aCP_b$. In this note we prove that given three arbitrary positive real numbers $u, v, w$, there exists a unique point $P \in T$ such that

$$\Delta_A(P) : \Delta_B(P) : \Delta_C(P) = u : v : w.$$ 

To this end, we define

$$f(P) = \Delta_A(P) : \Delta_B(P) : \Delta_C(P).$$

This is the point of $T$ such that

$$\Delta[BCf(P)] = \Delta_A(P), \quad \Delta[CAf(P)] = \Delta_B(P), \quad \Delta[ABf(P)] = \Delta_C(P).$$

**Lemma 1.** If $P$ has homogeneous barycentric coordinates $x : y : z$ with reference to triangle $ABC$, then

$$f(P) = \frac{(y+z)(2x+y+z)}{x} : \frac{(x+y)(2y+z+x)}{y} : \frac{(x+y)(x+y+2z)}{z}.$$ 

**Proof.** If $P = x : y : z$, we have

$$\overrightarrow{AP_c} = \frac{yAB}{x+y}, \quad \overrightarrow{AP} = \frac{yAB + zAC}{x+y+z}, \quad \overrightarrow{AP_b} = \frac{zAC}{x+z},$$

so that

$$\Delta_a(P) = \Delta(AP_cP) + \Delta(APP_b) = \frac{y}{x+y+z} \left( \frac{1}{x+y} + \frac{1}{x+z} \right) \Delta(ABC).$$

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By cyclic permutations of \(x, y, z\), we get the values of \(\Delta_B(P)\) and \(\Delta_C(P)\), and the result follows. \(\square\)

We shall prove that \(f : \mathcal{T} \to \mathcal{T}\) is a bijection. We adopt the following notations.

(i) \(G_a, G_b, G_c\) are the vertices of the anticomplementary triangle. They are the images \(A, B, C\) under the homothety \(h(G, -2)\), \(G\) being the centroid of \(ABC\).

(ii) \(P^*\) denotes the isotomic conjugate of \(P\) with respect to \(ABC\). Its traces \(P^*_a, P^*_b, P^*_c\) on the sidelines of \(ABC\) are the reflections of \(P_a, P_b, P_c\) with respect to the midpoint of the corresponding side.

(iii) \([L]_\infty\) denotes the infinite point of a line \(L\).

**Proposition 2.** Let \(P = x : y : z\) and \(U = u : v : w\). The lines \(G_a P\) and \(P_a^* U\) are parallel if and only if \(P\) lies on the hyperbola \(\mathcal{H}_{a,U}\) through \(A, G_a, U_a^*\), the reflection of \(U_b^*\) in \(C\) and the reflection of \(U_c^*\) in \(B\).

**Proof.** As \(P_a^* = 0 : z : y\) and \([G_a P]_\infty = -(2x + y + z) : z + x : x + y\), the lines \(G_a P\) and \(P_a^* U\) are parallel if and only if

\[
\begin{align*}
\det([G_a P]_\infty, P_a^*, U) &= -(2x + y + z) \begin{vmatrix} 0 & z & y \ 1 & v & w \ u & v & w \end{vmatrix} \ 
&= x((u + v)y - (w + u)z) + (x + y + z)(vy - wz) \\
&= 0.
\end{align*}
\]

It is clear that \(h_{a,U}(P) = 0\) defines a conic \(\mathcal{H}_{a,U}\) through \(A = 1 : 0 : 0\), and the infinite points of the lines \(x = 0\) and \((u + v)y - (w + u)z = 0\). These are the lines \(BC\) and \(G_a U\). It is also easy to check that it contains the points \(G_a = -1 : 1 : 1\), \(U_a^* = 0 : w : v\), and

\[
\begin{align*}
U_{bc}^* &= -w : 0 : u + 2w, \\
U_{cb}^* &= -v : u + 2v : 0.
\end{align*}
\]

These latter two are respectively the reflections of \(U_b^*\) in \(C\) and \(U_c^*\) in \(B\). The conic \(\mathcal{H}_{a,U}\) is a hyperbola since the four points \(A, G_a, U_{bc}^*\) and \(U_{cb}^*\) do not fall on two lines. \(\square\)

By cyclic permutations of coordinates, we obtain two hyperbolae \(\mathcal{H}_{b,U}\) and \(\mathcal{H}_{c,U}\) defined by

\[
\begin{align*}
\det([G_b P]_\infty, P_b^*, U) &= 0, \\
\det([G_c P]_\infty, P_c^*, U) &= 0.
\end{align*}
\]

It is easy to check that if \(U = f(P)\), then

\[
\det([G_a P]_\infty, P_a^*, U) = h_{a,U}(P) = h_{c,U}(P) = 0.
\]

From this we obtain a very easy construction of the point \(f(P)\).
**Corollary 3.** The point $f(P)$ is the intersection of the lines through $P_a^*$, $P_b^*$ and $P_c^*$ parallel to $G_a P$, $G_b P$, $G_c P$ respectively. See Figure 1.

![Figure 1.](image)

**Proof.** The lines $G_a P$, $G_b P$, $G_c P$ are parallel to $P_a^* f(P)$, $P_b^* f(P)$, $P_c^* f(P)$ respectively. □

**Remarks.** (1) $\mathcal{H}_{a,U}$ degenerates if and only if $v = w$, i.e., when $U$ lies on the median $AG$. In this case, $\mathcal{H}_{a,U}$ is the union of the median $AG$ and of a line parallel to $BC$.

(2) $P$, $P^*$, $f(P)$ are collinear.

(3) As $h_{a,U}(P) + h_{b,U}(P) + h_{c,U}(P) = 0$, the three hyperbolae $\mathcal{H}_{a,U}$, $\mathcal{H}_{b,U}$, $\mathcal{H}_{c,U}$ are members of a pencil of conics. If $U \in \mathcal{T}$, the points $P$ for which $f(P) = U$ are their common points lying in $\mathcal{T}$.

**Lemma 4.** If $U \in \mathcal{T}$, $\mathcal{H}_{a,U}$ and $\mathcal{H}_{b,U}$ have a real common point in $\mathcal{T}$ and a real common point in $\mathcal{T}_A$, reflection in $A$ of the open angular sector bounded by the half lines $AB$ and $AC$.

**Proof.** Using the fact that $\mathcal{H}_{a,U}$ passes through $[BC]_\infty$, we can cut $\mathcal{H}_{a,U}$ by lines parallel to $BC$ to get a rational parametrization of $\mathcal{H}_{a,U}$. More precisely, let $B_t$ and $C_t$ be the images of $B$ and $C$ under the homothety $h(A, 1 - t)$. The point

$$(1 - \mu)B_t + \mu C_t = t : (1 - \mu)(1 - t) : \mu(1 - t)$$

lies on $\mathcal{H}_{a,U}$ if and only if

$$\mu = \mu_t = \frac{v + t(u + v)}{v + w + t(2u + v + w)}.$$
Let $P(t) = (1 - \mu t)B_t + \mu tC_t$. It has homogeneous barycentric coordinates

$t((v + w) + t(2u + v + w)) : (1 - t)(w + t(w + u)) : (1 - t)(v + t(u + v))$.

with coordinate sum is $(v + w) + t(2u + v + w)$.

If $t \geq 0$, we have $0 < \mu t < 1$. It follows that, for $0 < t < 1$, $P(t) \in \mathcal{T}$ and for $t > 1$, $P(t) \in \mathcal{T}_A$. Consider

$$\varphi(t) := \frac{h_{b,U}(P(t))}{(u + v + w)((v + w) + t(2u + v + w))^2}.$$  

More explicitly,

$$\varphi(t) = \frac{2(u + v)(u + w)(u + v + w)t^4 + \text{lower degree terms of } t}{(u + v + w)(v + w + t(2u + v + w))^2}.$$  

Clearly, $\varphi(0) = \frac{2uvw}{(v+w)(u+v+w)} > 0$ and $\varphi(1) = -\frac{u}{u+v+w} < 0$. Note also that $\varphi(+\infty) = +\infty$. As $\varphi$ is continuous for $t \geq 0$, the result follows. \qed

**Theorem 5.** If $U \in \mathcal{T}$, the three hyperbolas $H_{a,U}$, $H_{b,U}$, $H_{c,U}$ have four distinct real common points, exactly one of which lies in $\mathcal{T}$. This point is the only point $P \in \mathcal{T}$ satisfying $f(P) = U$.  

![Figure 2](image.png)
Proof. In a similar way as in Lemma 4, we can see that $H_{b,U}$ and $H_{c,U}$ have a common point in $T$ and a real common point in $T_B$ and that $H_{c,U}$ and $H_{a,U}$ have a real common point in $T$ and a real common point in $T_B$. As the four sets $T, T_A, T_B, T_C$ pairwise have empty intersection, it follows that $H_{a,U}, H_{b,U}, H_{c,U}$ have four real common points, one in each of $T, T_A, T_B$ and $T_C$. See Figure 2. □

Remark. (4) If $U \in T$, the points $P$ such that
\[
\Delta(AP_cP) + \Delta(AP_bP) + \Delta(BP_aP) + \Delta(BP_cP) + \Delta(CP_bP) + \Delta(CP_aP) = u : v : w
\]
are the four common points of $H_{a,U}, H_{b,U}$ and $H_{c,U}$.

Remark (2) shows that $f^{-1}(U)$ lies on the isotomic cubic with pivot $U$. Clearly, $f(G) = f^{-1}(G) = G$.

References


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