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On an Affine Variant of a Steinhaus Problem

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Abstract. Given a triangle ABC and three positive real numbers u, v, w, we prove that there exists a unique point P in the interior of the triangle, with cevian triangle $P_aP_bP_c$, such that the areas of the three quadrilaterals PP_bAP_c, PP_cBP_a , PP_aCP_b are in the ratio $u : v : w$. We locate P as an intersection of three hyperbolas.

In this note we study a variation of the theme of [2], a generalization of a problem initiated by H. Steinhaus on partition of a triangle (see [1]). Given a triangle ABC with interior $\mathscr{T}$, and a point $P \in \mathcal{T}$ with cevian triangle $P_aP_bP_c$, we denote by $\Delta_A(P)$, $\Delta_B(P)$, $\Delta_C(P)$ the areas of the oriented quadrilaterals PP_bAP_c, PP_cBP_a, PP_aCP_b. In this note we prove that given three arbitrary positive real numbers u, v, w, there exists a unique point $P \in \mathscr{T}$ such that

$$\Delta_A(P) : \Delta_B(P) : \Delta_C(P) = u : v : w.$$

To this end, we define

$$f(P) = \Delta_A(P) : \Delta_B(P) : \Delta_C(P).$$

This is the point of $\mathscr{T}$ such that

$$\Delta[BCf(P)] = \Delta_A(P), \quad \Delta[CAf(P)] = \Delta_B(P), \quad \Delta[ABf(P)] = \Delta_C(P).$$

Lemma 1. *If P has homogeneous barycentric coordinates $x : y : z$ with reference to triangle ABC, then*

$$f(P) = \frac{(y+z)(2x+y+z)}{x} : \frac{(z+x)(2y+z+x)}{y} : \frac{(x+y)(x+y+2z)}{z}.$$

Proof. If $P = x : y : z$, we have

$$\overrightarrow{AP_c} = \frac{y\overrightarrow{AB}}{x+y}, \quad \overrightarrow{AP} = \frac{y\ \overrightarrow{AB} + z\ \overrightarrow{AC}}{x+y+z}, \quad \overrightarrow{AP_b} = \frac{z\overrightarrow{AC}}{x+z},$$

so that

$$\Delta_a(P) = \Delta(AP_cP) + \Delta(APP_b) = \frac{y\ z}{x+y+z}(\frac{1}{x+y} + \frac{1}{x+z})\ \Delta(ABC).$$

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By cyclic permutations of x, y, z, we get the values of $\Delta_B(P)$ and $\Delta_C(P)$, and the result follows. □

We shall prove that $f: \mathscr{T} \to \mathscr{T}$ is a bijection. We adopt the following notations.

(i) G_a, G_b, G_c are the vertices of the anticomplementary triangle. They are the images A, B, C under the homothety $\mathsf{h}(G, -2)$, G being the centroid of ABC.

(ii) P^* denotes the isotomic conjugate of P with respect to ABC. Its traces P_a^*, P_b^*, P_c^* on the sidelines of ABC are the reflections of P_a, P_b, P_c with respect to the midpoint of the corresponding side.

(iii) $[L]_\infty$ denotes the infinite point of a line L.

Proposition 2. *Let $P = x : y : z$ and $U = u : v : w$. The lines G_aP and P_a^*U are parallel if and only if P lies on the hyperbola $\mathscr{H}_{a,U}$ through A, G_a, U_a^*, the reflection of U_b^* in C and the reflection of U_c^* in B.*

Proof. As $P_a^* = 0 : z : y$ and $[G_aP]_\infty = -(2x+y+z) : z+x : x+y$, the lines G_aP and P_a^*U are parallel if and only if

$$\begin{aligned} h_{a,U}(P) :=&\ \det([G_aP]_\infty,\ P_a^*,\ U) \\ =& \begin{vmatrix} -(2x+y+z) & z+x & x+y \\ 0 & z & y \\ u & v & w \end{vmatrix} \\ =&\ x((u+v)y-(w+u)z)+(x+y+z)(vy-wz) \\ =&\ 0. \end{aligned}$$

It is clear that $h_{a,U}(P) = 0$ defines a conic $\mathscr{H}_{a,U}$ through $A = 1 : 0 : 0$, and the infinite points of the lines $x = 0$ and $(u+v)y-(w+u)z = 0$. These are the lines BC and G_aU. It is also easy to check that it contains the points $G_a = -1 : 1 : 1$, $U_a^* = 0 : w : v$, and

$$\begin{aligned} U_{bc}^* &:= -w : 0 : u+2w, \\ U_{cb}^* &:= -v : u+2v : 0. \end{aligned}$$

These latter two are respectively the reflections of U_b^* in C and U_c^* in B. The conic $\mathscr{H}_{a,U}$ is a hyperbola since the four points A, G_a, U_{bc}^* and U_{cb}^* do not fall on two lines. □

By cyclic permutations of coordinates, we obtain two hyperbolae $\mathscr{H}_{b,U}$ and $\mathscr{H}_{c,U}$ defined by

$$\begin{aligned} h_{b,U}(P) &:= \det([G_bP]_\infty,\ P_b^*,\ U) = 0, \\ h_{c,U}(P) &:= \det([G_cP]_\infty,\ P_c^*,\ U) = 0. \end{aligned}$$

It is easy to check that if $U = f(P)$, then

$$h_{a,U}(P) = h_{b,U}(P) = h_{c,U}(P) = 0.$$

From this we obtain a very easy construction of the point $f(P)$.

Corollary 3. *The point $f(P)$ is the intersection of the lines through P_a^*, P_b^* and P_c^* parallel to G_aP, G_bP, G_cP respectively.* See Figure 1.

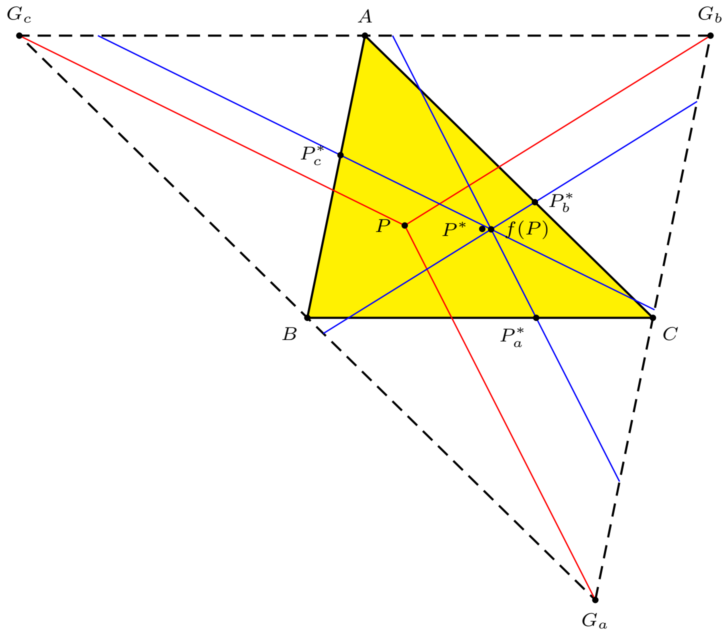

Figure 1.

Proof. The lines G_aP, G_bP, G_cP are parallel to $P_a^*f(P)$, $P_b^*f(P)$, $P_c^*f(P)$ respectively. □

Remarks. (1) $\mathscr{H}_{a,U}$ degenerates if and only if $v = w$, *i.e.*, when U lies on the median AG. In this case, $\mathscr{H}_{a,U}$ is the union of the median AG and of a line parallel to BC.

(2) P, P^*, $f(P)$ are collinear.

(3) As $h_{a,U}(P) + h_{b,U}(P) + h_{c,U}(P) = 0$, the three hyperbolae $\mathscr{H}_{a,U}$, $\mathscr{H}_{b,U}$, $\mathscr{H}_{c,U}$ are members of a pencil of conics. If $U \in \mathscr{T}$, the points P for which $f(P) = U$ are their common points lying in $\mathscr{T}$.

Lemma 4. *If $U \in \mathscr{T}$, $\mathscr{H}_{a,U}$ and $\mathscr{H}_{b,U}$ have a real common point in $\mathscr{T}$ and a real common point in $\mathscr{T}_A$, reflection in A of the open angular sector bounded by the half lines AB and AC.*

Proof. Using the fact that $\mathscr{H}_{a,U}$ passes through $[BC]_\infty$, we can cut $\mathscr{H}_{a,U}$ by lines parallel to BC to get a rational parametrization of $\mathscr{H}_{a,U}$. More precisely, let B_t and C_t be the images of B and C under the homothety $\mathsf{h}(A, 1-t)$. The point

$$(1-\mu)B_t + \mu C_t = t : (1-\mu)(1-t) : \mu(1-t)$$

lies on $\mathscr{H}_{a,U}$ if and only if

$$\mu = \mu_t = \frac{v + t(u+v)}{v + w + t(2u + v + w)}.$$

Let $P(t) = (1 - \mu_t)B_t + \mu_t C_t$. It has homogeneous barycentric coordinates

$$t((v+w) + t(2u+v+w)) : (1-t)(w + t(w+u)) : (1-t)(v + t(u+v)).$$

with coordinate sum is $(v + w) + t(2u + v + w)$.

If $t \geq 0$, we have $0 < \mu_t < 1$. It follows that, for $0 < t < 1$, $P(t) \in \mathscr{T}$ and for $t > 1$, $P(t) \in \mathscr{T}_A$. Consider

$$\varphi(t) := \frac{h_{b,U}(P(t))}{(u+v+w)((v+w) + t(2u+v+w))^2}.$$

More explicitly,

$$\varphi(t) = \frac{2(u+v)(u+w)(u+v+w)t^4 + \text{lower degree terms of } t}{(u+v+w)(v+w+t(2u+v+w))^2}.$$

Clearly, $\varphi(0) = \frac{2vw}{(v+w)(u+v+w)} > 0$ and $\varphi(1) = -\frac{u}{u+v+w} < 0$. Note also that $\varphi(+\infty) = +\infty$. As φ is continuous for $t \geq 0$, the result follows. □

Theorem 5. *If $U \in \mathscr{T}$, the three hyperbolas $\mathscr{H}_{a,U}$, $\mathscr{H}_{b,U}$, $\mathscr{H}_{c,U}$ have four distinct real common points, exactly one of which lies in $\mathscr{T}$. This point is the only point $P \in \mathscr{T}$ satisfying $f(P) = U$.*

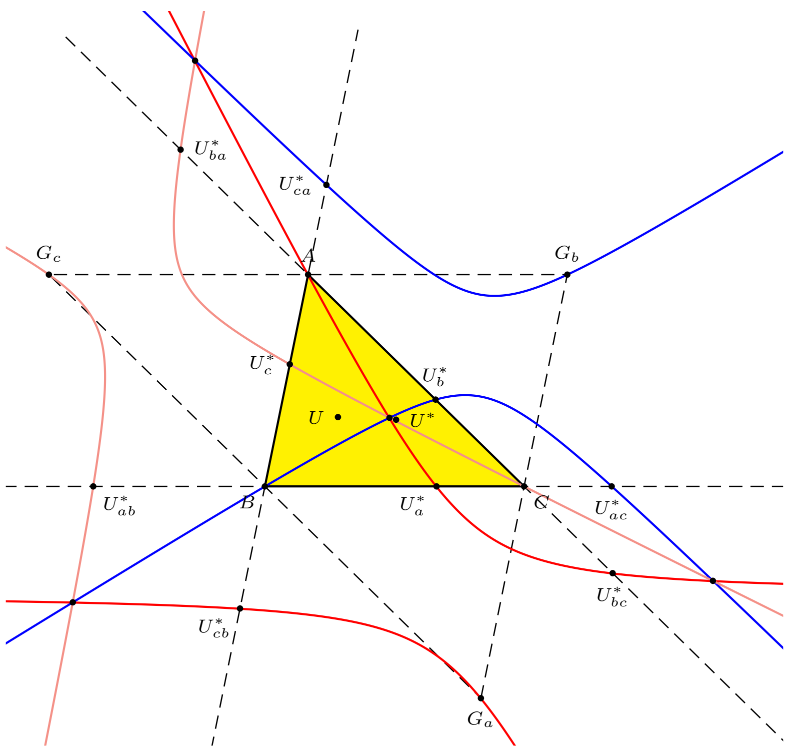

Figure 2.

Proof. In a similar way as in Lemma 4, we can see that $\mathscr{H}_{b,U}$ and $\mathscr{H}_{c,U}$ have a common point in $\mathscr{T}$ and a real common point in $\mathscr{T}_B$ and that $\mathscr{H}_{c,U}$ and $\mathscr{H}_{a,U}$ have a real common point in $\mathscr{T}$ and a real common point in $\mathscr{T}_B$. As the four sets $\mathscr{T}$, T_A, T_B, T_C pairwise have empty intersection, it follows that $\mathscr{H}_{a,U}$, $\mathscr{H}_{b,U}$, $\mathscr{H}_{c,U}$ have four real common points, one in each of $\mathscr{T}$, $\mathscr{T}_A$, $\mathscr{T}_B$ and $\mathscr{T}_C$. See Figure 2. □

Remark. (4) If $U \in \mathscr{T}$, the points P such that

$$\Delta(AP_cP) + \Delta(APP_b) : \Delta(BP_aP) + \Delta(BPP_c) : \Delta(CP_bP) + \Delta(CPP_a) = u : v : w$$

are the four common points of $\mathscr{H}_{a,U}$, $\mathscr{H}_{b,U}$ and $\mathscr{H}_{c,U}$.

Remark (2) shows that $f^{-1}(U)$ lies on the isotomic cubic with pivot U. Clearly, $f(G) = f^{-1}(G) = G$.

References

[1] A. Tyszka, Steinhaus' problem on partition of a triangle, *Forum Geom.*, 7(2007) 181–185.

[2] J.-P. Ehrmann, Constructive solution of a generalization of Steinhaus' problem on partition of a triangle, *Forum Geom.*, 7 (2007) 187–190.

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