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Two Triads of Congruent Circles from Reflections

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Abstract. Given a triangle, we construct two triads of congruent circles through the vertices, one associated with reflections in the altitudes, and the other reflections in the angle bisectors.

1. Reflections in the altitudes

Given triangle ABC with orthocenter H, let B_a and C_a be the reflections of B and C in the line AH. These are points on the sideline BC so that $\mathbf{BC_a} = \mathbf{CB_a}$. Similarly, consider the reflections C_b, A_b of C, A respectively in the line BH, and A_c, B_c of A, B in the line CH.

Theorem 1. *The circles AC_bB_c, BA_cC_a, and CB_aA_b are congruent.*

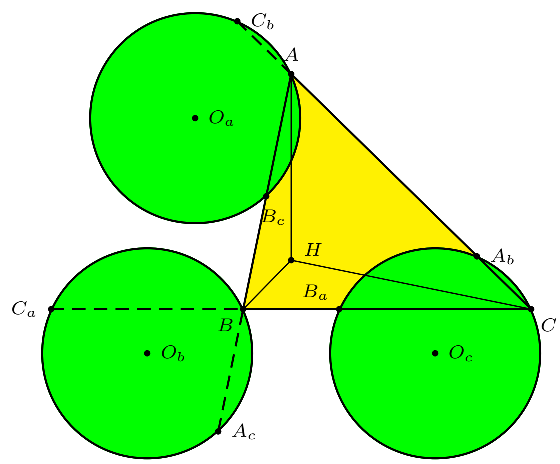

Figure 1.

Proof. Let O be the circumcenter of triangle ABC, and X its reflection in the A-altitude. This is the circumcenter of triangle AB_aC_a, the reflection of triangle ABC in its A-altitude. See Figure 2. It follows that H lies on the perpendicular bisector of OX, and $HX = OH$. Similarly, if Y and Z are the reflections of O in the lines BH and CH respectively, then $HY = HZ = OH$. It follows that O, X, Y, Z are concyclic, and H is the center of the circle containing them. See Figure 3.

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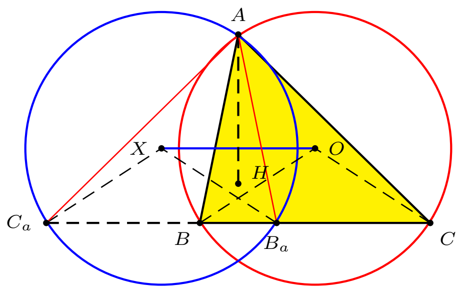

Figure 2

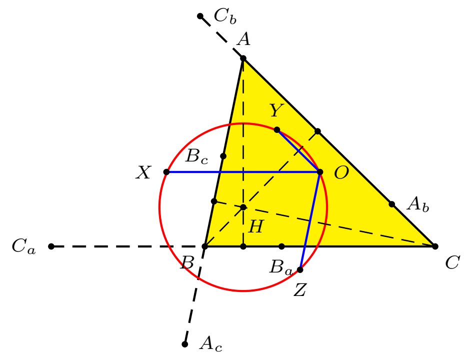

Figure 3

Let O be the circumcenter of triangle ABC. Note the equalities of vectors

$$\mathbf{OX} = \mathbf{BC_a} = \mathbf{CB_a},$$
$$\mathbf{OY} = \mathbf{CA_b} = \mathbf{AC_b},$$
$$\mathbf{OZ} = \mathbf{AB_c} = \mathbf{BA_c}.$$

The three triangles AC_bB_c, BA_cC_a, and CB_aA_b are the translations of OYZ by **OA**, OZX by **OB**, and OXY by **OC** respectively.

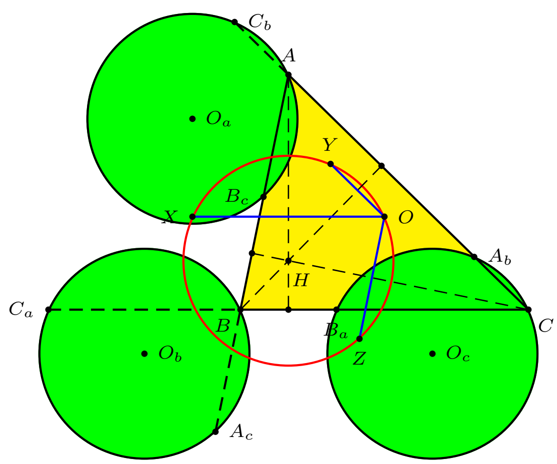

Figure 4.

Therefore, the circumcircles of the three triangles are all congruent and have radius OH. Their centers are the translations of H by the three vectors. $\square$

2. Reflections in the angle bisectors

Let I be the incenter of triangle ABC. Consider the reflections of the vertices in the angle bisectors: B'_a, C'_a of B, C in AI, C'_b, A'_b of C, A in BI, and A'_c, B'_c of A, B in CI. See Figure 5.

Theorem 2. *The circles $AC'_bB'_c$, $BA'_cC'_a$, and $CB'_aA'_b$ are congruent.*

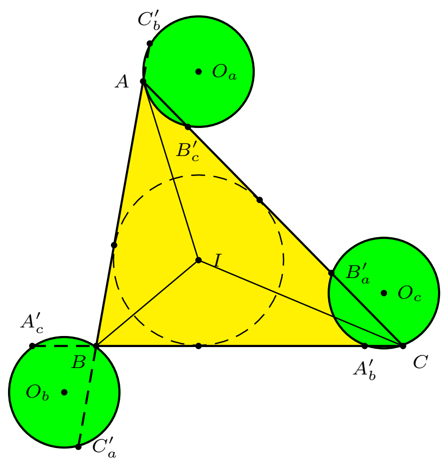

Figure 5.

Proof. Consider the reflections B''_c, C''_b of B'_c, C'_b in AI, C''_a, A''_c of C'_a, A'_c in BI, and A''_b, B''_a of A'_b, B'_a in CI. See Figure 6.

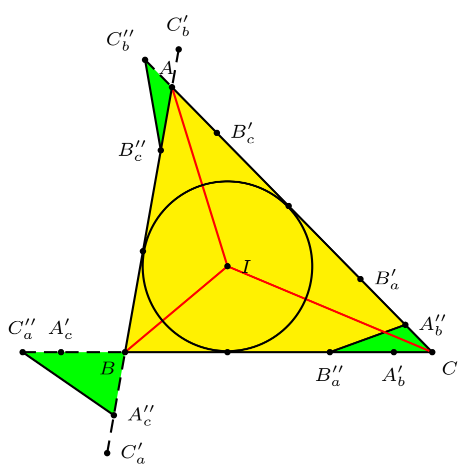

Figure 6

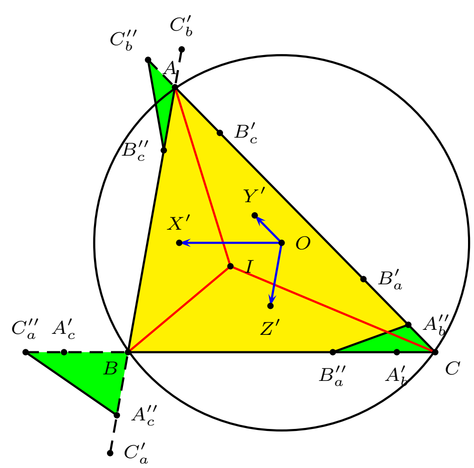

Figure 7

Note the equalities of vectors

$$\mathbf{BC''_a} = \mathbf{CB''_a}, \qquad \mathbf{CA''_b} = \mathbf{AC''_b}, \qquad \mathbf{AB''_c} = \mathbf{BA''_c}.$$

With the circumcenter O of triangle ABC, these define points X', Y', Z' such that

$$\begin{aligned} \mathbf{OX'} =& \mathbf{BC''_a} = \mathbf{CB''_a}, \\ \mathbf{OY'} =& \mathbf{CA''_b} = \mathbf{AC''_b}, \\ \mathbf{OZ'} =& \mathbf{AB''_c} = \mathbf{BA''_c}. \end{aligned}$$

The triangles $AC''_bB''_c$, $BA''_cC''_a$ and $CB''_aA''_b$ are the translations of $OY'Z'$, $OZ'X'$ and $OX'Y'$ by the vectors $\mathbf{OA}$, $\mathbf{OB}$ and $\mathbf{OC}$ respectively. See Figure 7.

Note, in Figure 8, that $OX'C''_aC$ is a symmetric trapezoid and $IC''_a = IC'_a = IC$. It follows that triangles IC''_aX' and ICO are congruent, and $IX' = IO$. Similarly, $IY' = IO$ and $IZ' = IO$. This means that the four points O, X', Y', Z' are on a circle center I. See Figure 9. The circumcenters O''_a, O''_b, O''_c of the triangles $AC''_bB''_c$, $BA''_cC''_a$ and $CB''_aA''_b$ are the translations of I by these vectors. These circumcircles are congruent to the circle $I(O)$.

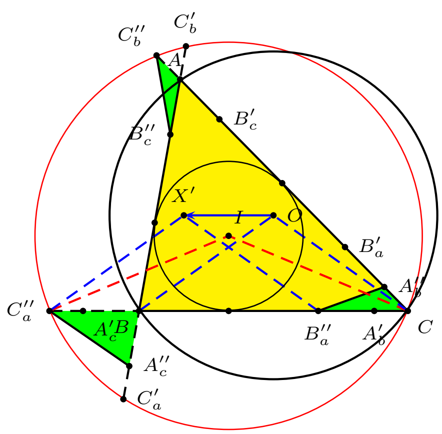

Figure 8

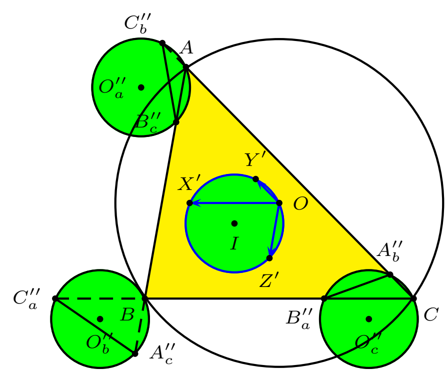

Figure 9

The segments AO''_a, BO''_b and CO''_c are parallel and equal in lengths. The triangles $AC'_bB'_c$, $BA'_cC'_a$ and $CB'_aA'_b$ are the reflections of $AC''_bB''_c$, $BA''_cC''_a$ and $CB''_aA''_b$ in the respective angle bisectors. See Figure 10. It follows that their circumcircles are all congruent to $I(O)$. □

Let O'_a, O'_b, O'_c be the circumcenters of triangles $AC'_bB'_c$, $BA'_cC'_a$ and $CB'_aA'_b$ respectively. The lines AO'_a and AO''_a are symmetric with respect to the bisector of angle A. Since AO''_a, BO''_b and CO''_c are parallel to the line OI, the reflections in the angle bisectors concur at the isogonal conjugate of the infinite point of OI. This is a point P on the circumcircle. It is the triangle center X_{104} in [1].

Finally, since $IO''_a = IO''_b = IO''_c$, we also have $IO'_a = IO'_b = IO'_c$. The 6 circumcenters all lie on the circle, center I, radius R.

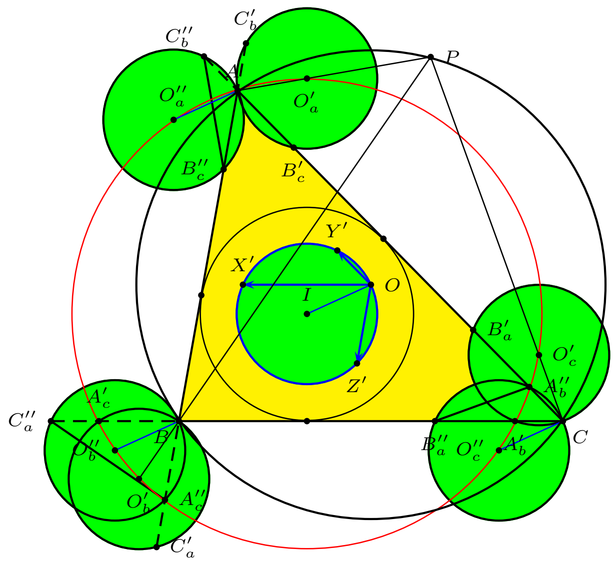

Figure 10.

To conclude this note, we establish an interesting property of the centers of the circles in Theorem 2.

Proposition 3. *The lines O'_aI, O'_bI and O'_cI are perpendicular to BC, CA and AB respectively.*

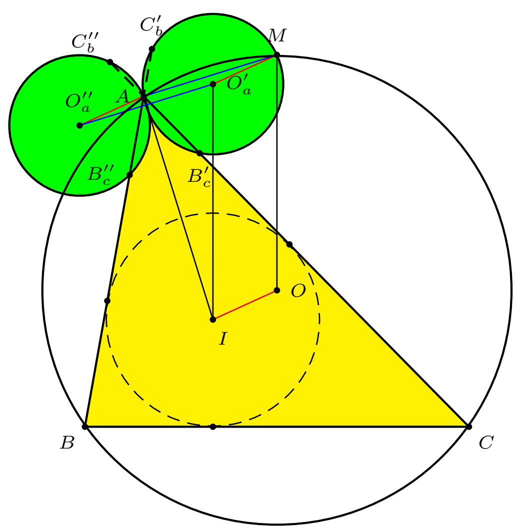

Figure 11.

Proof. It is enough to prove that for the line O'_aI. The other two cases are similar.

Let M be the intersection (other than A) of the circle (O'_a) with the circumcircle of triangle ABC. Since $IO'_a = OM$ (circumradius) and $O'_aM = IO$, O'_aMOI is a parallelogram. This means that $\mathbf{O'_aM} = \mathbf{IO} = \mathbf{O''_aA}$, and $AMO'_aO''_a$ is also a parallelogram. From this we conclude that AM, being parallel to $O''_aO'_a$, is perpendicular to the bisector AI. Thus, M is the midpoint of the arc BAC, and MO is perpendicular to BC. Since $\mathbf{O'_aI} = \mathbf{MO}$, the line O'_aI is also perpendicular to BC. □

Since the six circles (O'_a) and (O''_a) etc are congruent (with common radius OI) and their centers are all at a distance R from I, it is clear that there are two circles, center I, tangent to all these circles. These two circles are tangent to the circumcircle, the point of tangency being the intersection of the circumcircle with the line OI. These are the triangle centers X_{1381} and X_{1382} of [1].

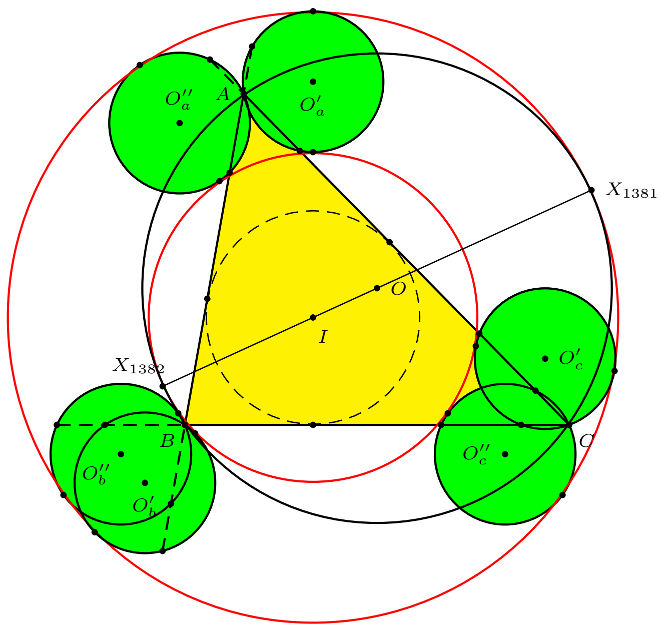

Figure 12.

References

[1] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

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