Two Triads of Congruent Circles from Reflections

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Abstract. Given a triangle, we construct two triads of congruent circles through the vertices, one associated with reflections in the altitudes, and the other reflections in the angle bisectors.

1. Reflections in the altitudes

Given triangle $ABC$ with orthocenter $H$, let $B_a$ and $C_a$ be the reflections of $B$ and $C$ in the line $AH$. These are points on the sideline $BC$ so that $BC_a = CB_a$. Similarly, consider the reflections $C_b, A_b$ of $C, A$ respectively in the line $BH$, and $A_c, B_c$ of $A, B$ in the line $CH$.

Theorem 1. The circles $AC_bB_c, BA_cC_a,$ and $CB_aA_b$ are congruent.

Proof. Let $O$ be the circumcenter of triangle $ABC$, and $X$ its reflection in the $A$-altitude. This is the circumcenter of triangle $AB_aC_a$, the reflection of triangle $ABC$ in its $A$-altitude. See Figure 2. It follows that $H$ lies on the perpendicular bisector of $OX$, and $HX = OH$. Similarly, if $Y$ and $Z$ are the reflections of $O$ in the lines $BH$ and $CH$ respectively, then $HY = HZ = OH$. It follows that $O, X, Y, Z$ are concyclic, and $H$ is the center of the circle containing them. See Figure 3.
Let \( O \) be the circumcenter of triangle \( ABC \). Note the equalities of vectors

\[
\begin{align*}
OX &= BC_a = CB_a, \\
OY &= CA_b = AC_b, \\
OZ &= AB_c = BA_c.
\end{align*}
\]

The three triangles \( AC_b B_c, BA_c C_a, \) and \( CB_a A_b \) are the translations of \( OYZ \) by \( OA, OZX \) by \( OB \), and \( OXY \) by \( OC \) respectively.

Therefore, the circumcircles of the three triangles are all congruent and have radius \( OH \). Their centers are the translations of \( H \) by the three vectors. \( \square \)
2. Reflections in the angle bisectors

Let $I$ be the incenter of triangle $ABC$. Consider the reflections of the vertices in the angle bisectors: $B_a'$, $C_a'$ of $B$, $C$ in $AI$, $C_b'$, $A_b'$ of $C$, $A$ in $BI$, and $A_c'$, $B_c'$ of $A$, $B$ in $CI$. See Figure 5.

**Theorem 2.** The circles $AC_b'B_c'$, $BA_c'C_a'$, and $CB_a'A_b'$ are congruent.

**Proof.** Consider the reflections $B_b''$, $C_b''$ of $B_b'$, $C_b'$ in $AI$, $C_a''$, $A_a''$ of $C_a'$, $A_a'$ in $BI$, and $A_c''$, $B_c''$ of $A_c'$, $B_c'$ in $CI$. See Figure 6.
Note the equalities of vectors
\[ BC''_a = CB''_a, \quad CA''_b = AC''_b, \quad AB''_c = BA''_c. \]
With the circumcenter \( O \) of triangle \( ABC \), these define points \( X', Y', Z' \) such that
\[
OX' = BC''_a = CB''_a, \\
OY' = CA''_b = AC''_b, \\
OZ' = AB''_c = BA''_c.
\]
The triangles \( AC''_b B''_c, BA''_c C''_a \) and \( CB''_a A''_c \) are the translations of \( OY' Z', OZ' X' \) and \( OX' Y' \) by the vectors \( OA, OB \) and \( OC \) respectively. See Figure 7.

Note, in Figure 8, that \( OX'C''_a C \) is a symmetric trapezoid and \( I C''_a = IC' = IC \). It follows that triangles \( IC''_a X' \) and \( ICO \) are congruent, and \( IX' = IO \).

Similarly, \( IY' = IO \) and \( IZ' = IO \). This means that the four points \( O, X', Y', Z' \) are on a circle center \( I \). See Figure 9. The circumcenters \( O''_a, O''_b, O''_c \) of the triangles \( AC''_b B''_c, BA''_c C''_a \) and \( CB''_a A''_c \) are the translations of \( I \) by these vectors. These circumcircles are congruent to the circle \( I(O) \).

\[ \text{Figure 8} \quad \text{Figure 9} \]

The segments \( AO''_a, BO''_b \) and \( CO''_c \) are parallel and equal in lengths. The triangles \( AC''_b B''_c, BA''_c C''_a \) and \( CB''_a A''_c \) are the reflections of \( AC''_b B''_c, BA''_c C''_a \) and \( CB''_a A''_c \) in the respective angle bisectors. See Figure 10. It follows that their circumcircles are all congruent to \( I(O) \).

Let \( O'_a, O'_b, O'_c \) be the circumcenters of triangles \( AC'_b B'_c, BA'_c C'_a \) and \( CB'_a A'_b \) respectively. The lines \( AO''_a \) and \( AO''_a \) are symmetric with respect to the bisector of angle \( A \). Since \( AO''_a, BO''_b \) and \( CO''_c \) are parallel to the line \( OF \), the reflections in the angle bisectors concur at the isogonal conjugate of the infinite point of \( OF \). This is a point \( P \) on the circumcircle. It is the triangle center \( X_{104} \) in [1].

Finally, since \( IO''_a = IO''_b = IO''_c \), we also have \( IO'_a = IO'_b = IO'_c \). The 6 circumcenters all lie on the circle, center \( I \), radius \( R \).
To conclude this note, we establish an interesting property of the centers of the circles in Theorem 2.

**Proposition 3.** The lines $O'_a I$, $O'_b I$ and $O'_c I$ are perpendicular to $BC$, $CA$ and $AB$ respectively.
Proof. It is enough to prove that for the line $O_a'I$. The other two cases are similar.

Let $M$ be the intersection (other than $A$) of the circle $(O_a')$ with the circumcircle of triangle $ABC$. Since $IO_a'O_a = OM$ (circumradius) and $O_a'M = IO = O_a'A$, and $AMO_a'O_a''$ is also a parallelogram. From this we conclude that $AM$, being parallel to $O_a'O_a''$, is perpendicular to the bisector $AI$. Thus, $M$ is the midpoint of the arc $BAC$, and $MO$ is perpendicular to $BC$. Since $O_a'I = MO$, the line $O_a'I$ is also perpendicular to $BC$.

Since the six circles $(O_a')$ and $(O_a'')$ etc are congruent (with common radius $OI$) and their centers are all at a distance $R$ from $I$, it is clear that there are two circles, center $I$, tangent to all these circles. These two circles are tangent to the circumcircle, the point of tangency being the intersection of the circumcircle with the line $OI$. These are the triangle centers $X_{1381}$ and $X_{1382}$ of [1].

References


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