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A Short Trigonometric Proof of the Steiner-Lehmus Theorem

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Abstract. We give a short trigonometric proof of the Steiner-Lehmus theorem.

The well known Steiner-Lehmus theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles. Unlike its trivial converse, this challenging statement has attracted a lot of attention since 1840, when Professor Lehmus of Berlin wrote to Sturm asking for a purely geometrical proof. Proofs by Rougevain, Steiner, and Lehmus himself appeared in the following few years. Since then, a great number of people, including several renowned mathematicians, took interest in the problem, resulting in as many as 80 different proofs. Extensive histories are given in [14], [15], [16], and [21], and biographies and lists of references can be found in [33], [37], and [19]. More references will be referred to later when we discuss generalizations and variations of the theorem.

In this note, we present a new trigonometric proof of the theorem. Compared with the existing proofs, such as the one given in [17, pp. 194–196], it is also short and simple. It runs as follows.

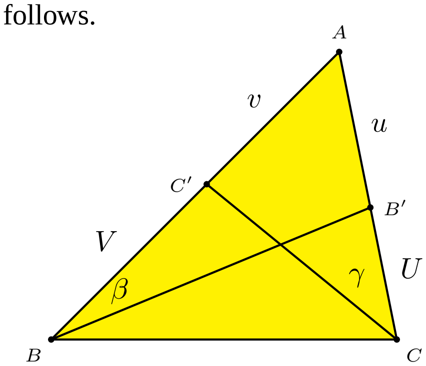

Figure 1

Let BB' and CC' be the respective internal angle bisectors of angles B and C in triangle ABC, and let a, b and c denote the sidelengths in the standard order. As shown in Figure 1, we set

$$B = 2\beta,\ C = 2\gamma,\quad u = AB',\ U = B'C,\quad v = AC',\ V = C'B.$$

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We shall see that the assumptions $BB' = CC'$ and $C > B$ (and hence $c > b$) lead to the contradiction that

$$\frac{b}{u} < \frac{c}{v}, \ \frac{b}{u} > \frac{c}{v}. \tag{1}$$

Geometrically, this means that the line $B'C'$ intersects both rays BC and CB.

To achieve (1), we use the law of sines, the angle bisector theorem, and the identity $\sin 2\theta = 2 \sin\theta \cos\theta$ to obtain

$$\frac{b}{u} - \frac{c}{v} = \frac{u+U}{u} - \frac{v+V}{v} = \frac{U}{u} - \frac{V}{v} = \frac{a}{c} - \frac{a}{b} < 0, \tag{2}$$

$$\begin{aligned} \frac{b}{u} \div \frac{c}{v} &= \frac{b}{c}\,\frac{v}{u} = \frac{\sin B}{\sin C}\,\frac{v}{u} = \frac{2\cos\beta\sin\beta}{2\cos\gamma\sin\gamma}\,\frac{v}{u} = \frac{\cos\beta}{\cos\gamma}\,\frac{\sin\beta}{u}\,\frac{v}{\sin\gamma} \\ &= \frac{\cos\beta}{\cos\gamma}\,\frac{\sin A}{BB'}\,\frac{CC'}{\sin A} = \frac{\cos\beta}{\cos\gamma} > 1. \end{aligned} \tag{3}$$

Clearly (2) and (3) lead to the contradiction (1).

No new proofs of the Steiner-Lehmus theorem seem to have appeared in the past several decades, and attention has been focused on generalizations, variations, and certain foundational issues. Instead of taking angle bisectors, one may take r-sectors, i.e., cevians that divide the angles internally in the ratio $r : 1 - r$ for $r \in (0, 1)$. Then the result still holds; see [35], [15, X, p. 311], [36], and more recently, [5], [2], and [10]. In fact, the result still holds in absolute (or neutral) geometry; see [15, X, p. 311] and the references therein, and more recently [6, Exercise 7, p. 9; solution, p. 420] and [19, Exercise 15, p. 119]. One may also consider external angle bisectors. Then one sees that the equality of two external angle bisectors (and similarly the equality of one internal and one external angle bisectors) does not imply isoscelessness. This is considered in [16], [22], [23], and more recently in [11]; see also [30] and the references therein. The situation in spherical geometry was also considered by Steiner; see [16, IX, p. 310].

Variations on the Steiner-Lehmus theme have become popular in the past few decades with much of the contribution due to the late C. F. Parry. Here, one starts with a center P of triangle ABC, not necessarily the incenter, and lets the cevians AA', BB', CC' through P intersect the circumcircle of ABC at A^*, B^*, C^*, respectively. The classical Steiner-Lehmus theorem deals with the case when P is the incenter and considers the assumption $BB' = CC'$. One may start with any center and consider any of the assumptions $BB' = CC'$, $BB^* = CC^*$, $A'B' = A'C'$, $A^*B^* = A^*C^*$, etc. Such variations and others have appeared in [27], [28], [29], [34], [3], [12], [32], [31], [1], and [26, Problem 4, p. 31], and are surveyed in [13]. Some of these variations have been investigated in higher dimensions in [7] and interesting results were obtained. However, the generalization of the classical Steiner-Lehmus theorem to higher dimensions remains open: We still do not know what degree of regularity a d-simplex must enjoy so that two or even all the internal angle bisectors of the corner angles are equal. This problem is raised at the end of [7].

The existing proofs of the Steiner-Lehmus theorem are all indirect (many being proofs by contradiction or *reductio ad absurdum*) or use theorems that do not have direct proofs. The question, first posed by Sylvester in [36], whether there is a direct proof of the Steiner-Lehmus theorem is still open, and Sylvester's conjecture (and semi-proof) that no such proof exists seems to be commonly accepted; see the refutation made in [20] of the allegedly direct proof given in [24], and compare to [8], where we are asked on p. 58 (Problem 16) to *give a direct proof of the Steiner-Lehmus theorem*, and where such a *a proof* is given on p. 390 using Stewart's theorem. An interesting forum discussion can also be visited at [9]. We would like here to raise the question whether one can provide a direct proof of the following weaker version of the Steiner-Lehmus theorem: *If the three internal angle bisectors of the angles of a triangle are equal, then the triangle is equilateral.*

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