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On the Parry Reflection Point

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Abstract. We give a synthetic proof of C. F. Parry's theorem that the reflections in the sidelines of a triangle of three parallel lines through the vertices are concurrent if and only if they are parallel to the Euler line, the point of concurrency being the Parry reflection point. We also show that the Parry reflection point is common to a triad of circles associated with the tangential triangle and the triangle of reflections (of the vertices in their opposite sides). A dual result is also given.

1. The Parry reflection point

Theorem 1 (Parry). *Suppose triangle ABC has circumcenter O and orthocenter H. Parallel lines α, β, γ are drawn through the vertices A, B, C, respectively. Let α', β', γ' be the reflections of α, β, γ in the sides BC, CA, AB, respectively. These reflections are concurrent if and only if α, β, γ are parallel to the Euler line OH. In this case, their point of concurrency P is the reflection of O in E, the Euler reflection point.*

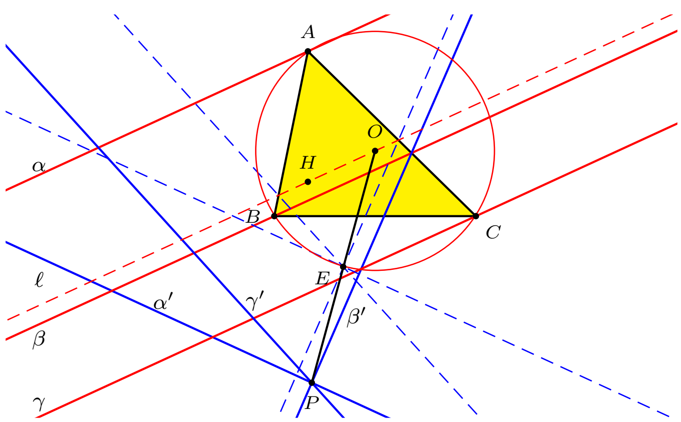

Figure 1.

We give a synthetic proof of this beautiful theorem below. C. F. Parry proposed this as a problem in the AMERICAN MATHEMATICAL MONTHLY, which was subsequently solved by R. L. Young using complex coordinates [6]. The point P in question is called the Parry reflection point. It appears as the triangle center X_{399}

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in [5]. The Euler reflection point E, on the other hand, is the point on the circumcircle which is the point of concurrency of the reflections of the Euler line in the sidelines. See Figure 1. It appears as X_{110} in [5]. The existence of E is justified by another elegant result on reflections of lines, which we use to deduce Theorem 1.

Theorem 2 (Collings)**.** *Let ℓ be a line in the plane of a triangle ABC. Its reflections in the sidelines BC, CA, AB are concurrent if and only if ℓ passes through the orthocenter H of ABC. In this case, their point of concurrency lies on the circumcircle.*

Synthetic proofs of Theorem 2 can be found in [1] and [3].

We denote by A', B', C' the reflections of A, B, C in their opposite sides, and by $A_{\rm t}B_{\rm t}C_{\rm t}$ the tangential triangle of ABC.

Theorem 3. *The circumcircles of triangles $A_{\rm t}B'C'$, $B_{\rm t}C'A'$ and $C_{\rm t}A'B'$ are concurrent at Parry's reflection point P.* See Figure 2.

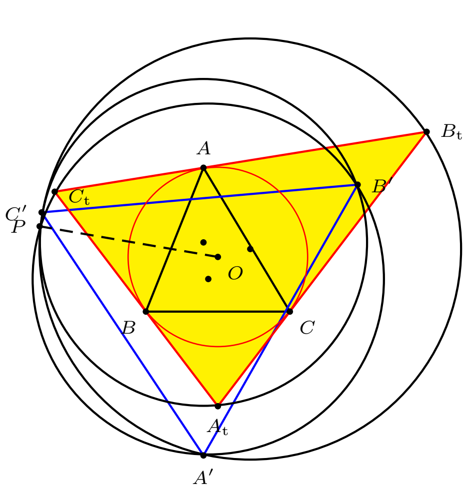

Figure 2

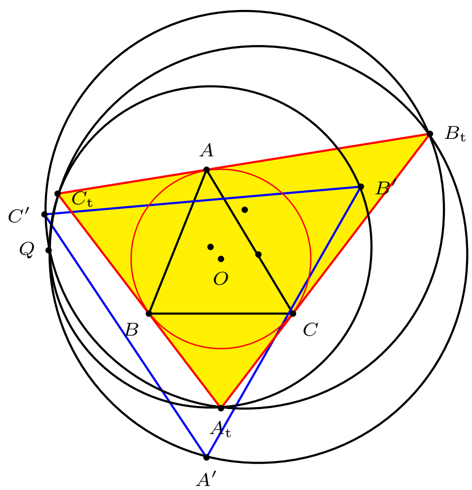

Figure 3

Theorem 4. *The circumcircles of triangles $A'B_{\rm t}C_{\rm t}$, $B'C_{\rm t}A_{\rm t}$ and $C'A_{\rm t}B_{\rm t}$ have a common point Q.* See Figure 3.

2. Proof of Theorem 1

Let $A_1B_1C_1$ be the image of ABC under the homothety $\mathsf{h}(O,2)$. The orthocenter H_1 of $A_1B_1C_1$ is the reflection of O in H, and is on the Euler line of triangle ABC.

Consider the line ℓ through H parallel to the given lines α, β, γ. Let M be the midpoint of BC, and $M_1 = \mathsf{h}(O,2)(M)$ on the line B_1C_1. The line AH intersects

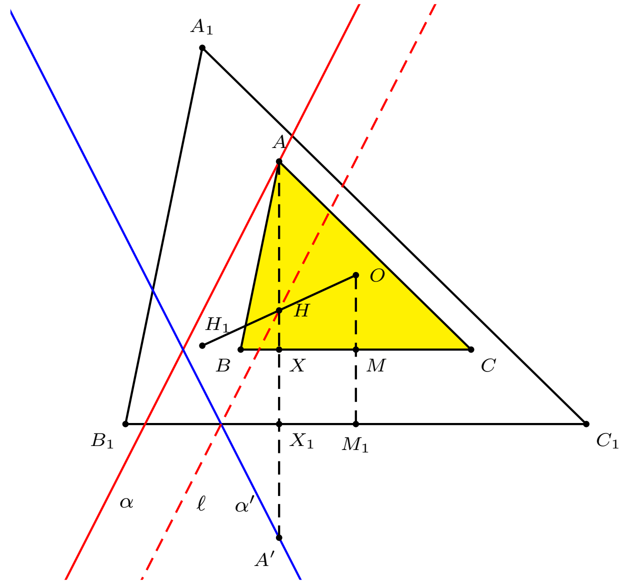

Figure 4.

BC and B_1C_1 at X and X_1 respectively. Note that the reflection of H in B_1C_1 is the reflection D of A in BC since $AH = 2 \cdot OM$ and

$$\begin{aligned} HA' =& AA' - AH = 2(AX - OM) = 2(AH + HX - OM) \\ =& 2(HX + OM) = 2(HX + XX_1) = 2HX_1. \end{aligned}$$

Therefore, α' coincides with the reflection of ℓ in the sides B_1C_1. Similarly, β' and γ' coincide with the reflections of ℓ in C_1A_1 and A_1B_1. By Theorem 2, the lines α', β', γ' are concurrent if and only if ℓ passes through the orthocenter H_1. Since H also lies on ℓ, this is the case when ℓ is the Euler line of triangle ABC, which is also the Euler line of triangle $A_1B_1C_1$. In this case, the point of concurrency is the Euler reflection point of $A_1B_1C_1$, which is the image of E under the homothety $\mathsf{h}(O, 2)$.

3. Proof of Theorem 3

We shall make use of the notion of directed angles $(\ell_1, \ \ell_2)$ between two lines ℓ_1 and ℓ_2 as the angle of rotation (defined modulo π) that will bring ℓ_1 to ℓ_2 in the same orientation as ABC. For the basic properties of directed angles, see [4, §§16–19].

Let α, β, γ be lines through the vertices A, B, C, respectively parallel to the Euler line. By Theorem 1, their reflections α', β', γ' in the sides BC, CA, AB pass through the Parry reflection point P.

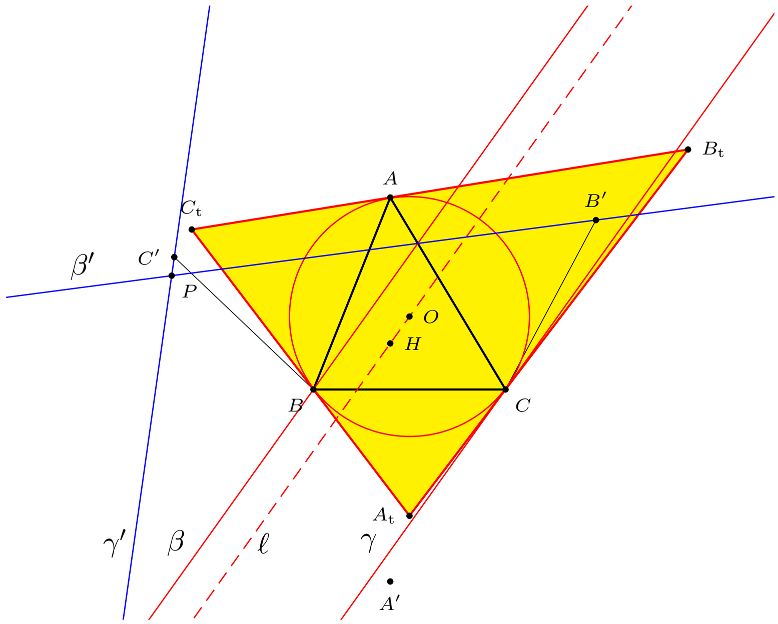

Figure 5

Now, since α, β, γ are parallel,

$$\begin{aligned}
(PB',\ PC') &= (\beta', \gamma') \\
&= (\beta', BC) + (BC, \gamma') \\
&= -(\beta', B'C) - (BC', \gamma') \quad \text{because of symmetry in } AC \\
&= (B'C,\ \beta) + (\gamma,\ BC') \\
&= (B'C,\ \beta) + (\beta,\ BC') \\
&= (B'C,\ BC') \\
&= (B'C,\ AC) + (AC,\ BC') \\
&= (AC,\ BC) + (AC,\ BC') \quad \text{because of symmetry in } AC \\
&= (AC,\ AB) + (AB,\ BC) + (AC,\ AB) + (AB,\ BC') \\
&= 2(AC,\ AB) \quad \text{because of symmetry in } AB \\
&= (OC,\ OB) \\
&= (A_tC,\ A_tB).
\end{aligned}$$

Since $A_tB = A_tC$ and $BC' = BC = B'C$, we conclude that the triangles A_tBC' and A_tCB' are directly congruent. Hence, $(A_tB',\ A_tC') = (A_tC,\ A_tB)$. This gives $(PB',\ PC') = (A_tB',\ A_tC')$, and the points P, A_t, B', C' are concyclic. The circle $A_tB'C'$ contains the Parry reflection point, so do the circles $B_tC'A'$ and $C_tA'B'$.

4. Proof of Theorem 4

Invert with respect to the Parry point P. By Theorem 3, the circles $A_tB'C'$, $B_tC'A'$, $C_tA'B'$ are inverted into the three lines bounding triangle $A'^*B'^*C'^*$. Here, A'^*, B'^*, C'^* are the inversive images of A', B', C' respectively. Since the points A_t^*, B_t^*, C_t^* lie on the lines $B'^*C'^*$, $C'^*A'^*$, $A'^*B'^*$, respectively, by Miquel's theorem, the circumcircles of triangles $A_t^*B'^*C'^*$, $B_t^*C'^*A'^*$, $C_t^*A'^*B'^*$ have a common point; so do their inversive images, the circles $A_tB'C'$, $B_tC'A'$, $C_tA'B'$. This completes the proof of Theorem 4.

The homogenous barycentric coordinates of their point of concurrency Q were computed by Javier Francisco Garcia Capitán [2] with the aid of Mathematica.

Added in proof. After the completion of this paper, we have found that the points P and Q are concyclic with the circumcenter O and the orthocenter H. See Figure 6. Paul Yiu has confirmed this by computing the coordinates of the center of the circle of these four points:

$$(a^2(b^2-c^2)(a^8(b^2+c^2)-a^6(4b^4+3b^2c^2+4c^4)+2a^4(b^2+c^2)(3b^4-2b^2c^2+3c^4)$$
$$-a^2(4b^8-b^6c^2+b^4c^4-b^2c^6+4c^8)+(b^2-c^2)^2(b^2+c^2)(b^4+c^4)))$$
$$:\cdots:\cdots),$$

where the second and third coordinates are obtained by cyclic permutations of a, b, c.

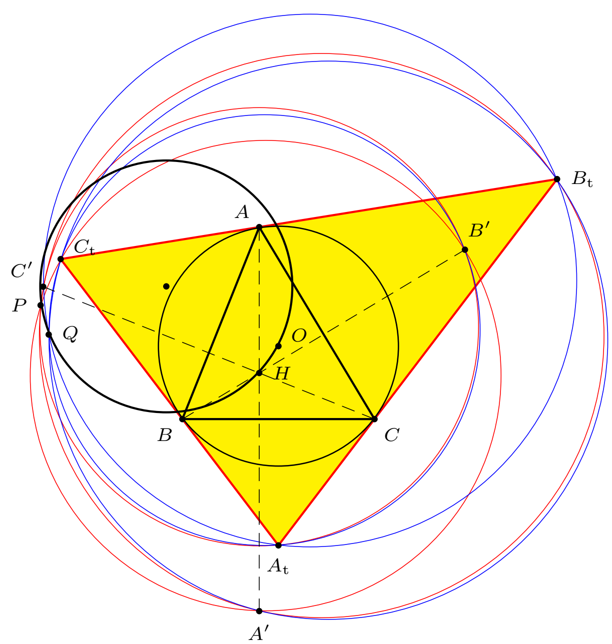

Figure 6.

For completeness, we record the coordinates of Q given by Garcia Capitán:

$$Q = (a^2 \sum_{k=0}^{10} a^{2(10-k)} f_{2k,a}(b,c) : b^2 \sum_{k=0}^{10} b^{2(10-k)} f_{2k,b}(c,a) : c^2 \sum_{k=0}^{10} c^{2(10-k)} f_{2k,c}(a,b)),$$

where

$$\begin{aligned}
f_{0,a}(b,c) &= 1,\\
f_{2,a}(b,c) &= -6(b^2+c^2)\\
f_{4,a}(b,c) &= 2(7b^4+12b^2c^2+7c^4),\\
f_{6,a}(b,c) &= -2(b^2+c^2)(7b^4+10b^2c^2+7c^4),\\
f_{8,a}(b,c) &= b^2c^2(18b^4+25b^2c^2+18c^4),\\
f_{10,a}(b,c) &= (b^2+c^2)(14b^8-15b^6c^2+8b^4c^4-15b^2c^6+14c^8),\\
f_{12,a}(b,c) &= -14b^{12}+b^{10}c^2+5b^8c^4-2b^6c^6+5b^4c^8+b^2c^{10}-14c^{12},\\
f_{14,a}(b,c) &= (b^2-c^2)^2(b^2+c^2)(6b^8+2b^6c^2+5b^4c^4+2b^2c^6+6c^8),\\
f_{16,a}(b,c) &= -(b^2-c^2)^2(b+c)^2(b^{12}-2b^{10}c^2-b^8c^4-6b^6c^6-b^4c^8-2b^2c^{10}+c^{12}),\\
f_{18,a}(b,c) &= -b^2c^2(b^2-c^2)^4(b^2+c^2)(3b^4+b^2c^2+3c^4),\\
f_{20,a}(b,c) &= b^2c^2(b^2-c^2)^6(b^2+c^2)^2.
\end{aligned}$$

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