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Construction of Malfatti Squares

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Abstract. We give a very simple construction of the Malfatti squares of a triangle, and study the condition when all three Malfatti squares are inside the given triangle. We also give an extension to the case of rectangles.

1. Introduction

The Malfatti squares of a triangle are the three squares each with two adjacent vertices on two sides of the triangle and the two remaining adjacent vertices from those of a triangle in its interior. We borrow this terminology from [3] (see also [1, p.48]) where the lengths of the sides of the Malfatti squares are stated. In Figure 1, the Malfatti squares of triangle ABC are $B'C'Z_aY_a$, $C'A'X_bZ_b$ and $A'B'Y_cX_c$. We shall call $A'B'C'$ the Malfatti triangle of ABC, and present a simple construction of $A'B'C'$ from a few common triangle centers of ABC. Specifically, we shall make use of the isogonal conjugate of the Vecten point of ABC. [1] This is a point on the Brocard axis, the line joining the circumcenter O and the symmedian point K.

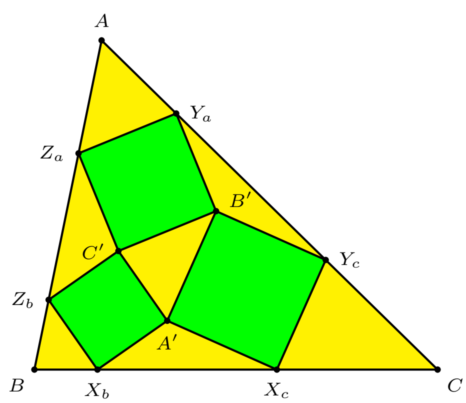

Figure 1.

Theorem 1. *Let P be the isogonal conjugate of the Vecten point of triangle ABC. The vertices of the Malfatti triangle are the intersections of the lines joining the centroid G to the pedals of the symmedian point K and the corresponding vertices to the pedals of P on the opposite sides of ABC.* See Figure 2.

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[1]The Vecten point of a triangle is the perspector of the (triangle whose vertices are) the centers of the squares erected externally on the sides. It appears as X_{485} in [6]. See Figure 6. Its isogonal conjugate appears as X_{371}, and is also called the Kenmotu point. It is associated with the construction of a triad of congruent squares in a triangle. In [5, p.268] the expression for the edge length of the squares should be reduced by a factor $\sqrt{2}$. A correct expression appears in [6] and [2, p.94].

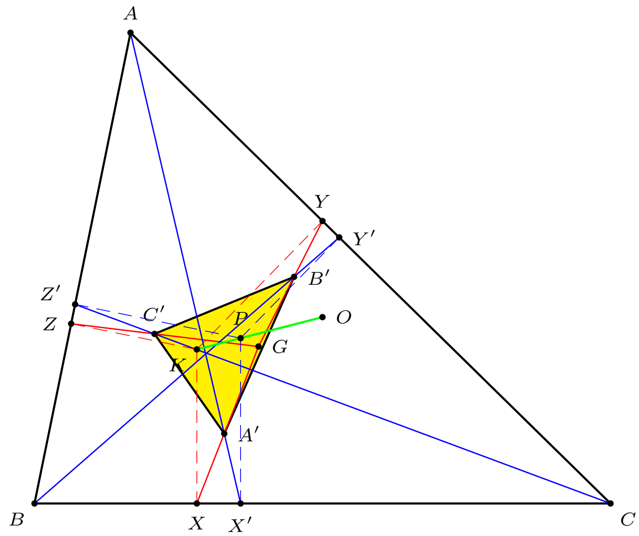

Figure 2.

2. Notations

We adopt the following notations. For a triangle of sidelengths a, b, c, let S denote *twice* the area of the triangle, and

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2}.$$

These satisfy

$$S_BS_C + S_CS_A + S_AS_B = S^2.$$

More generally, for an arbitrary angle θ, $S_\theta = S \cdot \cot\theta$. In particular,

$$S_A + S_B + S_C = \frac{a^2+b^2+c^2}{2} = S_\omega,$$

where ω is the Brocard angle of triangle ABC.

3. The triangle of medians

Given a triangle ABC with sidelengths a, b, c, let m_a, m_b, m_c denote the lengths of the medians. By the Apollonius theorem, these are given by

$$\begin{aligned} m_a^2 =& \frac{1}{4}(2b^2+2c^2-a^2), \\ m_b^2 =& \frac{1}{4}(2c^2+2a^2-b^2), \\ m_c^2 =& \frac{1}{4}(2a^2+2b^2-c^2). \end{aligned} \tag{1}$$

There is a triangle whose sidelengths are m_a, m_b, m_c. See Figure 3A. We call this the triangle of medians of ABC. The following useful lemma can be easily established.

Lemma 2. *Two applications of the triangle of medians construction gives a similar triangle of similarity factor* $\frac{3}{4}$. See Figure 3B.

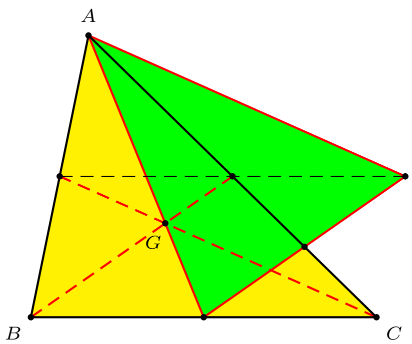

Figure 3A

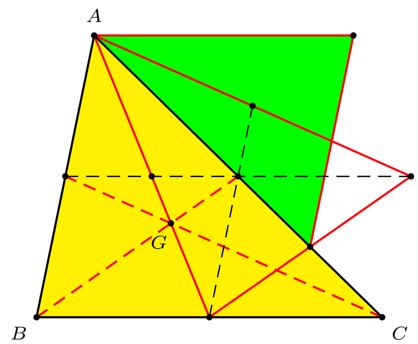

Figure 3B

We present an interesting example of a triangle similar to the triangle of medians which is useful for the construction of the Malfatti triangle.

Lemma 3. *The pedal triangle of the symmedian point is similar to the triangle of medians, the similarity factor being* $\tan\omega$.

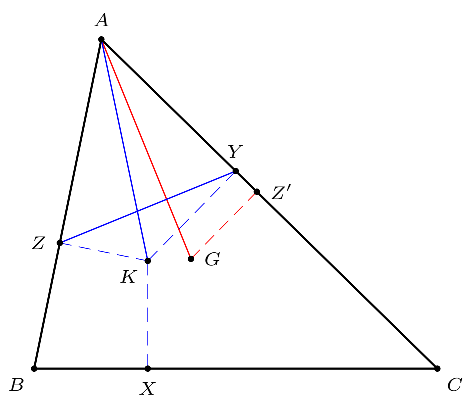

Figure 4

Proof. Since $S = bc\sin A$, the distance from the centroid G to AC is clearly $\frac{S}{3b}$. That from the symmedian point K to AB is

$$\frac{c^2}{a^2+b^2+c^2}\cdot\frac{S}{c} = \frac{S}{2S_\omega}\cdot c.$$

Since K and G are isogonal conjugates,

$$AK = AG\cdot\frac{\frac{S}{2S_\omega}\cdot c}{\frac{S}{3b}} = \frac{2}{3}m_a\cdot\frac{3bc}{2S_\omega} = \frac{bc}{S_\omega}\cdot m_a.$$

This is a diameter of the circle through A, K, and its pedals on AB and AC. It follows that the distance between the two pedals is

$$\frac{bc}{S_\omega} \cdot m_a \cdot \sin A = \frac{S}{S_\omega} \cdot m_a = \tan\omega \cdot m_a.$$

From this, it is clear that the pedal triangle is similar to the triangle of medians, the similarity factor being $\tan\omega$. □

Remark. The triangle of medians of ABC has the same Brocard angle as ABC.

Proposition 4. *Let P be a point with pedal triangle XYZ in ABC. The lines through A, B, C perpendicular to the sides YZ, ZX, XY concur at the isogonal conjugate of P.*

We shall also make use of the following characterization of the symmedian (Lemoine) point of a triangle.

Theorem 5 (Lemoine)**.** *The symmedian point is the unique point which is the centroid of its own pedal triangle.*

4. Proof of Theorem 1

Consider a triangle ABC with its Malfatti squares. Complete the parallelogram $A'B'A^*C'$. See Figure 5. Note that triangles $A'X_bX_c$ and $C'A'A^*$ are congruent. Therefore, $A'A^*$ is perpendicular to BC. Note that this line contains the centroid G' of triangle $A'B'C'$. Similarly, if we complete parallelograms $B'C'B^*A'$ and $C'A'C^*B'$, the lines $B'B^*$ and $C'C^*$ contain G' and are perpendicular to CA and AB respectively.

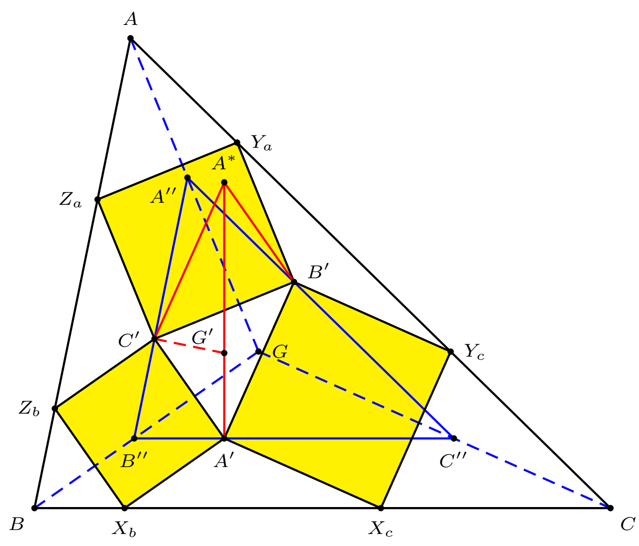

Figure 5.

Consider $A'B'C'$ as the pedal triangle of G' in a triangle $A''B''C''$ homothetic to ABC. By Lemoine's theorem, G' is the symmedian point of $A''B''C''$. Since

$A''B''C''$ is homothetic to ABC, $A'B'C'$ is homothetic to the pedal triangle of the symmedian point K of ABC.

In Figure 5, triangle $A''B'C'$ is the image of AY_aZ_a under the translation by the vector $\mathbf{Y_aB'} = \mathbf{Z_aC'} = \mathbf{AA''}$. This means that the line AA'' is perpendicular to $B'C'$, and to the A-side of the pedal triangle of K. Similarly, BB'' and CC'' are perpendicular to B- and C-sides of the same pedal triangle. By Proposition 4, the lines AA'', BB'', CC'' concur at the isogonal conjugate of K. This means that triangles $A''B''C''$ and ABC are homothetic at the centroid G of triangle ABC, and the sides of the Malfatti squares are parallel and perpendicular to the corresponding medians.

Denote by λ the homothetic ratio of $A''B''C''$ and ABC . This is also the homothetic ratio of the Malfatti triangle $A'B'C'$ and the pedal triangle of K. In Figure 5, $BX_b + X_cC = B''C'' = \lambda a$. Also, by Lemmas 2 and 3,

$$\begin{aligned} X_bX_c = A'A^* = {} & 2\lambda \cdot A\text{−median of pedal triangle of } K \\ & = 2\lambda \cdot \tan\omega \cdot A\text{−median of triangle of medians of } ABC \\ & = 2\lambda \cdot \tan\omega \cdot \frac{3}{4}a = \frac{3}{2}\lambda \cdot \tan\omega \cdot a. \end{aligned}$$

Since $BX_b + X_bX_c + X_cC = BX$, we have $\lambda(1 + \frac{3}{2}\tan\omega) = 1$ and

$$\lambda = \frac{2}{2 + 3\tan\omega} = \frac{2S_\omega}{3S + 2S_\omega}.$$

Let $\mathsf{h}(G, \lambda)$ be the homothety with center G and ratio λ. Since G' is the symmedian point of $A''B''C''$,

$$G' = \mathsf{h}(G, \lambda) = \lambda K + (1 - \lambda)G = \frac{1}{3S + 2S_\omega}(3S \cdot G + 2S_\omega \cdot K).$$

It has homogeneous barycentric coordinates $(a^2 + S : b^2 + S : c^2 + S)$. [2]

To compute the coordinates of the vertices of the Malfatti triangle, we make use of the pedals of the symmedian point K on the sidelines. The pedal on BC is the point

$$X = \frac{1}{2S_\omega}((S_A + 2S_C)B + (S_A + 2S_B)C).$$

A' is the point dividing the segment GX in the ratio $GA' : A'X = S_\omega : 3S$.

$$\begin{aligned} A' = {} & \frac{1}{3S + 2S_\omega}(3S \cdot G + 2S_\omega \cdot X) \\ = {} & \frac{1}{3S + 2S_\omega}\left(S \cdot A + (S + S_A + 2S_C)B + (S + S_A + 2S_B)C\right). \end{aligned}$$

[2]For a construction of G', see Proposition 6.

Similarly, we have B' and C'. In homogeneous barycentric coordinates, these are

$$\begin{aligned}A' =&(S : S+S_A+2S_C : S+S_A+2S_B),\\ B' =&(S+S_B+2S_C : S : S+S_B+2S_A),\\ C' =&(S+S_C+2S_B : S+S_C+2S_A : S).\end{aligned}$$

The lines AA', BB', CC' intersect the sidelines BC, CA, AB respectively at the points

$$\begin{aligned}X' =&(0 : S+S_A+2S_C : S+S_A+2S_B),\\ Y' =&(S+S_B+2S_C : 0 : S+S_B+2S_A),\\ Z' =&(S+S_C+2S_B : S+S_C+2S_A : 0).\end{aligned} \tag{2}$$

We show that these three intersections are the pedals of a specific point

$$P = (a^2(S_A+S) : b^2(S_B+S) : c^2(S_C+S)).$$

In absolute barycentric coordinates,

$$P = \frac{1}{2S(S+S_\omega)}\left((a^2(S_A+S)A + b^2(S_B+S)B + c^2(S_C+S)C\right).$$

The infinite point of the perpendiculars to BC being $-a^2\cdot A + S_C\cdot B + S_B\cdot C$, the perpendicular from P to BC contains the point

$$\begin{aligned}& P + \frac{S_A+S}{2S(S+S_\omega)}(-a^2\cdot A + S_C\cdot B + S_B\cdot C)\\ = & \frac{1}{2S(S+S_\omega)}\left((b^2(S_B+S)+S_C(S_A+S))B + (c^2(S_C+S)+S_B(S_A+S))C\right)\\ = & \frac{1}{2(S+S_\omega)}\left((S+S_A+2S_C)B + (S+S_A+2S_B)C\right).\end{aligned}$$

This is the point X' whose homogeneous coordinates are given in (2) above. Similarly, the pedals of P on the other two lines CA and AB are the points Y' and Z' respectively.

These lead to a simple construction of the vertex A', as the intersection of the lines GX and the line joining A to the pedal of P on BC. This completes the proof of Theorem 1.

Remark. Apart from A', B', C', the vertices of the Malfatti squares on the sidelines are

$$\begin{aligned}X_b &= (0 : 3S+S_A+2S_C : S_A+2S_B), & X_c &= (0 : S_A+2S_C : 3S+S_A+2S_B),\\ Y_c &= (S_B+2S_C : 0 : 3S+2S_A+S_B), & Y_a &= (3S+S_B+2S_C : 0 : 2S_A+S_B),\\ Z_a &= (3S+2S_B+S_C : 2S_A+S_C : 0), & Z_b &= (2S_B+S_C : 3S+2S_A+S_C : 0).\end{aligned}$$

5. An alternative construction

The vertices of the Malfatti triangle $A'B'C'$ are the intersections of the perpendiculars from G' to the sidelines of triangle ABC with the corresponding lines joining G to the pedals of K on the sidelines. A simple construction of G' would lead to the Malfatti triangle easily. Note that G' divides GK in the ratio

$$GG' : G'K = a^2 + b^2 + c^2 : 3S.$$

On the other hand, the point P is the isogonal conjugate of the Vecten point

$$V = \left(\frac{1}{S_A + S} : \frac{1}{S_B + S} : \frac{1}{S_C + S}\right).$$

As such, it can be easily constructed, as the intersection of the perpendiculars from A, B, C to the corresponding sides of the pedal triangles of V. See Figure 6. It is a point on the Brocard axis, dividing OK in the ratio

$$OP : PK = a^2 + b^2 + c^2 : 2S.$$

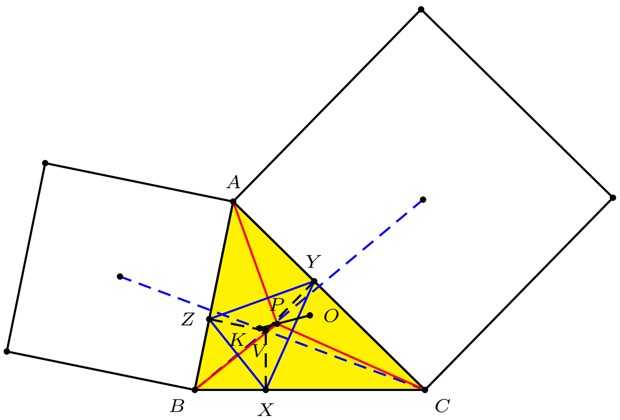

Figure 6.

This leads to a simple construction of the point G'.

Proposition 6. *G' is the intersection of GK with HP, where H is the orthocenter of triangle ABC.*

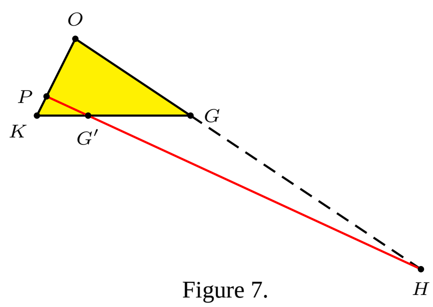

Figure 7.

Proof. Apply Menelaus' theorem to triangle OGK with transversal HP, noting that $OH : HG = 3 : -2$. See Figure 7. $\square$

6. Some observations

6.1. *Malfatti squares not in the interior of given triangle.* Sokolowsky [3] mentions the possibility that the Malfatti squares need not be contained in the triangle. Jean-Pierre Ehrmann pointed out that even the Malfatti triangle may have a vertex outside the triangle. Figure 8 shows an example in which both B' and Y_a are outside the triangle.

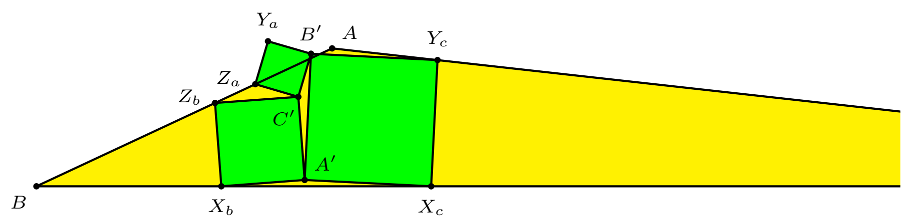

Figure 8.

Proposition 7. *At most one of the vertices the Malfatti triangle and at most one of the vertices of the Malfatti squares on the sidelines can be outside the triangle.*

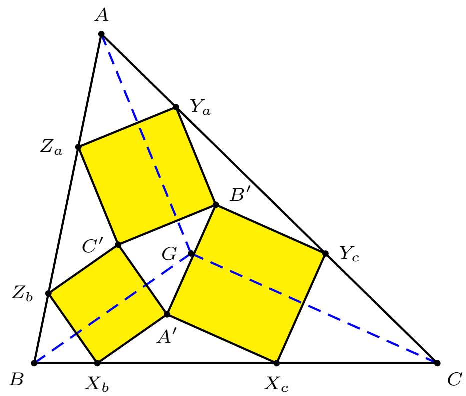

Figure 9.

Proof. If Y_a lies outside triangle ABC, then $\angle AZ_aC' < \frac{\pi}{2}$, and $\angle Z_bZ_aC' > \frac{\pi}{2}$. Since Z_aC' is parallel to AG, $\angle BAG = \angle Z_bZ_aC'$ is obtuse. Under the same hypothesis, if B' and C are on opposite sides of AB, then $\angle AZ_aC' < \frac{\pi}{4}$, and $\angle BAG > \frac{3\pi}{4}$.

Similarly, if any of Z_a, Z_b, X_b, X_c, Y_c lies outside the triangle, then correspondingly, $\angle CAG$, $\angle CBG$, $\angle ABG$, $\angle ACG$, $\angle BCG$ is obtuse. Since at most one of these angles can be obtuse, at most one of the six vertices on the sides and at most one of A', B', C' can be outside triangle ABC. $\square$

6.2. *A locus problem.* François Rideau [8] asked, given B and C, for the locus of A for which the Malfatti squares of triangle ABC are in the interior of the triangle. Here is a simple solution. Let M be the midpoint of BC, P the reflection of C in B, and Q that of B in C. Consider the circles with diameters PB, BM, MC, CQ, and the perpendiculars ℓ_P and ℓ_Q to BC at P and Q. See Figure 10.

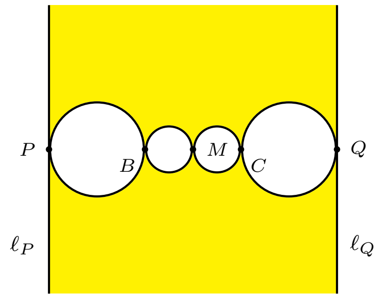

Figure 10.

For an arbitrary point A, consider ABC with centroid G.
(i) $\angle ABG$ is obtuse if A is inside the circle with diameter PB;
(ii) $\angle BAG$ is obtuse if A is inside the circle with diameter BM;
(iii) $\angle CAG$ is obtuse if A is inside the circle with diameter MC;
(iv) $\angle ACG$ is obtuse if A is inside the circle with diameter CQ;
(v) $\angle CBG$ is obtuse if A is on the side of ℓ_P opposite to the circles;
(vi) $\angle BCG$ is obtuse if A is on the side of ℓ_Q opposite to the circles.

Therefore, the locus of A for which the Malfatti squares of triangle ABC are in the interior of the triangle is the region between the lines ℓ_P and ℓ_Q with the four disks excised.

A similar reasoning shows that the locus of A for which the vertices A', B', C' of the Malfatti triangle of ABC are in the interior of triangle ABC is the shaded region in Figure 11.

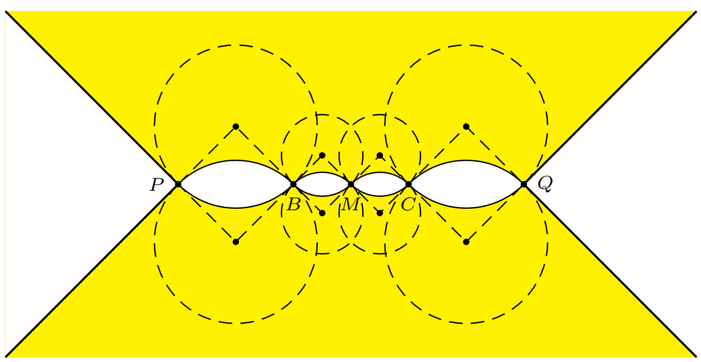

Figure 11.

7. Generalization

We present a generalization of Theorem 1 in which the Malfatti squares are replaced by rectangles of a specified shape. We say that a rectangle constructed on a side of triangle ABC has shape θ if its center is the apex of the isosceles triangle constructed on that side with base angle θ. We assume $0 < \theta < \frac{\pi}{2}$ so that the apex is on the opposite side of the corresponding vertex of the triangle. It is well known that for a given θ, the centers of the three rectangles of shape θ erected on the sides are perspective with ABC at the Kiepert perspector

$$K(\theta) = \left(\frac{1}{S_A + S_\theta} : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta}\right).$$

The isogonal conjugate of $K(\theta)$ is the point

$$K^*(\theta) = (a^2(S_A + S_\theta) : b^2(S_B + S_\theta) : c^2(S_C + S_\theta))$$

on the Brocard axis dividing the segment OK in the ratio $\tan\omega\tan\theta : 1$.

Theorem 8. *For a given θ, let $A(\theta)$ be the intersection of the lines joining* (i) *the centroid G to the pedal of the symmedian point K on BC,* (ii) *the vertex A to the pedal of $K^*(\theta)$ on BC. Analogously construct points $B(\theta)$ and $C(\theta)$. Construct rectangles of shape θ on the sides of $A(\theta)B(\theta)C(\theta)$. The remaining vertices of these rectangles lie on the sidelines of triangle ABC.*

Figure 12 illustrates the case of the isodynamic point J. The Malfatti rectangles $B'C'Z_aY_a$, $C'A'X_bZ_b$ and $A'B'Y_cX_c$ have shape $\frac{\pi}{3}$, *i.e.*, lengths and widths in the ratio $\sqrt{3} : 1$.

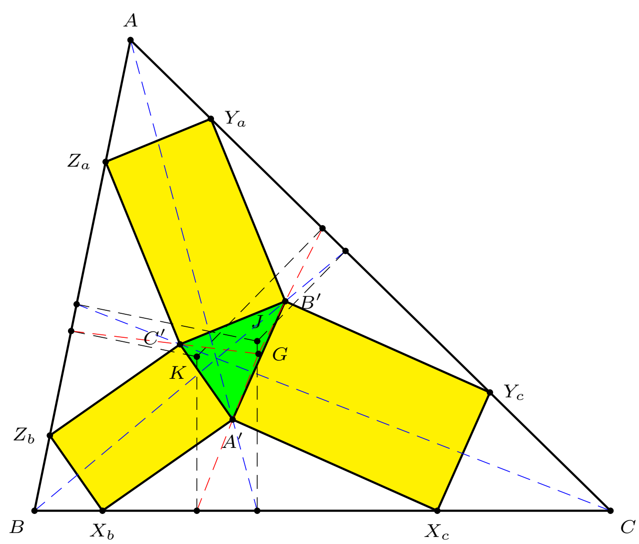

Figure 12.

The same reasoning in §6 shows that exactly one of the six vertices on the sidelines is outside the triangle if and only if a median makes an obtuse angle with an adjacent side. If this angle exceeds $\frac{\pi}{2} + \theta$, the corresponding vertex of Malfatti triangle is also outside ABC.

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