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Haruki's Lemma and a Related Locus Problem

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Abstract. In this paper we investigate the nature of the constant in Haruki's Lemma and study a related locus problem.

1. Introduction

In his papers [2, 3], Ross Honsberger mentions a remarkably beautiful lemma that he accredits to Professor Hiroshi Haruki. The beauty and mystery of Haruki's lemma is in its apparent simplicity.

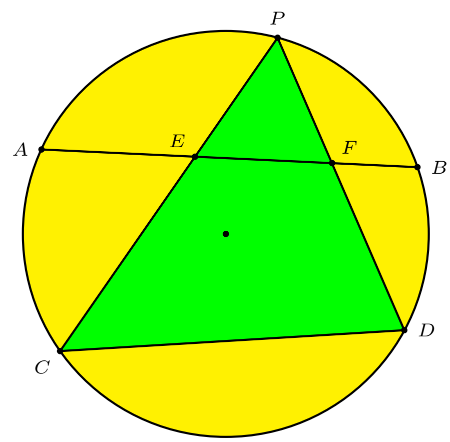

Figure 1. Haruki's lemma: $\frac{AE \cdot BF}{EF}$ = constant.

Lemma 1 (Haruki)**.** *Given two nonintersecting chords AB and CD in a circle and a variable point P on the arc AB remote from points C and D, let E and F be the intersections of chords PC, AB, and of PD, AB respectively. The value of* $\dfrac{AE \cdot BF}{EF}$ *does not depend on the position of P.*

A very intriguing statement indeed. It should be duly noted that Haruki's Lemma leads to an easy proof of the Butterfly Theorem; see [2], [3, pp.135–140]. The nature of the constant, however, remains unclear. By looking at it in more detail we shall discover some interesting results.

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2. Proof of Haruki's lemma

A good interactive visualisation and proof of Haruki's lemma can be found in [1]. Here we present the proof essentially as it appeared in [3]. The proof is quite ingenious and relies on the fact that the angle $\angle CPD$ is constant.

We begin by constructing a circumcirlce of triangle PED and define point G to be the intersection of this cirumcircle with the line AB. Note that $\angle EGD = \angle EPD$ as they are subtended by the same chord ED of the circumcircle of $\triangle PED$ and so these angles remain constant as P varies on the arc AB. Hence, for all positions of P, $\angle EGD$ remains fixed and, therefore, point G remains fixed on the line AB (See Figure 2). So $BG = constant$.

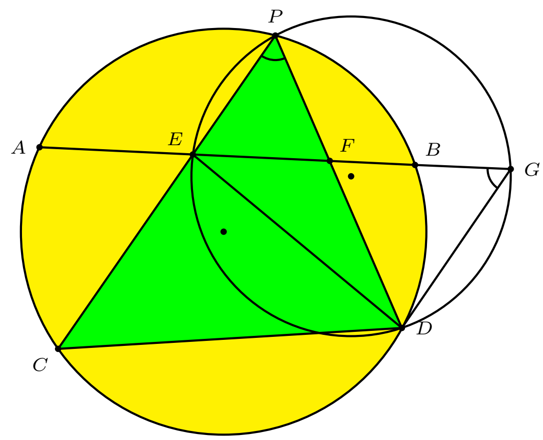

Figure 2. Point G is a fixed point on line AB.

Now, by applying the intersecting chords theorem to PD and AG in the two circles, we obtain the following:

$$AF \cdot FB = PF \cdot FD,$$
$$EF \cdot FG = PF \cdot FD.$$

From these, $(AE + EF) \cdot FB = EF \cdot (FB + BG)$, and $AE \cdot FB = EF \cdot BG$. Therefore, we have obtained $\dfrac{AE \cdot BF}{EF} = BG$, a constant. This completes the proof of Lemma 1.

Note that in the proof we could have used the circumcircle around $\triangle PFC$ instead of the one around $\triangle PED$.

3. An extension of Haruki's lemma

Haruki has apparently found the constant. However, finding it raises additional questions. Why is the ratio of distances that are bound to the circle (through points A, B, C, D, P) expressed by a constant that involves a point lying *outside* the circle? We explore the setup in Lemma 1. Consider an inversion with center P and

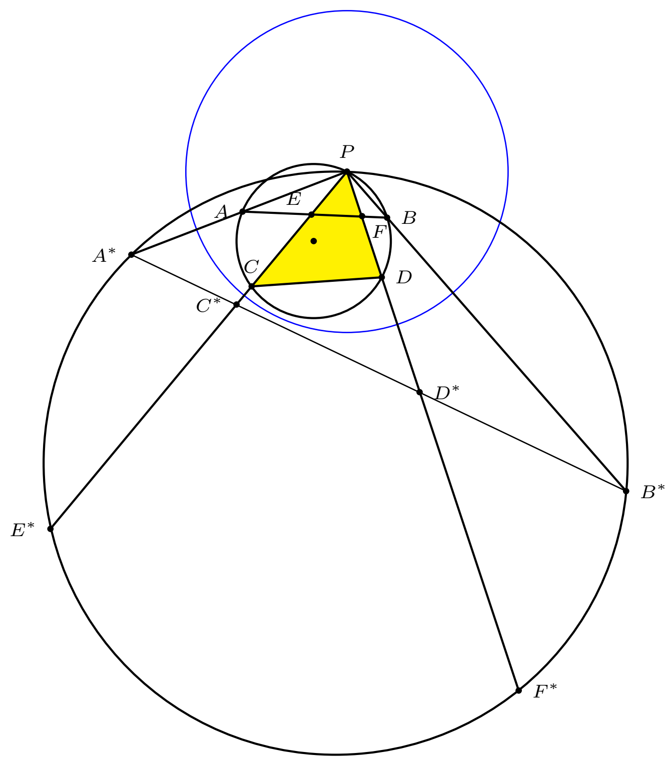

Figure 3. Applying inversion with center P

radius r that is bigger than the diameter of the circumcircle of $ABDC$ (See Figure 3).

Recall two basic facts about an inversion:
(a) It maps a line not through the center of inversion into a circle that goes through the center of inversion and vice versa.
(b) It maps the line that goes through the center of inversion into the same line.

Knowing these two facts, we can perform an inversion on the setup in Figure 1, the results of which are shown in Figure 3. We can see that the segments A^*E^*, B^*F^* and E^*F^* have taken the place of the segments AC, BD, CD. We shall use this hint to deduce the following extension of Haruki's Lemma.

Lemma 2. *Given two nonintersecting chords AB and CD in a circle and a variable point P on the arc AB remote from points C and D, let E and F be the intersections of chords PC, AB, and of PD, AB respectively. The following equalities hold:*

$$\frac{AE \cdot BF}{EF} = \frac{AC \cdot BD}{CD}, \tag{1}$$

$$\frac{AF \cdot BE}{EF} = \frac{AD \cdot BC}{CD}. \tag{2}$$

Proof. (1) Following the notation and proof of Lemma 1, we have $\frac{AE \cdot BF}{EF} = BG$. It remains to show that $BG = \frac{AC \cdot BD}{CD}$, or, equivalently,

$$\frac{BG}{BD} = \frac{AC}{CD}. \tag{3}$$

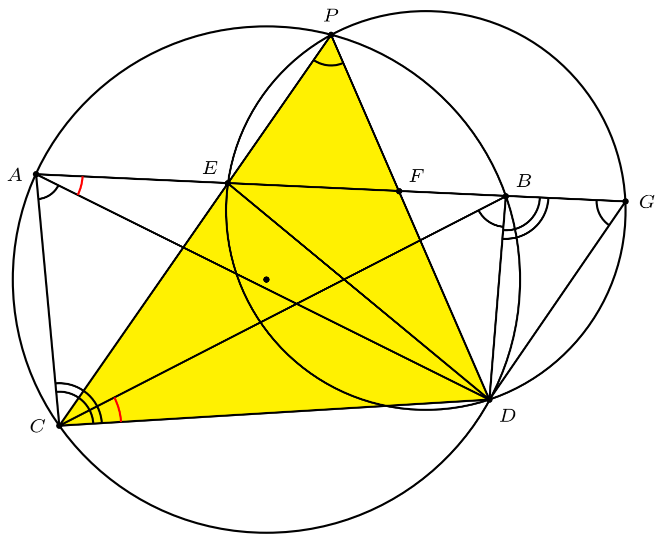

Figure 4. Triangles ACD and GBD are similar, as are AGD and CBD

Note that in Figure 4, $\angle CAD = \angle CPD = \angle EPD = \angle EGD$. Since $ABDC$ is a cyclic quadrilateral, we have $\angle ACD = \angle DBG$. This means that the triangles ACD and GBD are similar, thus yielding (3), and therefore (1).

For (2) we note that $\angle DCB = \angle DAB$. Also $\angle CBD = \angle CPD = \angle EPD = \angle EGD$, thus we get $\triangle AGD \sim \triangle CBD$ yielding:

$$\frac{AG}{AD} = \frac{BC}{CD} \Rightarrow AG = \frac{AD \cdot BC}{CD}.$$

However, $AF \cdot BE = (AE + EF) \cdot (EF + BF) = AE \cdot BF + AB \cdot EF$. We obtain, by using Lemma 1,

$$\frac{AF \cdot BE}{EF} = \frac{AE \cdot BF}{EF} + AB = BG + AB = AG = \frac{AD \cdot BC}{CD}.$$

□

Note that by switching the position of points C and D we effectively switch points E and F, thus equations (1) and (2) are equivalent. It may seem surprising; but the statement of Lemma 2 holds even for intersecting chords AB and CD and for any point P on the circle for which the points E and F are defined.

Theorem 3. *Given two distinct chords AB and CD in a circle and a point P on that circle distinct from A and B, let E and F be the intersections of the line AB with the lines PC and PD respectively. The equalities* (1) *and* (2) *hold.*

We leave the proof to the reader as an exercise. All that is necessary is to consider the different cases for the relative positions of A, B, C, D, P and to apply the ideas in the proofs of Lemmas 1 and 2, i.e. finding the point G as the intersection of the circumcircle of either $\triangle PED$ or $\triangle PFC$ with AB and then looking for similar triangles. Note that the point P may coincide with either C or D. In this case, by the line PC or PD we would mean the tangent to the circle at C or D.

4. A locus problem

Theorem 3 settles the case when points A, B, C, D lie on a circle. But what happens when points A, B, C, D do not belong to the same circle? Can we still find points P that will satisfy equation (1) or (2)? This gives rise to the following locus problem.

Problem. Given the points A, B, C, D find the locus $\mathcal{L}_1$ (respectively $\mathcal{L}_2$) of all points P that satisfy (1) (respectively (2)), where points E and F are the intersections of lines PC and AB, PD and AB respectively.

To investigate the loci $\mathcal{L}_1$ and $\mathcal{L}_2$, we begin with a result about the possibility of a point P belonging to both $\mathcal{L}_1$ and $\mathcal{L}_2$.

Lemma 4. *If there is a point P satisfying both* (1) *and* (2)*, then A, B, C, D are concyclic.*

Proof. First of all, points A, B, E, F are collinear, hence, they satisfy Euler's distribution theorem (See [4, p.3] and [5]), *i.e.*, if A, B, E, F are in this order, then, $AF \cdot BE + AB \cdot EF = AE \cdot BF$. Dividing through by EF, we obtain

$$\frac{AF \cdot BE}{EF} + AB = \frac{AE \cdot BF}{EF},$$

and so, by the fact that point P satisfies equations (1) and (2), we have:

$$\frac{AD \cdot BC}{CD} + AB = \frac{AC \cdot BD}{CD}.$$

Now multiplying by CD yields

$$AD \cdot BC + AB \cdot CD = AC \cdot BD,$$

which, by Ptolemy's inequality (See [6]), means that points A, B, C, D are concyclic with points A, C separating points B, D on the circle. The relative positions of A, B, E, F will influence the relative positions of points A, B, C, D on the circle. Similar argument can be applied to establish the validity of the statement of this lemma no matter the position of points A, B, E, F. $\square$

This lemma is interesting in the way it ties the "linear" Euler's equality, Ptolemy's inequality together with the extension of Haruki's lemma.

5. Barycentric coordinates

In order to find the loci $\mathcal{L}_1$ and $\mathcal{L}_2$ for the general position of points A, B, C, D we make use of the notion of homogeneous barycentric coordinates. Given a reference triangle ABC, any three numbers x, y, z proportional to the signed areas of oriented triangles PBC, PCA, PAB form a set of *homogeneous barycentric coordinates* of P, written as $(x : y : z)$.

With reference to triangle ABC, the absolute barycentric coordinates of the vertices are obviously $A(1 : 0 : 0)$, $B(0 : 1 : 0)$ and $C(0 : 0 : 1)$. We shall make use of the following basic property of barycentric coordinates.

Lemma 5. *Let P be point with homogeneous barycentric coordinates $(x : y : z)$ with reference to triangle ABC. The line AP intersects BC at a point X with coordinates $(0 : y : z)$, which divides BC in the ratio $BX : XC = z : y$. Similarly, BP intersects CA at $Y(x : 0 : z)$ such that $CY : YA = x : z$ and CP intersects AB at $Z(x : y : 0)$ such that $AZ : ZB = y : x$.*

Assume that D and P have barycentric coordinates $D(u : v : w)$ and $P(x : y : z)$. It is our aim to compute the coordinates of points E and F.

When there is no danger of confusion, we shall represent a line $p\alpha + q\beta + r\gamma = 0$ by $(p : q : r)$. The intersection of two lines $(p : q : r)$ and $(s : t : u)$ is the point

$$\left(\begin{vmatrix} q & r \\ t & u \end{vmatrix} : \begin{vmatrix} r & p \\ u & s \end{vmatrix} : \begin{vmatrix} p & q \\ s & t \end{vmatrix} \right).$$

This same expression also gives the line through the two points with homogeneous barycentric coordinates $(p : q : r)$ and $(s : t : u)$.

6. Solution of the locus problem

From the above formula we compute the coordinates of the lines AB, PC and PD:

Line	Coordinates
AB	$(0 : 0 : 1)$
PC	$(-y : x : 0)$
PD	$\left(\begin{vmatrix} y & z \\ v & w \end{vmatrix} : \begin{vmatrix} z & x \\ w & u \end{vmatrix} : \begin{vmatrix} x & y \\ u & v \end{vmatrix} \right)$

From these we obtain the coordinates of E and F:

$$E \left(\begin{vmatrix} x & 0 \\ 0 & 1 \end{vmatrix} : \begin{vmatrix} 0 & -y \\ 1 & 0 \end{vmatrix} : \begin{vmatrix} -y & x \\ 0 & 0 \end{vmatrix} \right) = (x : y : 0),$$

$$F \left(\begin{vmatrix} z & x \\ w & u \end{vmatrix} : \begin{vmatrix} z & y \\ w & v \end{vmatrix} : 0 \right) = (uz - wx : vz - wy : 0).$$

Assume $BC = a$, $CA = b$, and $AB = c$. Also, $AD = a'$, $BD = b'$, and $CD = c'$. These are also fixed quantities. From the coordinates of E and F, we obtain, by Lemma 5, the following *signed* lengths:

$$AE = \frac{y}{x+y} \cdot c, \qquad EB = \frac{x}{x+y} \cdot c;$$

$$AF = \frac{vz-wy}{z(u+v)-w(x+y)} \cdot c, \quad FB = \frac{uz-wx}{z(u+v)-w(x+y)} \cdot c.$$

Consequently,

$$EF = EB - FB = \frac{z(vx - uy)}{(x+y)(z(u+v) - w(x+y))} \cdot c.$$

Now we determine the loci $\mathcal{L}_1$ and $\mathcal{L}_2$.

Theorem 6. *Given the points A, B, C, D and a point P, define points E and F as the intersections of lines PC and AB, PD and AB respectively.*
(a) *The locus $\mathcal{L}_1$ of points P satisfying* (1) *is the union of two circumconics of $ABCD$ given by the equations*

$$(cc' + \varepsilon bb')uyz - \varepsilon bb'vzx - cc'wxy = 0, \quad \varepsilon = \pm 1. \tag{4}$$

(b) *The locus $\mathcal{L}_2$ of points P satisfying* (2) *is the union of two circumconics of $ABCD$ given by the equations*

$$\varepsilon aa'uyz + (cc' - \varepsilon aa')vzx - cc'wxy = 0, \quad \varepsilon = \pm 1. \tag{5}$$

Proof. In terms of signed lengths, (1) and (2) should be interpreted as $AE \cdot BF \cdot CD = \varepsilon \cdot AC \cdot BD \cdot EF$ and $AF \cdot BE \cdot CD = \varepsilon \cdot AD \cdot BC \cdot EF$ for $\varepsilon = \pm 1$. The results follow from direct substitutions. It is easy to see that the conics represented by (4) and (5) all contain the points A, B, C, D, with barycentric coordinates $(1:0:0)$, $(0:1:0)$, $(0:0:1)$, $(u:v:w)$ respectively. □

7. Constructions

Theorem 6 tells us that the loci in question are each a union of two conics, each containing the four given points A, B, C, D. In order to construct these conics, we would need to find a fifth point on each of them. The following proposition helps with this problem.

Proposition 7. *The four intersections of the bisectors of angles ABD, ACD, and the four intersections of the bisectors of angles CAB and CDB are points on $\mathcal{L}_1$.*

Proof. First of all, it is routine to verify that for $P = (x : y : z)$, we have

$$[AEP] \cdot [BFP] \cdot [CDP] = [ACP] \cdot [BDP] \cdot [EFP], \tag{6}$$

where $[XYZ]$ denotes the signed area of the oriented triangle XYZ. Let d_{XY} be the distance from P to the line XY. In terms of distances, the relation (6) becomes

$$(AE \cdot d_{AE})(BF \cdot d_{BF})(CD \cdot d_{CD}) = (AC \cdot d_{AC})(BD \cdot d_{BD})(EF \cdot d_{EF}).$$

From this it is clear that (1) is equivalent to

$$d_{AE} \cdot d_{BF} \cdot d_{CD} = d_{AC} \cdot d_{BD} \cdot d_{EF}. \tag{7}$$

Since AE, BF, EF are the same line AB, this condition can be rewritten as

$$d_{AB} \cdot d_{CD} = d_{AC} \cdot d_{BD}. \tag{8}$$

If P is an intersection of the bisectors of angles ABD and ACD, then $d_{AB} = d_{BD}$ and $d_{AC} = d_{CD}$. On the other hand, if P is an intersection of the bisectors of angles CAB and CDB, then $d_{AC} = d_{AB}$ and $d_{CD} = d_{BD}$. In both cases, (7) is satisfied, showing that P is a point on the locus $\mathcal{L}_1$. □

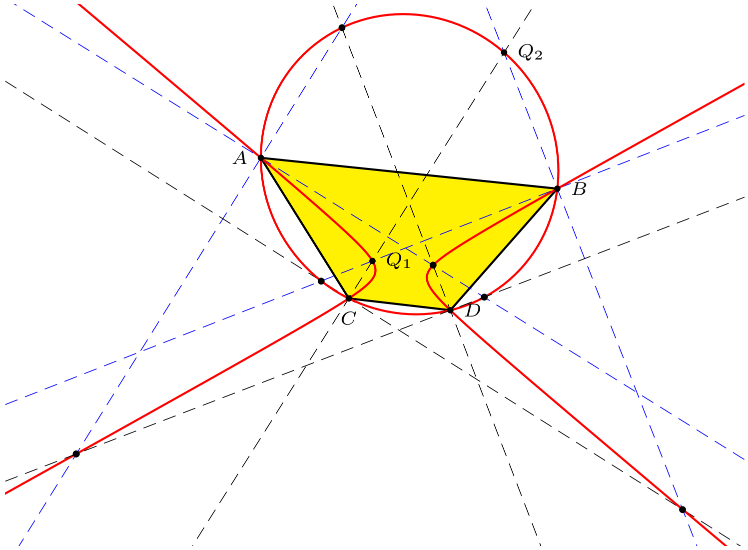

Figure 5. The locus $\mathcal{L}_1$

Let Q_1 be the intersection of the internal bisectors of angles ABD and ACD, and Q_2 as the intersection of the external bisector of angle ABD and the internal bisector of angle ACD. See Figure 5. Since Q_1, Q_2 and C are collinear, the points Q_1 and Q_2 must lie on distinct conics of $\mathcal{L}_1$.

Similarly, the locus $\mathcal{L}_2$ also contains the four intersections of the bisectors of angles BAD and BCD, and the four from angles ABC and ADC. Let Q_3 be the intersection of the internal bisectors and Q_4 the intersection of the internal bisector of BAD and the external bisector of angle BCD. See Figure 6. The points Q_3 and Q_4 are on different conics of $\mathcal{L}_2$.

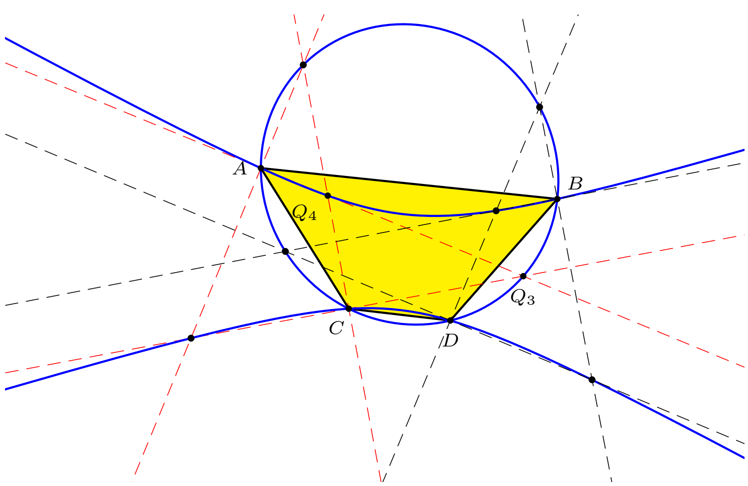

Figure 6. The locus $\mathcal{L}_2$

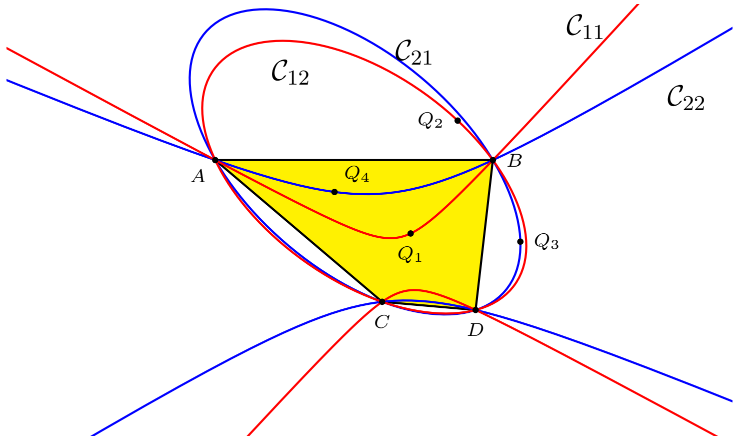

Figure 7.

Figure 7 shows the four conics, with $\mathcal{C}_{1,1}$, $\mathcal{C}_{1,2}$ forming $\mathcal{L}_1$ and $\mathcal{C}_{2,1}$, $\mathcal{C}_{2,2}$ forming $\mathcal{L}_2$.

Corollary 8. (a) *When points A, B, C, D all belong to the same circle $\mathcal{C}$, then one of the conics from $\mathcal{L}_1$ and one from $\mathcal{L}_2$ coincide with $\mathcal{C}$.*

(b) *If for some point P, (1) and (2) are both satisfied, then the points A, B, C, D, P are concyclic.*

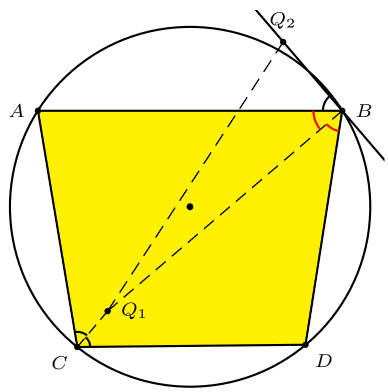

Figure 8.

Proof. (a) Assume Q_1 not on the circle $\mathcal{C}$. Suppose we have the situation as in Figure 8. In other cases the reasoning is similar. It is easy to see that $\angle ABQ_2 = \angle ACQ_2$ as Q_2 belongs to the external bisector of the angle ABD. This means that the points A, B, C and Q_2 are concyclic. But Q_2 lies on one of the conics from $\mathcal{L}_1$, therefore, this conic is actually a circle. Similarly, one can show that one of the conics from $\mathcal{L}_2$ coincides with $\mathcal{C}$. This proves (a).

(b) follows directly from (a) and Lemma 4. □

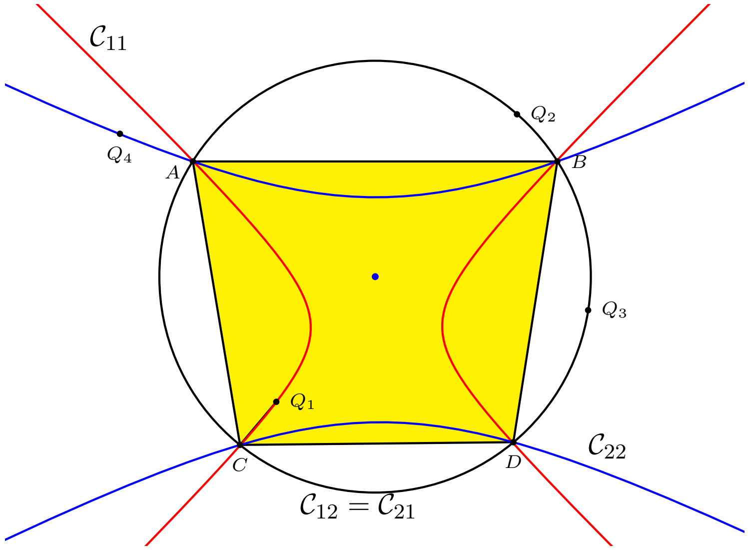

Figure 9. Loci $\mathcal{L}_1$ and $\mathcal{L}_2$ for cyclic quadrilateral $ABDC$

Theorem 3 together with Lemma 4 and part (b) of Corollary 8 provide us with the criteria for five points A, B, C, D and P to be concyclic. The case when $ABCD$ is a cyclic quadrilateral is depicted in Figure 9.

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