Haruki’s Lemma and a Related Locus Problem

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Abstract. In this paper we investigate the nature of the constant in Haruki’s Lemma and study a related locus problem.

1. Introduction

In his papers [2, 3], Ross Honsberger mentions a remarkably beautiful lemma that he accredits to Professor Hiroshi Haruki. The beauty and mystery of Haruki’s lemma is in its apparent simplicity.

![Figure 1. Haruki’s lemma: \( \frac{AE \cdot BF}{EF} \) = constant.]

Lemma 1 (Haruki). Given two nonintersecting chords \( AB \) and \( CD \) in a circle and a variable point \( P \) on the arc \( AB \) remote from points \( C \) and \( D \), let \( E \) and \( F \) be the intersections of chords \( PC, AB \), and of \( PD, AB \) respectively. The value of \( \frac{AE \cdot BF}{EF} \) does not depend on the position of \( P \).

A very intriguing statement indeed. It should be duly noted that Haruki’s Lemma leads to an easy proof of the Butterfly Theorem; see [2], [3, pp.135–140]. The nature of the constant, however, remains unclear. By looking at it in more detail we shall discover some interesting results.

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2. Proof of Haruki’s lemma

A good interactive visualisation and proof of Haruki’s lemma can be found in [1]. Here we present the proof essentially as it appeared in [3]. The proof is quite ingenious and relies on the fact that the angle \(\angle CPD\) is constant.

We begin by constructing a circumcircle of triangle \(PED\) and define point \(G\) to be the intersection of this circumcircle with the line \(AB\). Note that \(\angle EGD = \angle EPD\) as they are subtended by the same chord \(ED\) of the circumcircle of \(\triangle PED\) and so these angles remain constant as \(P\) varies on the arc \(AB\). Hence, for all positions of \(P\), \(\angle EGD\) remains fixed and, therefore, point \(G\) remains fixed on the line \(AB\) (See Figure 2). So \(BG = \text{constant}\).

![Figure 2. Point G is a fixed point on line AB.](image)

Now, by applying the intersecting chords theorem to \(PD\) and \(AG\) in the two circles, we obtain the following:

\[
AF \cdot FB = PF \cdot FD,
\]
\[
EF \cdot FG = PF \cdot FD.
\]

From these, \((AE + EF) \cdot FB = EF \cdot (FB + BG)\), and \(AE \cdot FB = EF \cdot BG\). Therefore, we have obtained \(\frac{AE \cdot BF}{EF} = BG\), a constant. This completes the proof of Lemma 1.

Note that in the proof we could have used the circumcircle around \(\triangle PFC\) instead of the one around \(\triangle PED\).

3. An extension of Haruki’s lemma

Haruki has apparently found the constant. However, finding it raises additional questions. Why is the ratio of distances that are bound to the circle (through points \(A, B, C, D, P\)) expressed by a constant that involves a point lying outside the circle? We explore the setup in Lemma 1. Consider an inversion with center \(P\) and
radius \( r \) that is bigger than the diameter of the circumcircle of \( ABDC \) (See Figure 3).

Recall two basic facts about an inversion:
(a) It maps a line not through the center of inversion into a circle that goes through the center of inversion and vice versa.
(b) It maps the line that goes through the center of inversion into the same line.

Knowing these two facts, we can perform an inversion on the setup in Figure 1, the results of which are shown in Figure 3. We can see that the segments \( A^*E^* \), \( B^*F^* \) and \( E^*F^* \) have taken the place of the segments \( AC, BD, CD \). We shall use this hint to deduce the following extension of Haruki’s Lemma.

**Lemma 2.** Given two nonintersecting chords \( AB \) and \( CD \) in a circle and a variable point \( P \) on the arc \( AB \) remote from points \( C \) and \( D \), let \( E \) and \( F \) be the intersections of chords \( PC, AB \), and of \( PD, AB \) respectively. The following equalities hold:

\[
\frac{AE \cdot BF}{EF} = \frac{AC \cdot BD}{CD}, \quad (1)
\]
\[
\frac{AF \cdot BE}{EF} = \frac{AD \cdot BC}{CD}. \quad (2)
\]
Proof. (1) Following the notation and proof of Lemma 1, we have \( \frac{AE \cdot BF}{EF} = BG \). It remains to show that \( BG = \frac{AC \cdot BD}{CD} \), or, equivalently,

\[
\frac{BG}{BD} = \frac{AC}{CD}.
\]

(3)

Note that in Figure 4, \( \angle CAD = \angle CPD = \angle EPD = \angle EGD \). Since \( ABDC \) is a cyclic quadrilateral, we have \( \angle ACD = \angle DBG \). This means that the triangles \( ACD \) and \( GBD \) are similar, thus yielding (3), and therefore (1).

For (2) we note that \( \angle DCB = \angle DAB \). Also \( \angle CBD = \angle CPD = \angle EPD = \angle EGD \), thus we get \( \triangle AGD \sim \triangle CBD \) yielding:

\[
\frac{AG}{AD} = \frac{BC}{CD} \Rightarrow AG = \frac{AD \cdot BC}{CD}.
\]

However, \( AF \cdot BE = (AE + EF) \cdot (EF + BF) = AE \cdot BF + AB \cdot EF \). We obtain, by using Lemma 1,

\[
\frac{AF \cdot BE}{EF} = \frac{AE \cdot BF}{EF} + AB = BG + AB = AG = \frac{AD \cdot BC}{CD}.
\]

\( \square \)

Note that by switching the position of points \( C \) and \( D \) we effectively switch points \( E \) and \( F \), thus equations (1) and (2) are equivalent. It may seem surprising; but the statement of Lemma 2 holds even for intersecting chords \( AB \) and \( CD \) and for any point \( P \) on the circle for which the points \( E \) and \( F \) are defined.

**Theorem 3.** Given two distinct chords \( AB \) and \( CD \) in a circle and a point \( P \) on that circle distinct from \( A \) and \( B \), let \( E \) and \( F \) be the intersections of the line \( AB \) with the lines \( PC \) and \( PD \) respectively. The equalities (1) and (2) hold.
We leave the proof to the reader as an exercise. All that is necessary is to consider the different cases for the relative positions of \(A, B, C, D, P\) and to apply the ideas in the proofs of Lemmas 1 and 2, i.e. finding the point \(G\) as the intersection of the circumcircle of either \(\triangle PED\) or \(\triangle PFC\) with \(AB\) and then looking for similar triangles. Note that the point \(P\) may coincide with either \(C\) or \(D\). In this case, by the line \(PC\) or \(PD\) we would mean the tangent to the circle at \(C\) or \(D\).

4. A locus problem

Theorem 3 settles the case when points \(A, B, C, D\) lie on a circle. But what happens when points \(A, B, C, D\) do not belong to the same circle? Can we still find points \(P\) that will satisfy equation (1) or (2)? This gives rise to the following locus problem.

**Problem.** Given the points \(A, B, C, D\) find the locus \(L_1\) (respectively \(L_2\)) of all points \(P\) that satisfy (1) (respectively (2)), where points \(E\) and \(F\) are the intersections of lines \(PC\) and \(AB, PD\) and \(AB\) respectively.

To investigate the loci \(L_1\) and \(L_2\), we begin with a result about the possibility of a point \(P\) belonging to both \(L_1\) and \(L_2\).

**Lemma 4.** If there is a point \(P\) satisfying both (1) and (2), then \(A, B, C, D\) are concyclic.

**Proof.** First of all, points \(A, B, E, F\) are collinear, hence, they satisfy Euler’s distribution theorem (See [4, p.3] and [5]), i.e., if \(A, B, E, F\) are in this order, then, \(AF \cdot BE + AB \cdot EF = AE \cdot BF\). Dividing through by \(EF\), we obtain

\[
\frac{AF \cdot BE}{EF} + AB = \frac{AE \cdot BF}{EF},
\]

and so, by the fact that point \(P\) satisfies equations (1) and (2), we have:

\[
\frac{AD \cdot BC}{CD} + AB = \frac{AC \cdot BD}{CD}.
\]

Now multiplying by \(CD\) yields

\[
AD \cdot BC + AB \cdot CD = AC \cdot BD,
\]

which, by Ptolemy’s inequality (See [6]), means that points \(A, B, C, D\) are concyclic with points \(A, C\) separating points \(B, D\) on the circle. The relative positions of \(A, B, E, F\) will influence the relative positions of points \(A, B, C, D\) on the circle. Similar argument can be applied to establish the validity of the statement of this lemma no matter the position of points \(A, B, E, F\). \(\square\)

This lemma is interesting in the way it ties the “linear” Euler’s equality, Ptolemy’s inequality together with the extension of Haruki’s lemma.
5. Barycentric coordinates

In order to find the loci $L_1$ and $L_2$ for the general position of points $A$, $B$, $C$, $D$ we make use of the notion of homogeneous barycentric coordinates. Given a reference triangle $ABC$, any three numbers $x$, $y$, $z$ proportional to the signed areas of oriented triangles $PBC$, $PCA$, $PAB$ form a set of homogeneous barycentric coordinates of $P$, written as $(x : y : z)$.

With reference to triangle $ABC$, the absolute barycentric coordinates of the vertices are obviously $A(1 : 0 : 0)$, $B(0 : 1 : 0)$ and $C(0 : 0 : 1)$. We shall make use of the following basic property of barycentric coordinates.

**Lemma 5.** Let $P$ be point with homogeneous barycentric coordinates $(x : y : z)$ with reference to triangle $ABC$. The line $AP$ intersects $BC$ at a point $X$ with coordinates $(0 : y : z)$, which divides $BC$ in the ratio $BX : XC = z : y$. Similarly, $BP$ intersects $CA$ at $Y(x : 0 : z)$ such that $CY : YA = x : z$ and $CP$ intersects $AB$ at $Z(x : y : 0)$ such that $AZ : ZB = y : x$.

Assume that $D$ and $P$ have barycentric coordinates $D(u : v : w)$ and $P(x : y : z)$. It is our aim to compute the coordinates of points $E$ and $F$.

When there is no danger of confusion, we shall represent a line $pα + qβ + rγ = 0$ by $(p : q : r)$. The intersection of two lines $(p : q : r)$ and $(s : t : u)$ is the point

$$\left(\frac{q \cdot r \cdot u - r \cdot s \cdot t + s \cdot p \cdot t - t \cdot p \cdot u}{p \cdot s - q \cdot t}, \frac{r \cdot q \cdot u - r \cdot s \cdot t + s \cdot p \cdot t - t \cdot p \cdot u}{p \cdot s - q \cdot t}, \frac{q \cdot r \cdot u - r \cdot s \cdot t + s \cdot p \cdot t - t \cdot p \cdot u}{p \cdot s - q \cdot t}\right).$$

This same expression also gives the line through the two points with homogeneous barycentric coordinates $(p : q : r)$ and $(s : t : u)$.

6. Solution of the locus problem

From the above formula we compute the coordinates of the lines $AB$, $PC$ and $PD$:

<table>
<thead>
<tr>
<th>Line</th>
<th>Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>$(0 : 0 : 1)$</td>
</tr>
<tr>
<td>$PC$</td>
<td>$(-y : x : 0)$</td>
</tr>
<tr>
<td>$PD$</td>
<td>$\left(\begin{array}{ccc}y &amp; z &amp; x \ v &amp; w &amp; u \ u &amp; v &amp; u\end{array}\right)$</td>
</tr>
</tbody>
</table>

From these we obtain the coordinates of $E$ and $F$:

$$E\left(\begin{array}{ccc}0 & 0 & -y \\ 1 & 0 & x \\ 0 & 1 & 0\end{array}\right) = (x : y : 0),$$

$$F\left(\begin{array}{ccc}z & x & y \\ w & u & v \\ w & v & 0\end{array}\right) = (uz - wx : vz - wy : 0).$$

Assume $BC = a$, $CA = b$, and $AB = c$. Also, $AD = a'$, $BD = b'$, and $CD = c'$. These are also fixed quantities. From the coordinates of $E$ and $F$, we obtain, by Lemma 5, the following signed lengths:

$$AE = \frac{x - y}{x + y} \cdot c, \quad EB = \frac{z}{x + y} \cdot c;$$

$$AF = \frac{z}{z(u + v) - w(x + y)} \cdot c, \quad FB = \frac{uz - wx}{z(u + v) - w(x + y)} \cdot c.$$
Consequently,
\[ EF = EB - FB = \frac{z(vx - uy)}{(x + y)(z(u + v) - w(x + y))} \cdot c. \]

Now we determine the loci \( L_1 \) and \( L_2 \).

**Theorem 6.** Given the points \( A, B, C, D \) and a point \( P \), define points \( E \) and \( F \) as the intersections of lines \( PC \) and \( AB \), \( PD \) and \( AB \) respectively.

(a) The locus \( L_1 \) of points \( P \) satisfying (1) is the union of two circumconics of \( ABCD \) given by the equations
\[ (cc' + \varepsilon bb')uyz - \varepsilon bb'vzx - cc'wxy = 0, \quad \varepsilon = \pm 1. \] (4)

(b) The locus \( L_2 \) of points \( P \) satisfying (2) is the union of two circumconics of \( ABCD \) given by the equations
\[ \varepsilon aa'uyz + (cc' - \varepsilon aa')vzx - cc'wxy = 0, \quad \varepsilon = \pm 1. \] (5)

**Proof.** In terms of signed lengths, (1) and (2) should be interpreted as \( AE \cdot BF \cdot CD = \varepsilon \cdot AC \cdot BD \cdot EF \) and \( AF \cdot BE \cdot CD = \varepsilon \cdot AD \cdot BC \cdot EF \) for \( \varepsilon = \pm 1 \). The results follow from direct substitutions. It is easy to see that the conics represented by (4) and (5) all contain the points \( A, B, C, D \), with barycentric coordinates \((1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (u : v : w)\) respectively. □

7. Constructions

Theorem 6 tells us that the loci in question are each a union of two conics, each containing the four given points \( A, B, C, D \). In order to construct these conics, we would need to find a fifth point on each of them. The following proposition helps with this problem.

**Proposition 7.** The four intersections of the bisectors of angles \( ABD, ACD \), and the four intersections of the bisectors of angles \( CAB \) and \( CDB \) are points on \( L_1 \).

**Proof.** First of all, it is routine to verify that for \( P = (x : y : z) \), we have
\[ [AEP] \cdot [BFP] \cdot [CDP] = [ACP] \cdot [BDP] \cdot [EFP], \] (6)
where \([XYZ]\) denotes the signed area of the oriented triangle \(XYZ\). Let \( d_{XY} \) be the distance from \( P \) to the line \( XY \). In terms of distances, the relation (6) becomes
\[ (AE \cdot d_{AE})(BF \cdot d_{BF})(CD \cdot d_{CD}) = (AC \cdot d_{AC})(BD \cdot d_{BD})(EF \cdot d_{EF}). \]
From this it is clear that (1) is equivalent to
\[ d_{AE} \cdot d_{BF} \cdot d_{CD} = d_{AC} \cdot d_{BD} \cdot d_{EF}. \] (7)
Since \( AE, BF, EF \) are the same line \( AB \), this condition can be rewritten as
\[ d_{AB} \cdot d_{CD} = d_{AC} \cdot d_{BD}. \] (8)
If \( P \) is an intersection of the bisectors of angles \( ABD \) and \( ACD \), then \( d_{AB} = d_{BD} \) and \( d_{AC} = d_{CD} \). On the other hand, if \( P \) is an intersection of the bisectors of angles \( CAB \) and \( CDB \), then \( d_{AC} = d_{AB} \) and \( d_{CD} = d_{BD} \). In both cases, (7) is satisfied, showing that \( P \) is a point on the locus \( L_1 \). □
Let $Q_1$ be the intersection of the internal bisectors of angles $ABD$ and $ACD$, and $Q_2$ as the intersection of the external bisector of angle $ABD$ and the internal bisector of angle $ACD$. See Figure 5. Since $Q_1$, $Q_2$ and $C$ are collinear, the points $Q_1$ and $Q_2$ must lie on distinct conics of $\mathcal{L}_1$.

Similarly, the locus $\mathcal{L}_2$ also contains the four intersections of the bisectors of angles $BAD$ and $BCD$, and the four from angles $ABC$ and $ADC$. Let $Q_3$ be the intersection of the internal bisectors and $Q_4$ the intersection of the internal bisector of $BAD$ and the external bisector of angle $BCD$. See Figure 6. The points $Q_3$ and $Q_4$ are on different conics of $\mathcal{L}_2$. 

Figure 5. The locus $\mathcal{L}_1$

Figure 6. The locus $\mathcal{L}_2$
Figure 7 shows the four conics, with \( C_{1,1}, C_{1,2} \) forming \( \mathcal{L}_1 \) and \( C_{2,1}, C_{2,2} \) forming \( \mathcal{L}_2 \).

**Corollary 8.** (a) When points \( A, B, C, D \) all belong to the same circle \( C \), then one of the conics from \( \mathcal{L}_1 \) and one from \( \mathcal{L}_2 \) coincide with \( C \).

(b) If for some point \( P \), (1) and (2) are both satisfied, then the points \( A, B, C, D, P \) are concyclic.

**Proof.** (a) Assume \( Q_1 \) not on the circle \( C \). Suppose we have the situation as in Figure 8. In other cases the reasoning is similar. It is easy to see that \( \angle ABQ_2 = \angle ACQ_2 \) as \( Q_2 \) belongs to the external bisector of the angle \( ABD \). This means that the points \( A, B, C \) and \( Q_2 \) are concyclic. But \( Q_2 \) lies on one of the conics from \( \mathcal{L}_1 \), therefore, this conic is actually a circle. Similarly, one can show that one of the conics from \( \mathcal{L}_2 \) coincides with \( C \). This proves (a).

(b) follows directly from (a) and Lemma 4. \( \square \)
Theorem 3 together with Lemma 4 and part (b) of Corollary 8 provide us with the criteria for five points $A$, $B$, $C$, $D$ and $P$ to be concyclic. The case when $ABCD$ is a cyclic quadrilateral is depicted in Figure 9.

References


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