A Short Proof of Lemoine’s Theorem

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Abstract. We give a short proof of Lemoine’s theorem that the Lemoine point of a triangle is the unique point which is the centroid of its own pedal triangle.

Lemoine’s theorem states that the Lemoine (symmedian) point of a triangle is the unique point which is the centroid of its own pedal triangle. A proof of the fact that the Lemoine point has this property can be found in Honsberger [4, p.72]. The uniqueness part was conjectured by Clark Kimberling in the very first Hyacinthos message [6], and was subsequently confirmed by computations by Barry Wolk [7], Jean-Pierre Ehrmann [2], and Paul Yiu [8, §4.6.2]. Darij Grinberg [3] has given a synthetic proof. In this note we give a short proof by applying two elegant results on orthologic triangles.

Lemma 1. If \(P\) is a point in plane of triangle \(ABC\), with pedal triangle \(A'B'C'\), then the perpendiculars from \(A\) to \(B'C'\), from \(B\) to \(C'A'\), from \(C\) to \(A'B'\) are concurrent at \(Q\), the isogonal conjugate of \(P\).

This is quite well-known. See, for example, [5, Theorem 237]. Figure 1 shows that \(AP\) and the perpendicular from \(A\) to \(B'C'\) are isogonal with reference to \(A\). From this Lemma 1 follows. The next beautiful result, illustrated in Figure 2, is the main subject of [1].
Theorem 2 (Danneels and Dergiades). If triangles $ABC$ and $A'B'C'$ are orthologic with centers $P$, $P'$, with the perpendiculars from $A$, $B$, $C$ to $B'C'$, $C'A'$, $A'B'$ intersecting at $P$ and those from $A'$, $B'$, $C'$ to $BC$, $CA$, $AB$ intersecting at $P'$, then the barycentric coordinates of $P$ with reference to $ABC$ are equal to the barycentric coordinates of $P'$ with reference to $A'B'C'$.

Now we prove Lemoine’s theorem.

Let $K$ be the Lemoine (symmedian) point of triangle $ABC$, and $A'B'C'$ its pedal triangle. According to Lemma 1, the perpendiculars from $A$ to $B'C'$, from $B$ to $C'A'$, from $C$ to $A'B'$ are concurrent at the centroid $G$ of $ABC$. Now since $ABC$ and $A'B'C'$ are orthologic, with $G$ as one of the orthology centers, by Theorem 2, the perpendiculars from $A'$ to $BC$, from $B'$ to $CA$, from $C'$ to $AB$ are concurrent at the centroid $G'$ of $A'B'C'$. Hence, the symmedian point $K$ coincides with the centroid of its pedal triangle.

Conversely, let $P$ a point with pedal triangle $A'B'C'$, and suppose $P$ is the centroid of $A'B'C'$; it has homogeneous barycentric coordinates $(1 : 1 : 1)$ with reference to $A'B'C'$. Since $ABC$ and $A'B'C'$ are orthologic, by Theorem 2, we have that the perpendiculars from $A$ to $B'C'$, from $B$ to $C'A'$, from $C$ to $A'B'$ are concurrent at a point $Q$ with homogeneous barycentric coordinates $(1 : 1 : 1)$ with reference to $ABC$. This is the centroid $G$. By Lemma 1, this is also the isogonal conjugate of $P$. This shows that $P = K$, the Lemoine (symmedian) point.

This completes the proof of Lemoine’s theorem.

References


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