Forum Geometricorum
Volume 8 (2008) 99–101.

FORUM GEOM
ISSN 1534-1178

# Means as Chords

Francisco Javier García Capitán

**Abstract**. On the circumcircle of a right triangle, we display chords whose lengths are the quadratic, arithmetic, geometric, and harmonic means of the two shorter sides.

Given two positive numbers $a$ and $b$, the inequalities among their various means

$$\frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}}$$

are well known. In order, these are the harmonic, geometric, arithmetic, and quadratic means of $a$ and $b$. Nelsen [1] has presented several few geometric proofs (without words). In the same spirit, we exhibit these various means as chords of a circle constructed from two segments of lengths $a$ and $b$.

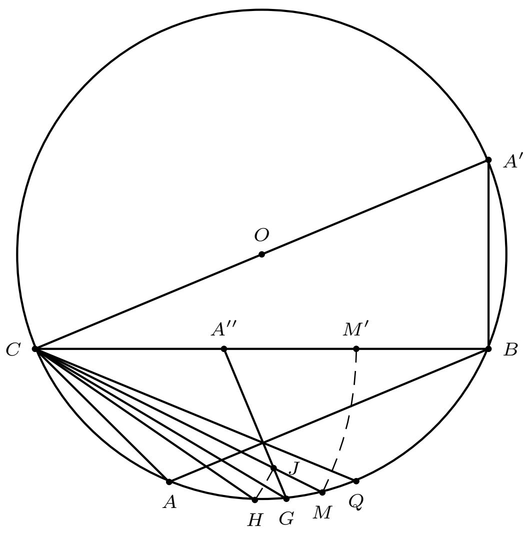


Figure 1

We shall assume $a \le b$, and begin with a right triangle $A'BC$ with $A'B = a$, $CB = b$ and a right angle at $B$. Construct
(1) the circumcircle of the triangle (with center at the midpoint $O$ of $CA'$,

Publication Date: April 28, 2008. Communicating Editor: Paul Yiu.
The author is grateful to Professor Paul Yiu for his suggestions for the improvement of this paper, and dedicates it to the memory of our friend Juan Carlos Salazar.

(2) points $A''$ on the segment $CB$ and and $A$ the circle respectively such that $CA = CA'' = A'B$ and $A$, $A'$ are on opposite sides of $CB$,
(3) the bisector $AQ$ of angle $ACB$ with $Q$ on the circle $(O)$,
(4) the midpoint $M'$ of $A''B$ and the point $M$ on the arc $CAB$ with $CM = CM'$,
(5) the perpendicular from $A''$ to $AB$ to intersect $AM$ at $J$ and the arc $CAB$ at $G$,
(6) the point $H$ on the arc $CAB$ such that $CH = CJ$.

**Proposition 1.** *For the two segments $CA$ and $CB$,*
(1) *$CQ$ is the quadratic mean,*
(2) *$CM$ is the arithmetic mean,*
(3) *$CG$ is the geometric mean,*
(4) *$CH$ is the harmonic mean.*

*Proof.* Note that the circle has radius $\frac{1}{2}\sqrt{a^2+b^2}$.

(1) Since $CA = A'B$, $CA'BA$ is an isosceles trapezoid, with $AB$ parallel to $CA'$. Since $CQ$ is the bisector of angle $ACB$, $Q$ is the midpoint of the arc $CAB$, and $OQ$ is perpendicular to $AB$. Hence, the radii $OQ$ and $OC$ are perpendicular to each other, and $CQ = \sqrt{2} \cdot OC = \frac{a^2+b^2}{2}$. This shows that $CQ$ is the quadratic mean of $a$ and $b$.

(2) $CM = CM' = \frac{1}{2}(a+b)$ is the arithmetic mean of $a$ and $b$.

(3) Let $A''G$ intersect $OA'$ at $L$. See Figure 2. Since $CA'$ is parallel to $AB$, $LG$ is perpendicular to $CA'$. From the similarity of the right triangles $CA''L$ and $CA'B$, we have $\frac{CL}{CA''} = \frac{CB}{CA'}$. In the right triangle $CA'G$, we have $CG^2 = CL \cdot CA' = CA'' \cdot CB = ab$. This shows that $CG$ is the geometric mean of $a$ and $b$.

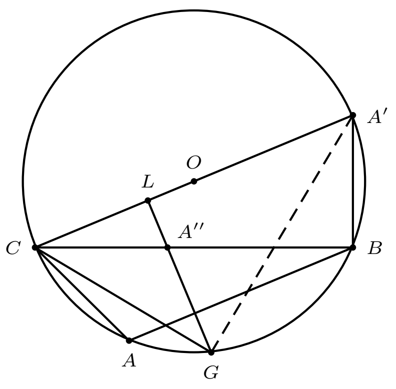


Figure 2

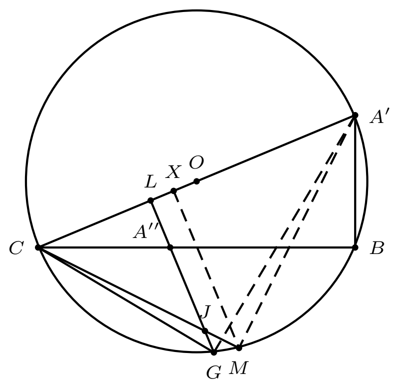


Figure 3

(4) Let the perpendicular from $M$ to $CA'$ intersect the latter at $X$. See Figure 3. From the similarity of triangles $CLJ$ and $CXM$, we have

$$CJ = CM \cdot \frac{CL}{CX} = CM \cdot \frac{CL \cdot CA'}{CX \cdot CA'} = CM \cdot \frac{CG^2}{CM^2} = \frac{CG^2}{CM} = \frac{2ab}{a+b}.$$

This shows that $CH = CJ$ is the harmonic mean of $a$ and $b$. □

We conclude with an interesting concurrency.

**Proposition 2.** *The lines $AB$, $CQ$, and $A''G$ are concurrent.*

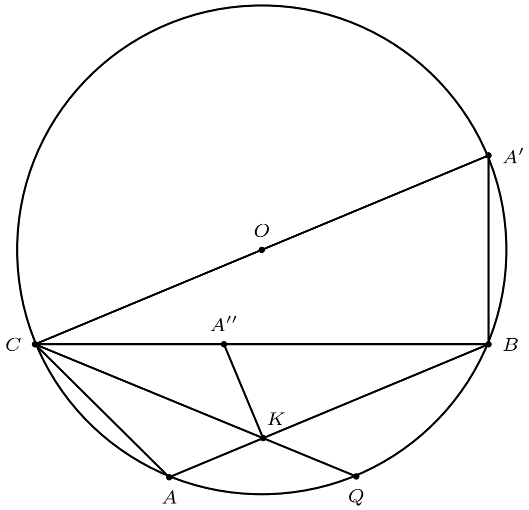


Figure 4

*Proof.* Let the bisector $CQ$ of angle $ACB$ intersect $AB$ at $K$. See Figure 4. Clearly, the triangles $ACK$ and $A''CK$ are congruent. Now,

$$\begin{aligned}\angle CA''K &= \angle CAK = \angle CAB \\ &= 180^\circ - \angle CA'B \qquad (C, A, B, A' \text{ concyclic}) \\ &= \angle ABA' \qquad\qquad (AB \text{ parallel to } CA') \\ &= \angle ABC + 90^\circ.\end{aligned}$$

It follows that $\angle A''KB = 90^\circ$, and $A''K$ is perpendicular to $AB$. This shows that $K$ lies on $A''G$. $\square$

**Reference**

[1] R. B. Nelsen, *Proofs Without Words*, MAA, 1994.

Francisco Javier García Capitán: Departamento de Matemáticas, I.E.S. Alvarez Cubero, Avda. Presidente Alcalá-Zamora, s/n, 14800 Priego de Córdoba, Córdoba, Spain

*E-mail address*: garciacapitan@gmail.com