A Condition for a Circumscribable Quadrilateral to be Cyclic

Mowaffaq Hajja

Abstract. We give a short proof of a characterization, given by M. Radić et al., of convex quadrilaterals that admit both an incircle and a circumcircle.

A convex quadrilateral is said to be cyclic if it admits a circumcircle (i.e., a circle that passes through the vertices); it is said to be circumscribable if it admits an incircle (i.e., a circle that touches the sides internally). A quadrilateral is bicentric if it is both cyclic and circumscribable. For basic properties of these quadrilaterals, see [7, Chapter 10, pp. 146–170]. One of the two main theorems in [5], namely Theorem 1 (p. 35), can be stated as follows:

Theorem. Let $ABCD$ be a circumscribable quadrilateral with diagonals $AC$ and $BD$ of lengths $u$ and $v$ respectively. Let $a$, $b$, $c$, and $d$ be the lengths of the tangents from the vertices $A$, $B$, $C$, and $D$ (see Figure 1). The quadrilateral $ABCD$ is cyclic if and only if $u v = a + c b + d$.

In this note, we give a proof that is much simpler than the one given in [5]. Our proof actually follows immediately from the three very simple lemmas below, all under the same hypothesis of the Theorem. Lemma 1 appeared as a problem in the MONTHLY [6] and Lemma 2 appeared in the solution of a quickie in the MAGAZINE [3], but we give proofs for the reader’s convenience. Lemma 3 uses Lemma 2 and gives formulas for the lengths of the diagonals of a circumscribable quadrilateral counterpart to those for cyclic quadrilaterals as given in [1], [7, §10.2, p. 148], and other standard textbooks.

Publication Date: May 1, 2008. Communicating Editor: Paul Yiu.

The author would like to thank Yarmouk University for supporting this work and Mr. Esam Darabseh for drawing the figures.
Lemma 1. ABCD is cyclic if and only if $ac = bd$.

Proof. Let $ABCD$ be any convex quadrilateral, not necessarily admitting an incircle, and let its vertex angles be $2A$, $2B$, $2C$, and $2D$. Then $A$, $B$, $C$, and $D$ are acute, and $A + B + C + D = 180^\circ$. We shall show that

$$ABCD \text{ is cyclic } \iff \tan A \tan C = \tan B \tan D. \quad (1)$$

If $ABCD$ is cyclic, then $A + C = B + D = 90^\circ$, and $\tan A \tan C = \tan B \tan D$, each being equal to 1. Conversely, if $ABCD$ is not cyclic, then one may assume that $A + C > 90^\circ$ and $B + D < 90^\circ$. From

$$0 > \tan(A + C) = \frac{\tan A + \tan C}{1 - \tan A \tan C},$$

and the fact that $A$ and $C$ are acute, we conclude that $\tan A \tan C > 1$. Similarly $\tan B \tan D < 1$, and therefore $\tan A \tan C \neq \tan B \tan D$. This proves (1).

The result follows by applying (1) to the given quadrilateral, and using $\tan A = r/a$, etc., where $r$ is the radius of the incircle (as shown in Figure 2). \hfill \Box

Lemma 2. The radius $r$ of the incircle is given by

$$r^2 = \frac{bcd + acd + abd + abc}{a + b + c + d}. \quad (2)$$

Proof. Again, let the vertex angles of $ABCD$ be $2A$, $2B$, $2C$, and $2D$, and let

$$\alpha = \tan A, \quad \beta = \tan B, \quad \gamma = \tan C, \quad \delta = \tan D.$$ 

Let $\epsilon_1 = \sum \alpha$, $\epsilon_2 = \sum \alpha \beta$, $\epsilon_3 = \sum \alpha \beta \gamma$, and $\epsilon_4 = \alpha \beta \gamma \delta$ be the elementary symmetric polynomials in $\alpha$, $\beta$, $\gamma$, and $\delta$. By [4, §125, p. 132], we have

$$\tan(A + B + C + D) = \frac{\epsilon_1 - \epsilon_3}{1 - \epsilon_2 + \epsilon_4}.$$

Since $A + B + C + D = 180^\circ$, it follows that $\tan(A + B + C + D) = 0$ and hence $\epsilon_1 = \epsilon_3$, i.e.,

$$\frac{r}{a} + \frac{r}{b} + \frac{r}{c} + \frac{r}{d} = \frac{r^3}{bcd} + \frac{r^3}{acd} + \frac{r^3}{abd} + \frac{r^3}{abc},$$

and (2) follows. \hfill \Box

Lemma 3.

$$u^2 = \frac{a + c}{b + d}((a + c)(b + d) + 4bd), \quad \text{and} \quad v^2 = \frac{b + d}{a + c}((a + c)(b + d) + 4ac).$$

Proof. Again, let the vertex angles of $ABCD$ be $2A$, $2B$, $2C$, and $2D$. Then

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{a^2 - r^2}{a^2 + r^2} = \frac{a^2(a + b + c + d) - (bcd + acd + abd + abc)}{a^2(a + b + c + d) + (bcd + acd + abd + abc)}, \quad \text{by (2)}$$

$$= \frac{a^2(a + b + c + d) - (bcd + acd + abd + abc)}{(a + b)(a + c)(a + d)}. $$
Therefore

\[ v^2 = (a + b)^2 + (a + d)^2 - 2(a + b)(a + d) \cos 2A \]
\[ = (a + b)^2 + (a + d)^2 - \frac{2(a^2 + b + c + d)^2 - (bcd + acd + abd + abc)}{a + c} \]
\[ = \frac{b + d}{c + a} ((a + c)(b + d) + 4ac). \]

A similar formula holds for \( u \).

\[ \square \]

Proof of the main theorem. Using Lemmas 1 and 3 we see that

\[ ABCD \text{ is cyclic} \iff ac = bd, \text{ by Lemma 1} \]
\[ \iff (a + c)(b + d) + 4bd = (a + c)(b + d) + 4ac \]
\[ \iff \frac{u^2}{v^2} = \left( \frac{c + a}{b + d} \right)^2, \text{ by Lemma 3} \]
\[ \iff \frac{u}{v} = \frac{c + a}{b + d}, \]

as desired. This completes the proof of the main theorem.

Remarks. (1) As mentioned earlier, Theorem 1 is one of the two main theorems in [5]. The other theorem is similar and deals with those quadrilaterals that admit an excircle. Note that the terms chordal and tangential are used in that paper to describe what we referred to as cyclic and circumscriptible quadrilaterals.

(2) Let \( A_1 \ldots A_n \) be circumscriptible \( n \)-gon and let \( B_1, \ldots, B_n \) be the points where the incircle touches the sides \( A_1A_2, \ldots, A_nA_1 \). Let \( |A_iB_i| = a_i \) for \( i = 1, \ldots, n \). Theorem 2 states that if \( n = 4 \), then the polygon is cyclic if and only if \( a_1a_3 = a_2a_4 \). One wonders whether a similar criterion holds for \( n > 4 \).

(3) It is proved in [2] that if \( a_1, \ldots, a_n \) are any positive numbers, then there exists a unique circumscriptible \( n \)-gon \( A_1 \ldots A_n \) such that the points \( B_1, \ldots, B_n \) where the incircle touches the sides \( A_1A_2, \ldots, A_nA_1 \) have the property \( |A_iB_i| = a_i \) for \( i = 1, \ldots, n \). Thus one can, in principle, express all the elements of the circumscriptible polygon in terms of the parameters \( a_1, \ldots, a_n \). Instances of this, when \( n = 4 \), are found in Lemmas 2 and 3 where the inradius \( r \) and the lengths of the diagonals are so expressed. When \( n > 4 \), one can prove that \( r^2 \) is the unique positive zero of the polynomial

\[ \sigma_{n-1} - r^2\sigma_{n-3} + r^4\sigma_{n-5} - \cdots = 0, \]

where \( \sigma_1, \ldots, \sigma_n \) are the elementary symmetric polynomials in \( a_1, \ldots, a_n \), and where \( a_1, \ldots, a_n \) are as given in Remark 2. This is obtained in the same way we obtained (2) using the the formula

\[ \tan(A_1 + \cdots + A_n) = \frac{\varepsilon_1 - \varepsilon_3 + \varepsilon_5 - \cdots}{1 - \varepsilon_2 + \varepsilon_4 - \cdots}, \]

where \( \varepsilon_1, \ldots, \varepsilon_n \) are the elementary symmetric polynomials in \( \tan A_1, \ldots, \tan A_n \), and where \( A_1, \ldots, A_n \) are half the vertex angles of the polygon.
References


Mowaffaq Hajja: Mathematics Department, Yarmouk University, Irbid, Jordan

E-mail address: mhajja@yu.edu.jo, nowhajja@yahoo.com