Forum Geometricorum
Volume 8 (2008) 103–106.

FORUM GEOM
ISSN 1534-1178

A Condition for a Circumscriptible Quadrilateral to be Cyclic

Mowaffaq Hajja

Abstract. We give a short proof of a characterization, given by M. Radić et al, of convex quadrilaterals that admit both an incircle and a circumcircle.

A convex quadrilateral is said to be *cyclic* if it admits a circumcircle (*i.e.*, a circle that passes through the vertices); it is said to be *circumscriptible* if it admits an incircle (*i.e.*, a circle that touches the sides internally). A quadrilateral is *bicentric* if it is both cyclic and circumscriptible. For basic properties of these quadrilaterals, see [7, Chapter 10, pp. 146–170]. One of the two main theorems in [5], namely Theorem 1 (p. 35), can be stated as follows:

Theorem. *Let $ABCD$ be a circumscriptible quadrilateral with diagonals AC and BD of lengths u and v respectively. Let a, b, c, and d be the lengths of the tangents from the vertices A, B, C, and D* (see Figure 1). *The quadrilateral $ABCD$ is cyclic if and only if $\frac{u}{v} = \frac{a+c}{b+d}$.*

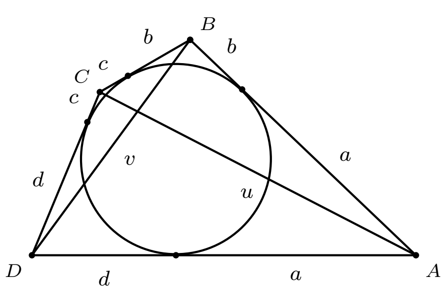

Figure 1

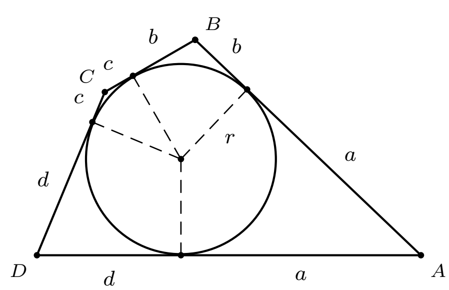

Figure 2

In this note, we give a proof that is much simpler than the one given in [5]. Our proof actually follows immediately from the three very simple lemmas below, all under the same hypothesis of the Theorem. Lemma 1 appeared as a problem in the MONTHLY [6] and Lemma 2 appeared in the solution of a quickie in the MAGAZINE [3], but we give proofs for the reader's convenience. Lemma 3 uses Lemma 2 and gives formulas for the lengths of the diagonals of a circumscriptible quadrilateral counterpart to those for cyclic quadrilaterals as given in [1], [7, § 10.2, p. 148], and other standard textbooks.

Publication Date: May 1, 2008. Communicating Editor: Paul Yiu.

The author would like to thank Yarmouk University for supporting this work and Mr. Esam Darabseh for drawing the figures.

Lemma 1. *$ABCD$ is cyclic if and only if $ac = bd$.*

Proof. Let $ABCD$ be any convex quadrilateral, not necessarily admitting an incircle, and let its vertex angles be $2A$, $2B$, $2C$, and $2D$. Then A, B, C, and D are acute, and $A + B + C + D = 180°$. We shall show that

$$ABCD \text{ is cyclic} \quad \Leftrightarrow \quad \tan A \tan C = \tan B \tan D. \tag{1}$$

If $ABCD$ is cyclic, then $A+C = B+D = 90°$, and $\tan A \tan C = \tan B \tan D$, each being equal to 1. Conversely, if $ABCD$ is not cyclic, then one may assume that $A + C > 90°$ and $B + D < 90°$. From

$$0 > \tan(A + C) = \frac{\tan A + \tan C}{1 - \tan A \tan C}$$

and the fact that A and C are acute, we conclude that $\tan A \tan C > 1$. Similarly $\tan B \tan D < 1$, and therefore $\tan A \tan C \neq \tan B \tan D$. This proves (1).

The result follows by applying (1) to the given quadrilateral, and using $\tan A = r/a$, etc., where r is the radius of the incircle (as shown in Figure 2). □

Lemma 2. *The radius r of the incircle is given by*

$$r^2 = \frac{bcd + acd + abd + abc}{a + b + c + d}. \tag{2}$$

Proof. Again, let the vertex angles of $ABCD$ be $2A$, $2B$, $2C$, and $2D$, and let

$$\alpha = \tan A,\ \beta = \tan B,\ \gamma = \tan C,\ \delta = \tan D.$$

Let $\varepsilon_1 = \sum \alpha$, $\varepsilon_2 = \sum \alpha\beta$, $\varepsilon_3 = \sum \alpha\beta\gamma$, and $\varepsilon_4 = \alpha\beta\gamma\delta$ be the elementary symmetric polynomials in α, β, γ, and δ. By [4, § 125, p. 132], we have

$$\tan(A + B + C + D) = \frac{\varepsilon_1 - \varepsilon_3}{1 - \varepsilon_2 + \varepsilon_4}.$$

Since $A+B+C+D = 180°$, it follows that $\tan(A+B+C+D) = 0$ and hence $\varepsilon_1 = \varepsilon_3$, *i.e.*,

$$\frac{r}{a} + \frac{r}{b} + \frac{r}{c} + \frac{r}{d} = \frac{r^3}{bcd} + \frac{r^3}{acd} + \frac{r^3}{abd} + \frac{r^3}{abc},$$

and (2) follows. □

Lemma 3.

$$u^2 = \frac{a+c}{b+d}\left((a+c)(b+d) + 4bd\right), \quad \textit{and} \quad v^2 = \frac{b+d}{a+c}\left((a+c)(b+d) + 4ac\right).$$

Proof. Again, let the vertex angles of $ABCD$ be $2A$, $2B$, $2C$, and $2D$. Then

$$\begin{aligned}
\cos 2A &= \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{a^2 - r^2}{a^2 + r^2} \\
&= \frac{a^2(a+b+c+d) - (bcd + acd + abd + abc)}{a^2(a+b+c+d) + (bcd + acd + abd + abc)}, \text{ by (2)} \\
&= \frac{a^2(a+b+c+d) - (bcd + acd + abd + abc)}{(a+b)(a+c)(a+d))}.
\end{aligned}$$

Therefore

$$\begin{aligned} v^2 &= (a+b)^2 + (a+d)^2 - 2(a+b)(a+d)\cos 2A \\ &= (a+b)^2 + (a+d)^2 - 2\frac{a^2(a+b+c+d) - (bcd+acd+abd+abc)}{a+c} \\ &= \frac{b+d}{c+a}\left((a+c)(b+d)+4ac\right). \end{aligned}$$

A similar formula holds for u. □

Proof of the main theorem. Using Lemmas 1 and 3 we see that

$$\begin{aligned} ABCD \text{ is cyclic} &\iff ac = bd, \text{ by Lemma 1} \\ &\iff (a+c)(b+d)+4bd = (a+c)(b+d)+4ac \\ &\iff \frac{u^2}{v^2} = \left(\frac{c+a}{b+d}\right)^2, \text{ by Lemma 3} \\ &\iff \frac{u}{v} = \frac{c+a}{b+d}, \end{aligned}$$

as desired. This completes the proof of the main theorem.

Remarks. (1) As mentioned earlier, Theorem 1 is one of the two main theorems in [5]. The other theorem is similar and deals with those quadrilaterals that admit an *excircle*. Note that the terms *chordal* and *tangential* are used in that paper to describe what we referred to as *cyclic* and *circumscriptible* quadrilaterals.

(2) Let $A_1 \dots A_n$ be circumscriptible n-gon and let $B_1, \dots, B_n$ be the points where the incircle touches the sides $A_1A_2, \dots, A_nA_1$. Let $|A_iB_i| = a_i$ for $i = 1, \dots, n$. Theorem 2 states that if $n = 4$, then the polygon is cyclic if and only if $a_1a_3 = a_2a_4$. One wonders whether a similar criterion holds for $n > 4$.

(3) It is proved in [2] that if $a_1, \dots, a_n$ are any positive numbers, then there exists a unique circumscriptible n-gon $A_1 \dots A_n$ such that the points $B_1, \dots, B_n$ where the incircle touches the sides $A_1A_2, \dots, A_nA_1$ have the property $|A_iB_i| = a_i$ for $i = 1, \dots, n$. Thus one can, in principle, express all the elements of the circumscriptible polygon in terms of the parameters $a_1, \dots, a_n$. Instances of this, when $n = 4$, are found in Lemms 2 and 3 where the inradius r and the lengths of the diagonals are so expressed. When $n > 4$, one can prove that r^2 is the unique positive zero of the polynomial

$$\sigma_{n-1} - r^2\sigma_{n-3} + r^4\sigma_{n-5} - \dots \cdots = 0,$$

where $\sigma_1, \dots, \sigma_n$ are the elementary symmetric polynomials in $a_1, \dots, a_n$, and where $a_1, \dots, a_n$ are as given in Remark 2. This is obtained in the same way we obtained (2) using the the formula

$$\tan(A_1 + \cdots + A_n) = \frac{\varepsilon_1 - \varepsilon_3 + \varepsilon_5 - \dots}{1 - \varepsilon_2 + \varepsilon_4 - \dots},$$

where $\varepsilon_1, \dots, \varepsilon_n$ are the elementary symmetric polynomials in $\tan A_1, \dots, \tan A_n$, and where $A_1, \dots, A_n$ are half the vertex angles of the polygon.

References

[1] C. Alsina and R. B. Nelson, On the diagonals of a cyclic quadrilateral, *Forum Geom.*, 7 (2007) 147–149.

[2] D. E. Gurarie and R. Holzsager, Problem 10303, *Amer. Math. Monthy*, 100 (1993) 401; solution, *ibid.*, 101 (1994) 1019–1020.

[3] J. P. Hoyt, Quickie Q 694, *Math. Mag.*, 57 (1984) 239; solution, *ibid.*, 57 (1984) 242.

[4] S. L. Loney, Plane Trigonometry, S. Chand & Company Ltd, New Delhi, 1996.

[5] M. Radić, Z. Kaliman, and V. Kadum, A condition that a tangential quadrilateral is also a chordal one, *Math. Commun.*, 12 (2007) 33–52.

[6] A. Sinefakupoulos, Problem 10804, *Amer. Math. Monthy*, 107 (2000) 462; solution, *ibid.*, 108 (2001) 378.

[7] P. Yiu, *Euclidean Geometry*, Florida Atlantic Univesity Lecture Notes, 1998, available at `http://www.math.fau.edu/Yiu/Geometry.html`.

Mowaffaq Hajja: Mathematics Department, Yarmouk University, Irbid, Jordan
E-mail address: `mhajja@yu.edu.jo, mowhajja@yahoo.com`