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Periodic Billiard Trajectories in Polyhedra

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Abstract. We consider the billiard map inside a polyhedron. We give a condition for the stability of the periodic trajectories. We apply this result to the case of the tetrahedron. We deduce the existence of an open set of tetrahedra which have a periodic orbit of length four (generalization of Fagnano's orbit for triangles), moreover we can study completely the orbit of points along this coding.

1. Introduction

We consider the billiard problem inside polyhedron. We start with a point of the boundary of the polyhedron and we move along a straight line until we reach the boundary, where there is reflection according to the mirror law. A famous example of a periodic trajectory is Fagnano's orbit: we consider an acute triangle and the foot points of the altitudes. Those points form a billiard trajectory which is periodic [1].

For the polygons some results are known. For example we know that there exists a periodic orbit in all rational polygons (the angles are rational multiples of π), and recently Schwartz has proved in [8] the existence of a periodic billiard orbit in every obtuse triangle with angle less than 100 degrees . A good survey of what is known about periodic orbits can be found in the article [4] by Gal'perin, Stëpin and Vorobets or in the book of Masur, Tabachnikov [6]. In this article they define the notion of stability: They consider the trajectories which remain periodic if we perturb the polygon. They find a combinatorial rule which characterize the stable periodic words. Moreover they find some results about periodic orbits in obtuse triangles.

The study of the periodic orbits has also been done by famous physicists. Indeed Glashow and Mittag prove that the billiard inside a triangle is equivalent to the system of three balls on a ring, [5]. Some others results can be found in the article of Ruijgrok and Rabouw [7]. In the polyhedral case much less is known. The result on the existence of periodic orbit in a rational polygon can be generalized, but it is less important, because the rational polyhedra are not dense in the set of polyhedra. There is no other general result, the only result concerns the example of the tetrahedron. Stenman [10] shows that a periodic word of length four exists in a regular tetrahedron.

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The aim of this paper is to find Fagnano's orbit in a regular tetrahedron and to obtain a rule for the stability of periodic words in polyhedra. This allows us to obtain a periodic orbit in each tetrahedron in a neighborhood of the regular one. Moreover we give examples which prove that the trajectory is not periodic in all tetrahedra, and we find bounds for the size of the neighborhood. In the last section we answer a question of Gal'perin, Krüger, Troubetzkoy [3] by an example of periodic word v with non periodic points inside its beam.

2. Statement of results

The definitions are given in the following sections as appropriate. In Section 4 we prove the following result. Consider a periodic billiard orbit coded by the word v. In §4.2, we derive a certain isometry S_v from the combinatorics of the path.

Theorem 1. *Let P be a polyhedron and v the prefix of a periodic word of period $|v|$ in P. If the period is an even number, and S_v is different from the identity, then v is stable. If the period is odd, then the word is stable if and only if S_v is constant as a function of P.*

In Section 5 we prove

Theorem 2. *Assume the billiard map inside the tetrahedron is coded by a, b, c, d.*
(1) *The word $abcd$ is periodic for all the tetrahedra in a neighborhood of the regular one.* (This orbit will be referred to as Fagnano's orbit).
(2) *In any right tetrahedron Fagnano's orbit does not exist. There exists an open set of obtuse tetrahedron where Fagnano's orbit does not exist.*

The last section of this article is devoted to the study of the first return map of the billiard trajectory.

3. Background

3.1. *Isometries.* We recall some usual facts about affine isometries of $\mathbb{R}^3$. A general reference is [1].

To an affine isometry a, we can associate an affine map f and a vector u such that: f has a fixed point or is equal to the identity, and such that $a = t_u \circ f = f \circ t_u$ where t_u is the translation of vector u. Then f can be seen as an element of the orthogonal group $O_3(\mathbb{R})$.

Definition. First assume that f belongs to $O_3(+)$, and is not equal to the identity. If u is not an eigenvector of f, then a is called an affine rotation. The axis of a is the set of invariants points. If u is an eigenvector of f, a is called a screw motion. In this case the axis of a is the axis of the affine rotation.

If f, in $O_2(-)$ or $O_3(-)$, is a reflection and u is an eigenvector of f with eigenvalue 1, then a is called a glide reflection.

We recall Rodrigue's formula which gives the axis and the angle of the rotation product of two rotations. It can be done by the following method.

Lemma 3 ([2])**.** *We assume that the two rotations are not equal to the identity, or to a rotation of angle π. Let θ and u be the angle and axis of the first rotation, and denote by t the vector $\tan\frac{\theta}{2}\cdot u$ and t' the associated vector for the second rotation. Then the product of the two rotations is given by the vector t'' such that*

$$t'' = \frac{1}{1 - t\cdot t'}(t + t' + t \wedge t').$$

3.2. *Combinatorics.* Let $\mathcal{A}$ be a finite set called the alphabet. By a language L over $\mathcal{A}$ we mean always a factorial extendable language. A language is a collection of sets $(L_n)_{n\geq 0}$ where the only element of L_0 is the empty word, and each L_n consists of words of the form $a_1a_2\dots a_n$ where $a_i \in \mathcal{A}$ such that
(i) for each $v \in L_n$ there exist $a, b \in \mathcal{A}$ with $av, vb \in L_{n+1}$, and
(ii) for all $v \in L_{n+1}$, if $v = au = u'b$ with $a, b \in \mathcal{A}$, then $u, u' \in L_n$.

If $v = a_1a_2\dots a_n$ is a word, then for all $i \leq n$, the word $a_1\dots a_i$ is called a prefix of v.

4. Polyhedral billiard

4.1. *Definition.* We consider the billiard map T inside a polyhedron P. Let $X \subset \partial P \times \mathbb{PR}^3$ consist of (m, θ) for which $m + \mathbb{R}^*\theta$ does not intersect ∂P on an edge. The map T is defined by the rule

$$T(m, \theta) = (m', \theta')$$

if and only if mm' is collinear with θ, where $\theta' = S\theta$ and S is the linear reflection over the face which contains m'.

We identify $\mathbb{PR}^3$ with the unit vectors of $\mathbb{R}^3$ in the preceding definition.

4.2. *Coding.* We code the trajectory by the letters from a finite alphabet where we associate a letter to each face.

We call s_i the reflection in the face i, S_i the linear reflection in this face. If we start with a point of direction θ which has a trajectory of coding $v = v_0\cdots v_{n-1}$ the image of θ is: $S_{v_{n-1}}...S_{v_1}\theta$. Indeed the trajectory of the point first meets the face v_1, then the face v_2 etc.

If it is a periodic orbit, it meets the face v_0 after the face v_{n-1} and we have: $S_{v_0}S_{v_{n-1}}\dots S_{v_1}\theta = \theta = S_v\theta$, S_v is the product of the S_i, and s_v the product of the s_i.

We recall a result of [3]: the word v is the prefix of a periodic word of period $|v|$ if and only if there exists a point whose orbit is periodic and has v as coding.

Remark. If a point is periodic, the initial direction is an eigenvector of the map S_v with eigenvalue 1. It implies that in $\mathbb{R}^3$, for a periodic word of odd period, S is a reflection.

Definition. Let v be a finite word. The beam associated to v is the set of (m, θ) where m is in the face v_o (respectively edge), θ a vector of $\mathbb{R}^3$ (respectively $\mathbb{R}^2$), such that the orbit of (m, θ) has a coding which begins with v. We denote it σ_v.

A vector u of $\mathbb{R}^3$ (respectively $\mathbb{R}^2$) is admissible for v, with base point m, if there exists a point m in the face (edge) v_0 such that (m,u) belongs to the beam of v.

Lemma 4. *Let s be an isometry of $\mathbb{R}^3$ not equal to a translation. Let S be the associated linear map and u the vector of translation. Assume s is either a screw motion or a glide reflection. Then the points n which satisfy $\overrightarrow{ns(n)} \in \mathbb{R}u$, are either on the axis of s (if S is a rotation), or on the plane of reflection. In this case the vector $\overrightarrow{ns(n)}$ is the vector of the glide reflection.*

Proof. We call θ the eigenspace of S related to the eigenvalue one. We have $s(n) = s(o) + S\overrightarrow{on}$ where o, the origin of the base will be chosen later. Elementary geometry yields $\overrightarrow{ns(n)} = (S - Id)X + Y$ (where $X = \overrightarrow{on}$, $Y = \overrightarrow{os(o)}$) is inside the space θ.

The map s has no fixed point by assumption, thus $\overrightarrow{ns(n)}$ is nonzero. The condition gives that $(S - Id)X + Y$ is an eigenvector of S associated to the eigenvalue one. Thus,

$$S((S - Id)X + Y) = (S - Id)X + Y,$$
$$(S - Id)^2X = -(S - I)Y. \tag{1}$$

We consider first the case $\det S > 0$. We choose o on the axis of s. Then θ is a line, we call the direction of the line by the same name. Since $\det S > 0$ we have $S \in O_3(+)$ and thus in an appropriate basis S has the form $\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$, where R is a matrix of rotation of $\mathbb{R}^2$. The equation (1) is equivalent to

$$(R - Id)^2X' = -(R - Id)Y',$$

where X' is the vector of $\mathbb{R}^2$ such that $X = \begin{pmatrix} X' \\ x \end{pmatrix}$ in this basis. Furthermore, since S is a screw motion with axis $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in these coordinates, Y has the following coordinates $\begin{pmatrix} Y' \\ y \end{pmatrix}$ where $Y' = 0$. Since $S \neq Id$, $R - Id$ is invertible and thus $X' = 0$. Thus the vectors X solutions of this equation are collinear with the axis.

Consider now the case $detS < 0$. By assumption S is a reflection, it implies that the eigenspace related to one is a plane. We will solve (1), keeping the notation $X = \begin{pmatrix} X' \\ x \end{pmatrix}$ and $Y = \begin{pmatrix} Y' \\ y \end{pmatrix}$.

We may assume that o is on the plane of reflection. Moreover we can choose the coordinates such that that this plane is orthogonal to the line $\mathbb{R}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. It implies that $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, and $y = 0$. The equation (1) becomes $4x = 0$. It implies that X is on the plane of reflection. Since s is a glide reflection, the last point becomes obvious. □

Proposition 5. *Let P a polyhedron, the following properties are equivalent.*
(1) *A word v is the prefix of a periodic word with period $|v|$.*
(2) *There exists $m \in v_0$ such that $\overrightarrow{s_v(m)m}$ is admissible with base point m for vv_0, and $\theta = \overrightarrow{s_v(m)m}$ is such that $S\theta = \theta$.*

Remark. Assume $|v|$ is even. In the polygonal case the matrix S_v can only be the identity, thus s_v is a translation. We see by unfolding that s_v can not have a fixed point, thus in the polyhedral case s_v is either a translation or a screw motion or a glide reflection. If we do not assume the admissibility in condition (2) it is not equivalent to condition (1) as can be seen in a obtuse triangle, or a right prism above the obtuse triangle and the word abc.

Proof of Proposition 5. First we claim the following fact. The vector connecting $T^{|v|}(m, \theta)$ to $s_v(m)$ is parallel to the direction of $T^{|v|}(m, \theta)$. For $|v| = 1$ if the billiard trajectory goes from (m, θ) to (m', θ') without reflection between, then the direction θ' is parallel to $\overrightarrow{s(m)m'}$, where s is the reflection over the face of m' (see Figure 1). Thus the claim follows combining this observation with an induction argument.

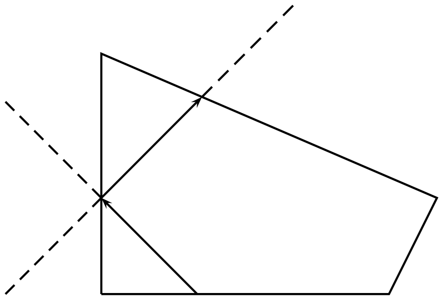

Figure 1. Billiard orbit and the associated map

Next assume (1). Then there exists (m, θ) periodic. We deduce that $S\theta = \theta$, moreover this direction is admissible. Then the claim implies that $\overrightarrow{s_v(m)m} = \theta$ and thus is admissible for vv_0.
Finally assume (2). First we consider the case where $S \neq Id$. Lemma 4 implies that m is on the axis of s if $|v|$ is even, otherwise on the plane of reflection. If $|v|$ is even then $\theta = \overrightarrow{s(m)m}$ is collinear to the axis of the screw motion. Since we have assumed $\overrightarrow{s_v(m)m}$ admissible we deduce that θ is admissible with base point m. If $|v|$ is odd then Lemma 4 implies that θ is the direction of the glide. The hypothesis implies that θ is admissible for v.

Now we prove that (m, θ) is a periodic trajectory. We consider the image $T^{|v|}(m, \theta)$. We denote this point (p, θ'). We have by hypothesis that p is in v_0. The above claim implies that $\overrightarrow{s_v(m)p}$ is parallel to the direction θ'. The equation $S\theta = \theta$ gives $\theta' = \theta$. Thus we have $\overrightarrow{s_v(m)m}$ is parallel to $\overrightarrow{s_v(m)p}$, since we do not consider direction included in a face of a polyhedron this implies $p = m$. Thus (m, θ) is a periodic point.

If $S = Id$, then s is a translation of vector $\overrightarrow{s_v(m)m} = u$. The vector u is admissible. Then we consider a point m on the face v_0 which is admissible. Then we show that (m, u) is a periodic point by the same argument related to the claim. □

Thus we have a new proof of the following result of [3].

Theorem 6. *Let v be a periodic word of even length. The set of periodic points in the face v_0 with code v and length $|v|$ can have two shapes. Either it is an open set or it is a point.*

If v is a periodic word of odd length, then the set of periodic points in the face v_0 with code v and period $|v|$ is a segment.

Proof. Let Π be a face of the polyhedron, and let $m \in \Pi$ be the starting point for a periodic billiard path. The first return map to m is an isometry of $\mathbb{R}^3$ that fixes both m and the direction u of the periodic billiard path.

Assume first $|v|$ is odd. Then the first return map is a reflection since it fixes a point. Then it fixes a plane Π'. Note that $u \in \Pi'$, and that the intersection $\Pi \cap \Pi'$ is a segment. Points in this segment sufficiently near m have a periodic orbit just as the one starting at v.

Assume now $|v|$ is even, we will use Proposition 5. If S_v is the identity, then the periodic points are the points such that the coding of the billiard orbit in the direction of the translation begins with v, otherwise there is a single point, at the intersection of the axis of s and v_0. However the set of points with code v is still an open set. □

Note that our proof gives an algorithm to locate this set in the face. We will use it in Section 6.

5. Stability

First of all we define the topology on the set of polyhedra with k vertices. As in the polygonal case we identify this set with $\mathbb{R}^{3(k-2)}$. But we remark the following fact. Consider a polyhedron P such that a face of P is not a triangle. Then we can find a perturbation of P, as small as we want, such that the new polyhedron has a different combinatorial type (*i.e.*, the numbers of vertices, edges and faces are different). In this case consider a triangulation of each face which does not add new vertices. Consider the set of all such triangulations of all faces. There are finitely many such triangulations. Each can be considered as a combinatorial type of the given polyhedron. Let $B(P, \varepsilon)$ be the ball of radius ε in $\mathbb{R}^{3(k-2)}$ of polyhedra Q. If P has a single combinatorial type, ε is chosen so small that all Q in the ball have the same combinatorial type. If P has several combinatorial types, then ε is taken so small that all Q have one of those combinatorial type. The definition of stability is now analogous to the definition in polygons. On the other hand, let v be a periodic word in P and g a piecewise similarity. Consider the polyhedron $g(P)$, and the same coding as in P. If v exists in $g(P)$ it is always a periodic word in $g(P)$. We note that the notion of periodicity only depends on the normal vectors to the planes of the faces.

Theorem 7. *Let P be a polyhedron and v the prefix of a periodic word of period $|v|$ in P.*
(1) *If the period is even, and S_v is different from the identity, then v is stable.*
(2) *If the period is odd, then the word is stable if and only if S_v is constant as a function of P.*

Remark. The second point has no equivalence in dimension two, since each element of $O(2,-)$ is a reflection. It is not the case for $O(3,-)$.

Proof of Theorem 7. First consider the case of period even. The matrix $S = S_v$ is not the identity, and $\theta = \theta_v$ is the eigenvector associated to the eigenvalue one. First note that by continuity v persists for sufficiently small perturbations of the polyhedron. Fix a perturbation and let $B = S_v^Q$ be the resulting rotation for the new polyhedron Q. We will prove that the eigenvalue of S is a continuous function of P. We take the reflections which appears in v two by two. The product of two of those reflections is a rotation. We only consider the rotations different of the identity. The axes of the rotations are continuous map as function of P since they are at the intersection of two faces. Then Rodrigue's formula implies that the axes of the rotation, product of two of those rotations, are continuous maps of the polyhedron, under the assumption that the rotation is not the identity (because t must be of non-zero norm). Since S^P is not equal to Id, there exists a neighborhood of P where $S^Q \neq Id$. It implies that the axis of S^P is a continuous function of P. Thus the two eigenvectors of B, S are near if B is sufficiently close to S. The direction θ was admissible for v, we know that the beam of v is an open set of the phase space [3], so we have for Q sufficiently close to P that α (the real eigenvector of B) is admissible for the same word. Moreover the foot points are not far from the initial points because they are on the axis of the isometries. Thus the perturbated word is periodic by Proposition 5.

If the length of v is odd, then Remark 4.2 implies that S is a reflection. We have two cases: either S_v is constant, or not. If it is not a constant function, then in any neighborhood there exists a polyhedron Q such that S_v^Q is different from a reflection. Then the periodic trajectory can not exist in Q. If S_v is constant, then it is always a reflection, and a similar argument to the even case shows that the plane of reflection of S is a continuous map of P. It completes the proof of Theorem 7. $\square$

Corollary 8. (1) *All the words of odd length are stable in a polygon.*
(2) *Consider a periodic billiard path in a right prism. Then its projection inside the polygonal basis is a billiard path. We denote the coding of the projected trajectory*

as the projected word. Assume that the projected word is not stable in the polygonal basis. Then the word is unstable.
(3) *All the words in the cube are unstable.*

Proof. (1) was already mentioned in [4]. The proof is the same as that of Theorem 7. Indeed is $|v|$ is odd then s has a real eigenvector, and we can apply the proof.

For (2) we begin with the period two trajectory which hits the top and the bottom of the prism. It is clearly unstable, for example we can change one face and keep the other. Let v be any other periodic word, and w the word corresponding to the projection of v to the base of the prism assumed to be unstable. We perturb a vertical face of the prism such that this face contains an edge which appears in the coding of w. The word v can not be periodic in this polyhedron by unstability of w.

For (3), let v be a periodic word, by preceding point its projection on each coordinate plane must be stable. But an easy computation shows that no word is stable in the square. □

We remark that the two and three dimensional cases are different for the periodic trajectories of odd length. They are all stable in one case, and all unstable in the second. Recently Vorobets has shown that if $S_v = Id$ then the word is not stable [11].

6. Tetrahedron

In the following two Sections we prove the following result.

Theorem 9. *Assume the billiard map inside the tetrahedron is coded by a, b, c, d.*
(1) *The word $abcd$ is periodic for all the tetrahedra in a neighborhood of the regular one.*
(2) *In any right tetrahedron Fagnano's orbit does not exist. There exists an open set of obtuse tetrahedron where Fagnano's orbit does not exist.*

Remark. Steinhaus in his book [9], cites Conway for a proof that $abcd$ is periodic in all tetrahedra, but our theorem gives a counter example. Moreover our proof gives an **algorithm** which find the coordinates of the periodic point, when it exists.

For the definition of obtuse tetrahedron, see Section 7.

We consider a regular tetrahedron. We can construct a periodic trajectory of length four, which is the generalization of Fagnano's orbit. To do this we introduce the appropriate coding (see Figure 2 in which the letter a is opposite to the vertex A, etc).

Lemma 10. *Let $ABCD$ be a regular tetrahedron, with the natural coding. If v is the word $adcb$, there exists a direction θ, there exists an unique point m such that (m, θ) is periodic and has v as prefix of its coding. Moreover m is on the altitude of the triangle BCD which starts at C.*

Remark. If we consider the word v^n, the preceding point m is the unique periodic point for v^n. Indeed the map s_{v^n} has the same axis as s_v, and we use Proposition 5.

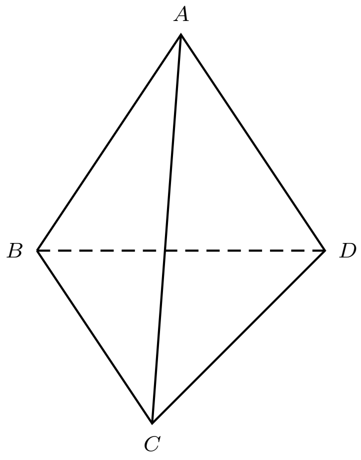

Figure 2. Coding of a tetrahedron

We use the following coordinates for two reasons. First these coordinates were used by Ruijgrok and Rabouw [7]. Secondly with these coordinates the matrix S_v has rational entries, and the computations seems more simples.

Proof. The lemma has already been proved in [10], but we rewrite it in a different form with the help of Proposition 5.

We have $S_v = S_a.S_b.S_c.S_d = R_{DC}.R_{AB}$ where R_{DC} is the linear rotation of axis DC, it is a product of the two reflections. We compute the real eigenvector of S_v, and we obtain the point m at the intersection of the axis of s and the face BCD. We consider an orthonormal base of $\mathbb{R}^3$ such that the points have the following coordinates (see [10]):

$$A = \frac{\sqrt{2}}{4}\begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \quad D = \frac{\sqrt{2}}{4}\begin{pmatrix}1\\-1\\1\end{pmatrix}, \quad C = \frac{\sqrt{2}}{4}\begin{pmatrix}1\\1\\-1\end{pmatrix}, \quad B = \frac{\sqrt{2}}{4}\begin{pmatrix}-1\\1\\1\end{pmatrix}.$$

The matrices of S_a, S_d, S_c, S_b are:

$$\frac{1}{3}\begin{pmatrix}1 & -2 & -2\\-2 & 1 & -2\\-2 & -2 & 1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1 & 2 & 2\\2 & 1 & -2\\2 & -2 & 1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1 & -2 & 2\\-2 & 1 & 2\\2 & 2 & 1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1 & 2 & -2\\2 & 1 & 2\\-2 & 2 & 1\end{pmatrix}.$$

From these we obtain

$$S = S_aS_bS_cS_d = \frac{1}{81}\begin{pmatrix}-79 & -8 & 16\\8 & 49 & 64\\-16 & 64 & -47\end{pmatrix}.$$

This has a real eigenvector $u = \frac{1}{\sqrt{5}}\begin{pmatrix}0\\2\\1\end{pmatrix}$. Now we compute the vector N such that $s(X) = SX + N$. To do this we use the relation $s(A) = s_a(A)$. s_a is the product of S_a and a translation of vector v. We obtain

$$v = \frac{\sqrt{2}}{6}\begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad s(A) = \frac{5\sqrt{2}}{12}\begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad N = \frac{\sqrt{2}}{81}\begin{pmatrix}16\\64\\34\end{pmatrix}.$$

We see that s is a screw motion. Finally we find the point at the intersection of the axis and the face a. The points of the axis verify the equation

$$SX + N = X + \lambda u.$$

where X are the coordinates of the point of the axis, and λ is a real number. The point m is on the face a if we have the dot product

$$\overrightarrow{Cm} \cdot (\overrightarrow{CB} \wedge \overrightarrow{CD}) = 0.$$

So X is the root of the system made by those two equations. The last equation gives $x + y + z = \frac{\sqrt{2}}{4}$. We obtain

$$m = \frac{\sqrt{2}}{20}\begin{pmatrix}2\\2\\1\end{pmatrix}.$$

We remark that $\overrightarrow{Cm} \cdot \overrightarrow{DB} = 0$ which proves that m is on the altitude of the triangle BCD. □

In fact there are six periodic trajectories of length four, one for each of the word

$$abcd,\ abdc,\ acbd,\ acdb,\ adbc,\ adcb.$$

The six orbits come in pairs which are related by the natural involution of direction reversal. Now we can ask the same question in a non regular tetrahedron. Applying Theorem 7 yield the first part of Theorem 9.

Now the natural question is to characterize the tetrahedron which contains this periodic word.

7. Stability for the tetrahedron

A tetrahedron is acute if and only if in each face the orthogonal projection of the other vertex is inside the triangle. It is a right tetrahedron if and only if there exists a vertex, where the three triangles are right triangles.

Proof of second part of Theorem 9. We consider a tetrahedron $ABCD$ with vertices

$$A = (0,0,0) \quad B = (a,0,0) \quad C = (0,b,0) \quad D = (0,0,1).$$

We study the word $v = abcd$. We have $S = S_a * S_d * S_c * S_b$. Since

$$S_b = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \qquad S_c = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}, \qquad S_d = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix},$$

we obtain $S = -S_a$. Thus S has 1 for eigenvalue, and the associated eigenvector is the normal vector to the plane a. We remark that $s(A) = s_a(A)$. The fact that

$S = -S_a$ implies that S is a rotation of angle π, thus s is the product of a rotation of angle π and a translation.

Consider the plane which contains A and orthogonal to the axis of S, let O the point of intersection. Then S is a rotation of angle π, thus O is the middle of $[AE]$, where E is given by $S(\overrightarrow{OE}) = \overrightarrow{OA}$. It implies that the middle M of the edge $[As(A)]$ is on the axis of s, see Figure 3.

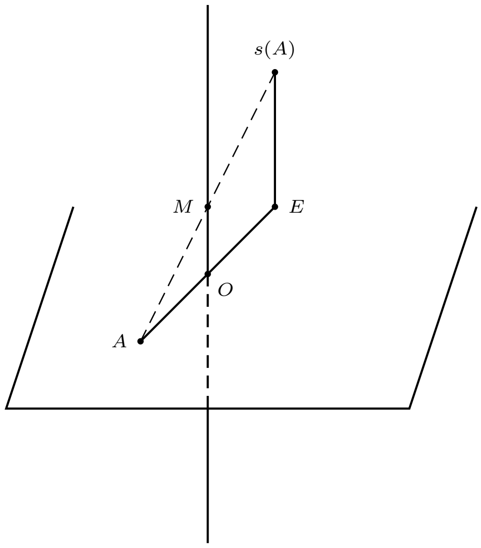

Figure 3. Screw motion associated to the word abcd

Clearly m is a point in the side ABC. If v is periodic then applying Proposition 5 yields that M is the base point of the periodic trajectory. Moreover, since the direction of the periodic trajectory is the normal vector to the plane a, we deduce that A is on the trajectory. So the periodic trajectory cannot exist.

Now we prove the second part of the theorem. We give an example of obtuse tetrahedron where Fagnano's orbit does not exist.

In this example the point on the initial face, which must be periodic see Proposition 5, is not in the interior of the triangle.

We consider the tetrahedron $ABCD$ with vertices

$$A(0,0,0), \quad B(2,0,0), \quad C(1,1,0), \quad D(3,2,1).$$

We study the word $v = abcd$. We obtain the matrix of S_v

$$\begin{pmatrix} \frac{1}{33} & \frac{8}{33} & \frac{32}{33} \\ \frac{104}{165} & \frac{-25}{33} & \frac{28}{165} \\ \frac{128}{165} & \frac{20}{33} & \frac{-29}{165} \end{pmatrix}.$$

Now s is the map $SX+N$ where $N = \begin{pmatrix} \frac{4}{11} \\ \frac{4}{11} \\ -\frac{12}{11} \end{pmatrix}$. S has eigenvector $u = \begin{pmatrix} \frac{9}{8} \\ \frac{1}{2} \\ 1 \end{pmatrix}$.

Now s is a screw motion and we find the point at the intersection of the axis of s and the face a by solving the system

$$Sm + N = m + \lambda u \tag{2}$$

$$\overrightarrow{Bm} \cdot n = 0. \tag{3}$$

This is equivalent to the system

$$\begin{pmatrix} S - Id & -u \\ n^t & 0 \end{pmatrix} \begin{pmatrix} m \\ \lambda \end{pmatrix} = \begin{pmatrix} -N \\ 2 \end{pmatrix}.$$

where $n = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$ is the normal vector to the face BCD.

We obtain the matrix

$$\begin{pmatrix} -\frac{32}{33} & \frac{8}{33} & \frac{32}{33} & -\frac{9}{8} \\ \frac{104}{165} & -\frac{58}{33} & \frac{28}{165} & -\frac{1}{2} \\ \frac{128}{165} & \frac{20}{33} & -\frac{194}{165} & -1 \\ 1 & 1 & -3 & 0 \end{pmatrix}$$

and

$$m = \begin{pmatrix} \frac{22}{161} \\ \frac{6}{23} \\ -\frac{86}{161} \end{pmatrix}.$$

But this point is not inside BCD. Moreover we see that this point is not on the altitude at BD which passes through C.

The tetrahedron is obtuse, due to the triangle ABD. The triangle BCD is acute, and the axis of s does not cut this face in the interior of the triangle.

Moreover we obtain that there exists a neighborhood of this tetrahedron, where Fagnano's word is not periodic. Indeed in a neighborhood the point m can not be in the interior of $ABCD$.

Remark. We can remark that our proof gives a criterion for the existence of a periodic billiard path of this type. One computes the axis of the screw motion, and finds if it intersects the relevant faces.

For a generic tetrahedron we can use it to know if there exists a Fagnano's orbit. But we have not find a good system of coordinates where the computations are easy. Thus we are not able to caracterize the tetrahedra with a Fagnano's orbit.

8. First return map

In this section we use the preceding example to study a related problem for periodic billiard paths. We answer to a question of Gal'perin, Krüger and Troubetzkoy [3] by an example of periodic word v with non periodic points inside its beam.

We consider the word $v = (abcd)^\infty$ and the set σ_v. The projection of this set on the face a is an open set. Each point in this open set return to the face a after three reflections. We study this return map and the set $\pi_a(\sigma_v)$. We consider the same basis as in Section 6. Moreover, in the face a we consider the following basis

$$\begin{pmatrix} \frac{\sqrt{2}}{4} \\ 0 \\ 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Theorem 11. *In the regular tetrahedron, consider the word $v = (abcd)^\infty$. Then the set $\pi_a(\sigma_v)$ is an open set. There exists only one point in this set with a periodic billiard orbit.*

The theorem of [3] explains that some such cases could appear, but there were no example before this result.

Theorem 11 means that for all point in $\pi_a(\sigma_v)$, except one, the billiard orbit is coded by a periodic word, but it is never a periodic trajectory. For the proof we make use of the following lemma.

Lemma 12. *In the regular tetrahedron, consider the word $v = (abcd)^\infty$. The first return map r on $\pi_a(\sigma_v)$ is given by*

$$r\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + B,$$

where

$$A = \frac{1}{81}\begin{pmatrix} -83 & 28 \\ -12 & -75 \end{pmatrix}, \qquad B = \frac{1}{81}\begin{pmatrix} -15 \\ 9 \end{pmatrix}.$$

The set $\pi_a(\sigma_v)$ is the interior of the biggest ellipse of center m related to the matrix A.

Proof. If m is a point of the face a, the calculation in Section 6 shows that

$$r(m) = \frac{1}{81}\begin{pmatrix} -79x - 8y + 16\sqrt{2} \\ 66x - 21y + 42z + \frac{3\sqrt{2}}{2} \\ 13x + 29y - 58z\frac{33\sqrt{2}}{12} \end{pmatrix}$$

Now we compute m and rm in the basis of the face a. We obtain the matrices A, B. $\square$

Proof of Theorem 11. We can verify that the periodic point $\frac{\sqrt{2}}{20}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ is fixed by r. Indeed in this basis, it becomes $\frac{\sqrt{2}}{20}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Now the orbit of a point under r

is contained on an ellipse related to the matrix A. This shows that the set $\pi_a(\sigma_v)$ is the biggest ellipse included in the triangle. And an obvious computation shows that only one point is fixed by r.

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