Forum Geometricorum
Volume 8 (2008) 131–140.

FORUM GEOM
ISSN 1534-1178

Another Variation on the Steiner-Lehmus Theme

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Abstract. Let the internal angle bisectors BB' and CC' of angles B and C of triangle ABC be extended to meet the circumcircle at B^* and C^*. The Steiner-Lehmus theorem states that if $BB' = CC'$, then $AB = AC$. In this article, we investigate those triangles for which $BB^* = CC^*$ and we address several issues that arise within this investigation.

1. Introduction

The celebrated Steiner-Lehmus theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles. In terms of triangle centers and cevians, it states that if two cevians through the *incenter* of a triangle are equal, then the triangle is isosceles. Variations on the theme can be obtained by replacing the incenter by any of the hundreds of centers known in the literature; see [6] and the website [7]. Other variations on this theme are obtained by letting the cevians of ABC through a center P meet the circumcircle of ABC at A^*, B^*, and C^* and asking whether the equality $BB^* = CC^*$ implies that $AB = AC$, where XY denotes the length of the line segment XY. This variation, together with several others, is investigated in [5] where it is proved that if P is the incenter, the orthocenter, or the Fermat-Torricelli point, then $BB^* = CC^*$ if and only if $AB = AC$ or $A = \frac{\pi}{3}$. When P is the centroid, the triangles for which $BB^* = CC^*$ are proved, in Theorem 9 below, to be the ones whose side lengths satisfy the relation $a^4 = b^4 - b^2c^2 + c^4$, a relation that has no geometric interpretation and cannot be fitted into a traditional geometry context such as Euclid's *Elements*.

Using geometric arguments, we show that if the centroid P of a scalene triangle ABC is such that $BB^* = CC^*$, then $\angle BAC$ must lie in the interval $[\frac{\pi}{3}, \frac{\pi}{2}]$ and that to every θ in $[\frac{\pi}{3}, \frac{\pi}{2}]$ there is essentially a unique scalene triangle with $\angle BAC = \theta$ and with $BB^* = CC^*$. The proof uses a generalization of Proposition 7 of Book III of Euclid's *Elements*, in brief Euclid III.7 [1], and deserves recording on its own.

Publication Date: June 16, 2008. Communicating Editor: Paul Yiu.

The first and second named authors are supported by a research grant from Yarmouk University and would like to express their thanks for this support. The authors would also like to thank the referee for his valuable remarks and for providing the construction given at the very end of this note, and to Mr. Essam Darabseh for drawing the figures.

[1] Throughout, the symbol Euclid $*.**$ designates Proposition $**$ of Book $*$ in Euclid's *Elements*.

2. Euclid III.7 and a generalization

Euclid III.7, not that well known, states that if Ω is a circle centered at O, if $M \neq O$ is a point inside Ω, and if the intersection of a ray MX with Ω is denoted by X', then
(i) the maximum value of MX' is attained when the ray MX passes through O and the minimum is attained when the ray MX is the opposite ray $-MO$,
(ii) as the ray MX rotates from the position MO to the opposite position $-MO$, the quantity MX' changes monotonically. We restate this proposition in Theorem 1 as a preparation for the generalization that is made in Theorem 5.

Theorem 1 (Euclid III.7)**.** *Let BC be a chord in a circle Ω, let M be the mid-point of BC, and let the line perpendicular to BC through M meet Ω at E and F. As a point P moves from E to F along the arc ECF of Ω, the length MP changes monotonically. It increases or decreases according as E is closer or farther than F from M.*

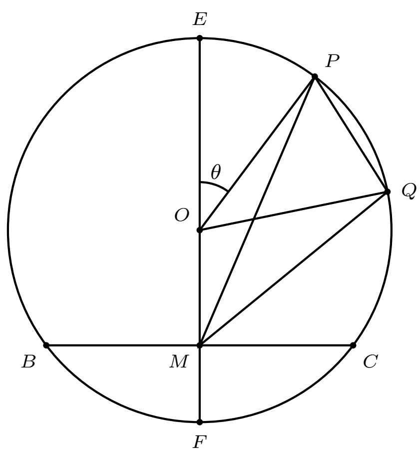

Figure 1.

Proof. Referring to Figure 1, we shall show that if $EM > MF$, *i.e.*, if the center O of Ω is between E and M, and if P and Q are any points on the arc ECF such that P is closer to E than Q, then $MP > MQ$. Under these assumptions,

$$\angle MQP > \angle OQP = \angle OPQ > \angle MPQ.$$

Thus $\angle MQP > \angle MPQ$, and therefore $MP > MQ$, as desired. □

Remark. The proof above uses the fairly simple-minded fact that in a triangle, the greater angle is subtended by the greater side. This is Euclid I.19. It is interesting that Euclid's proof uses the more sophisticated Euclid I.24. This theorem, referred to in [8, Theorem 6.3.9, page 140] as the *Open Mouth Theorem*, states that if triangles ABC and $A'B'C'$ are such that $AB = A'B'$, $AC = A'C'$, $\angle BAC > \angle B'A'C'$, then $BC > B'C'$. Quoting [8], this says that *the wider you open your mouth, the farther apart your lips are.* Although this follows immediately from the law of cosines, the intricate proofs given by Euclid and in [8] have the advantage of showing that the theorem is a theorem in neutral geometry.

Theorem 5 below generalizes Theorem 1. In fact Theorem 1 follows from Theorem 5 by taking BC to be a diameter of one of the circles Ω and Ω'. For the proof of Theorem 5, we need the following simple lemmas.

Lemma 2. *Let ABC be a triangle and let D and E be points on the sides AB and AC respectively* (see Figure 2). *Then $\frac{AD}{AB}$ is greater than, less than, or equal to $\frac{AE}{AC}$ according as $\angle ABC$ is greater than, less than, or equal to $\angle ADE$, respectively.*

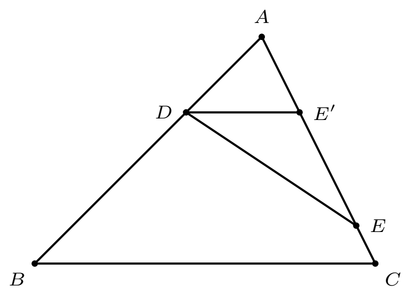

Figure 2

Proof. Let E' be the point on AC such that $\frac{AE'}{AC} = \frac{AD}{AB}$; *i.e.*, DE' is parallel to BC. If $\frac{AE}{AC} = \frac{AD}{AB}$, then $E' = E$ and $\angle ABC = \angle ADE$. If $\frac{AE}{AC} > \frac{AD}{AB}$, then E lies between E' and C, and $\angle ABC = \angle ADE' < \angle ADE$. Similarly for the case $\frac{AE}{AC} < \frac{AD}{AB}$. □

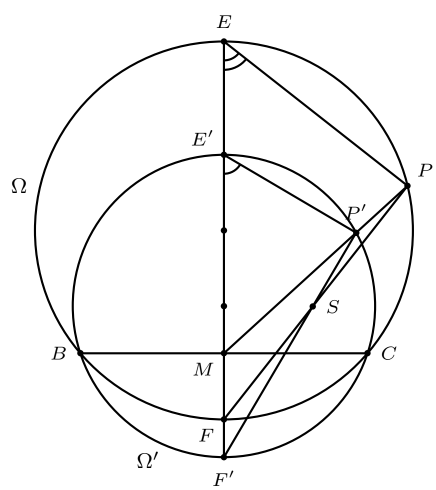

Figure 3

Lemma 3. *Two circles Ω and Ω' intersect at B and C, and the line perpendicular to BC through the midpoint M of BC meets Ω and Ω' at E and E', respectively, such that E' is inside Ω* (see Figure 3). *If P is any point on the arc ECF of Ω and if the ray MP meets Ω' at P', then $\frac{MP'}{MP} > \frac{ME'}{ME}$.*

Proof. Let S be the point of intersection of FP and $F'P'$. Since $\angle EPF = \frac{\pi}{2} = \angle E'P'F'$, it follows that $\angle ME'P' + \angle MF'P' = \frac{\pi}{2} = \angle MEP + \angle MFP$. But $\angle MFP > \angle MF'P'$, by the exterior angle theorem. Hence $\angle ME'P' > \angle MEP$. By Lemma 2, we have $\frac{MP'}{MP} > \frac{ME'}{ME}$, as desired. □

Lemma 4. *Let EBC be an isosceles triangle having $EB = EC$. Let M be the midpoint of BC and let E' be the circumcenter of EBC* (see Figure 4). *Then $\frac{ME'}{ME}$ is greater than, equal to, or less than $\frac{1}{3}$ according as $\angle BEC$ is less than, equal to, or greater than $\frac{\pi}{3}$, respectively.*

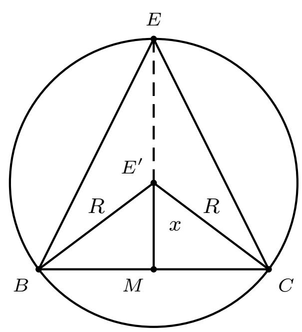

Figure 4.

Proof. Let $\theta = \angle BEC$, $x = ME'$, and let R be the circumradius of EBC. Then $\angle ME'C = \theta$ and

$$\frac{ME'}{ME} - \frac{1}{3} = \frac{x}{x+R} - \frac{1}{3} = \frac{R\cos\theta}{R\cos\theta + R} - \frac{1}{3} = \frac{2\cos\theta - 1}{3(\cos\theta + 1)}.$$

$\cos\theta$ is greater than, equal to, or less than $\frac{1}{2}$. □

Theorem 5. *Two circles Ω and Ω' intersect at B and C and the line perpendicular to BC through the midpoint M of BC meets Ω at E and F and meets Ω' at E' and F'. For every point P on Ω, let P' be the point where the ray MP meets Ω'. As a point P moves from E to F along the arc ECF, the ratio $\frac{MP'}{MP}$ changes monotonically. It decreases or increases according as E' is inside or outside Ω.*

Proof. Referring to Figure 5, suppose that E' lies inside Ω and let P and Q be two points on the arc ECF of Ω such that P is closer to E than Q. we are to show that $\frac{MP'}{MP} < \frac{MQ'}{MQ}$.

Extend QM to meet Ω at U and Ω' at U'. Let T be the point of intersection of EU and $E'U'$. Since the quadrilaterals $EPQU$ and $E'P'Q'U'$ are cyclic, it follows that

$$\angle UQP + \angle UEP = \pi = \angle U'Q'P' + \angle U'E'P'. \qquad (1)$$

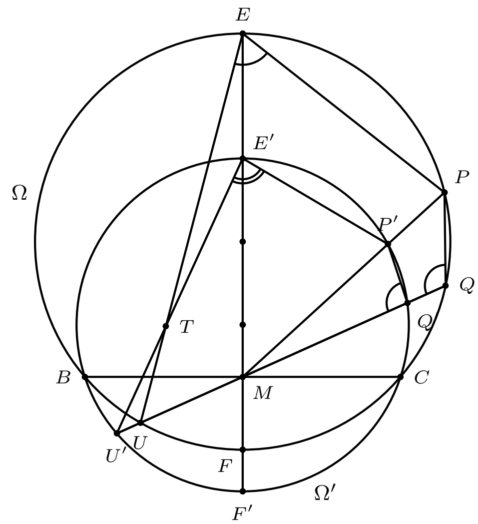

Figure 5

But

$$
\begin{aligned}
\angle U'E'P' &= \angle U'E'M + \angle ME'P' \\
&> \angle UEM + \angle ME'P' \text{ (by the exterior angle theorem)} \\
&> \angle UEM + \angle MEP \text{ (by Lemmas 3 and 2)} \\
&= \angle UEP.
\end{aligned}
$$

From this and (1) it follows that $\angle U'Q'P' > \angle UQP$. By Lemma 2, we conclude that $\frac{MP'}{MP} < \frac{MQ'}{MQ}$, as desired.

Note that if P is on the arc EC and Q is on the arc CF, then $\frac{MP'}{MP} < 1 < \frac{MQ'}{MQ}$. □

3. Conditions of equality of two chords through a given point

The next simple lemma exhibits the relation between two geometric properties of a point P inside a triangle ABC. It will be used in the proof of Theorem 9.

Lemma 6. *Let P be a point inside triangle ABC and let the rays BP and CP meet the circumcircle of ABC at B^* and C^* respectively* (see Figure 6). *Then*
(a) $BB^* = CC^*$ *if and only if* $PB = PC$ *or* $\angle BPC = 2\angle BAC$*;*
(b) $\angle BPC = 2\angle BAC \iff PB^* = PC \iff B^*C \parallel C^*B$.
Moreover, if P is the centroid, then
(c) $PB = PC \iff AB = AC \iff B^*C^* \parallel BC$.

Proof. (a) It is clear that

$$\begin{aligned} BB^* = CC^* &\iff \angle BAB^* = \angle CAC^* \text{ or } \angle BAB^* + \angle CAC^* = \pi \\ &\iff \angle CAB^* = \angle BAC^* \text{ or } \angle CAB^* + \angle BAC^* + 2\angle BAC = \pi \\ &\iff \angle CBB^* = \angle BCC^* \text{ or } \angle CBB^* + \angle BCC^* + 2\angle BAC = \pi \\ &\iff PB = PC \text{ or } \angle BPC = 2\angle BAC. \end{aligned}$$

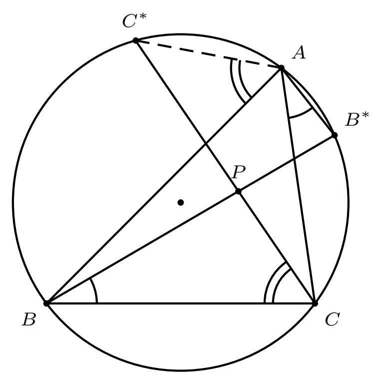

Figure 6.

(b) Also,

$$\begin{aligned} \angle BPC = 2\angle BAC &\iff \angle PB^*C + \angle PCB^* = 2\angle PB^*C \\ &\iff \angle PCB^* = \angle PB^*C \\ &\iff \angle PB^* = \angle PC. \end{aligned}$$

This proves the first part of (b). The implication $PB^* = PC \iff B^*C \parallel C^*B$ is easy.

(c) Let the lengths of the medians from B and C be β and γ, respectively. By Apollonius theorem, we have

$$\frac{b^2}{2} + 2\beta^2 = a^2 + c^2\,,\ \frac{c^2}{2} + 2\gamma^2 = a^2 + b^2.$$

The rest follows from the facts that $PB = \frac{2\beta}{3}$ and $PC = \frac{2\gamma}{3}$. □

4. Chords of circumcircle through the centroid

In Theorem 7, we focus on triangles ABC whose centroid G has the property that $\angle BGC = 2\angle BAC$. Interest in this property stems from Lemma 6. Note that Part (i) provides a solution of the problem in [4].

Theorem 7. (i) *If ABC is a triangle whose centroid G has the property that $\angle BGC = 2\,\angle BAC$, then $\frac{\pi}{3} \le \angle BAC < \frac{\pi}{2}$ with $\angle BAC = \frac{\pi}{3}$ if and only if ABC is equilateral.*

(ii) *If θ is any angle in the interval $(\frac{\pi}{3}, \frac{\pi}{2})$ and if BC is any line segment, then there is a triangle ABC, unique up to reflection about BC and about the perpendicular bisector of BC, having $\angle BAC = \theta$ and whose centroid G has the property $\angle BGC = 2\,\angle BAC$.*

Proof. (i) Let Ω be the circumcircle of ABC and let E' be its circumcenter. Let Ω' be the circumcircle of $E'BC$. Let M be the midpoint of BC and let the perpendicular bisector of BC meet Ω at E and F and meet Ω' at (E' and) F', where E is on the arc BAC of Ω (see Figure 7). Let $\angle BAC = \theta$, and let G be the centroid of ABC. Also, for every P on Ω, let P' be the point where the ray MP meets Ω'.

Suppose that $\angle BGC = 2\angle BAC$. Since $\angle BE'C = 2\angle BAC$, it follows that G lies on the arc $BE'C$ of Ω'. Also, G lies on the median AM of ABC. Therefore, G is the point A' where the ray MA meets Ω'. In particular, $\frac{MA'}{MA} = \frac{1}{3}$. As P moves from E to F along the arc ECF, the ratio $\frac{MP'}{MP}$ increases by Theorem 5. Therefore

$$\frac{ME'}{ME} \leq \frac{MA'}{MA} = \frac{1}{3}.$$

By Lemma 4, $\theta \geq \frac{\pi}{3}$, with equality if and only if $A = E$, or equivalently if and only if ABC is equilateral. The possibility that $\angle BAC \geq \frac{\pi}{2}$ is ruled out since it would lead to the contradiction $\angle BGC \geq \pi$.

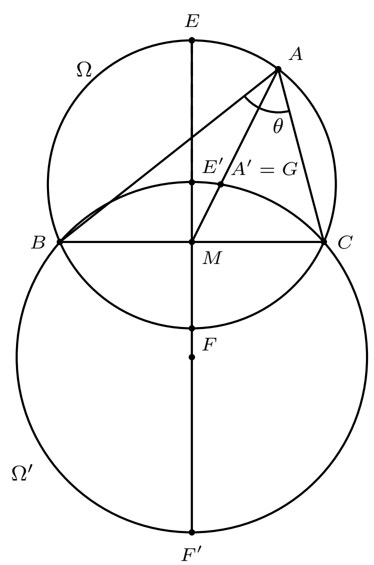

Figure 7

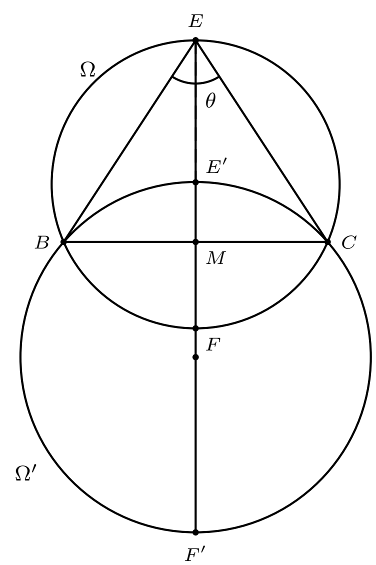

Figure 8

(ii) Suppose that θ is a given angle such that $\frac{\pi}{3} \leq \theta < \frac{\pi}{2}$ and that BC is a given segment. Let EBC be an isosceles triangle with $EB = EC$ and with $\angle BEC = \theta$. Let Ω be the circumcircle of EBC and let E' be its circumcenter. Let Ω' be the circumcircle of $E'BC$. Let M be the midpoint of BC and let the perpendicular bisector of BC meet Ω at (E and) F and meet Ω' at (E' and) F'; see Figure 8. For every P on Ω, we let P' be the point where the ray MP meets Ω'. Let $t = \frac{ME'}{ME}$. Since $\theta \geq \frac{\pi}{3}$, it follows from Lemma 4 that $t \leq \frac{1}{3}$. Also, $C' = C$ and $\frac{MC'}{MC} = 1$. Thus as P moves from E to C along one of the arcs EC of Ω, the ratio $\frac{MP'}{MP}$ increases from $t \leq \frac{1}{3}$ to 1. By continuity and the intermediate value theorem, there is a unique point A on that arc EC for which $\frac{MA'}{MA} = \frac{1}{3}$. If we think of MC as the x-axis and of ME as the y-axis, then the point A is the only point in the first quadrant for which ABC has the desired property. Points in the other quadrants are obtained by reflection about the x- and y-axes.

This is precisely the point A on the arc ECF for which A' is the centroid of ABC. This triangle ABC is the unique triangle (up to reflection about BC and about the perpendicular bisector of BC) whose vertex angle at A is θ and whose centroid G has the property that $\angle BGC = 2\angle BAC$. □

Theorem 9 characterizes those triangles whose centroid has the property $BB^* = CC^*$. For the proof, we need the following simple lemma.

Lemma 8. *Let ABC be a triangle with side-lengths a, b, and c* (in the standard order) *and with centroid G. Let the rays BG and CG meet the circumcircle of ABC at B^* and C^* respectively.*

$$BB^{*2} = \frac{(a^2 + c^2)^2}{2a^2 + 2c^2 - b^2}.$$

Proof. Let $m = BB'$, $x = BB^*$. By Apollonius' theorem, $m^2 = \frac{2(a^2+c^2)-b^2}{4}$. Since $BB'B^*$ and $AB'C$ are diagonals of a cyclic quadrilateral, $m(x - m) = \frac{b^2}{4}$. It follows that $mx = \frac{a^2+c^2}{2}$ and $x^2 = \frac{(a^2+c^2)^2}{4m^2} = \frac{(a^2+c^2)^2}{2a^2+2c^2-b^2}$. □

Theorem 9. *Let ABC be a triangle with side-lengths a, b, and c* (in the standard order) *and with centroid G. Let the rays BG and CG meet the circumcircle of ABC at B^* and C^*, respectively. If $b \neq c$, then the following are equivalent:*
(i) $BB^* = CC^*$,
(ii) $\angle BGC = 2\angle BAC$,
(iii) $a^4 = b^4 + c^4 - b^2c^2$.

Proof. Since $b \neq c$, it follows that $GB \neq GC$. By Lemma 6, (i) is equivalent to (ii). To see that (i) is equivalent to (iii), let $x = BB^*$, $y = CC^*$, and let $s = a^2 + b^2 + c^2$. By Lemma 8,

$$x^2 = \frac{(s - b^2)^2}{2s - 3b^2}, \qquad y^2 = \frac{(s - c^2)^2}{2s - 3c^2}.$$

Therefore

$$\begin{aligned} x = y &\iff \frac{(s-b^2)^2}{2s-3b^2} = \frac{(s-c^2)^2}{2s-3c^2} \\ &\iff (s^2 - 2b^2 s + b^4)(2s-3c^2) = (s^2 - 2c^2 s + c^4)(2s - 3b^2) \\ &\iff s^2(c^2-b^2) - 2s(c^2-b^2)(c^2+b^2) + 3c^2b^2(c^2-b^2) = 0 \\ &\iff s^2 - 2s(c^2+b^2) + 3c^2b^2 = 0 \qquad \text{(because } b \neq c\text{)} \\ &\iff (s-(c^2+b^2))^2 = (c^2+b^2)^2 - 3c^2b^2 \\ &\iff a^4 = c^4 + b^4 - c^2b^2, \end{aligned}$$

as claimed. □

It follows from [1, Theorem 2.3.3., page 83] (or [9, page 20]) that the only positive solutions of the diophantine equation

$$a^4 + b^4 - a^2b^2 = c^4 \tag{2}$$

are given by $a = b = c$. Thus there are no non-isosceles triangles ABC with integer side-lengths whose centroid G has the property $BB^* = CC^*$.

We end this note by a Euclidean construction, provided by a referee, of triangles ABC whose centroid has the property $BB^* = CC^*$. We start with any segment BC.

(i) Take any point A' on the major arc BA_0C of an equilateral triangle A_0BC.

(ii) Extend $A'C$ and $A'B$ to Y and Z respectively such that $CY = BZ = BC$.

(iii) Construct a circle with diameter $A'Z$ and the perpendicular at B to $A'Z$, intersecting the circle at B'

(iii$'$) Construct a circle with diameter $A'Y$ and the perpendicular at C to $A'Y$, intersecting the circle at C'

(iv) Construct the circles centered at B and C and passing through B' and C', respectively.

Letting A be a point of intersection of the two circles in (iv), one can verify that triangle ABC satisfies $BB^* = CC^*$.

In this regard, one may ask whether one can construct a triangle ABC having the property $BB^* = CC^*$ and having preassigned side BC and angle A (in $[\frac{\pi}{3}, \frac{\pi}{2}]$). The answer is affirmative as seen below.

Without loss of generality, assume $BC = 1$. Let $b = AC$, $c = AB$, and $t = \cos A$. We are to show that b and c are constructible. These are defined by

$$b^4 + c^4 - b^2c^2 = 1\ ,\ b^2 + c^2 = 2bct + 1.$$

Subtracting the square of the second from the first and simplifying, we obtain $bc = \frac{4t}{3-4t^2}$. Thus bc is constructible. Since $b^2 + c^2 = 2bct + 1$, it follows that $b^2 + c^2$ is constructible. Thus both b^2c^2 and $b^2 + c^2$ are constructible, and hence b^2 and c^2, being the zeros of $f(T) := T^2 - (b^2 + c^2)T + b^2c^2$, are constructible. This shows that b and c are constructible, as desired. The restriction $A \in [60^\circ, 90^\circ]$, i.e., $t \in \left[0, \frac{1}{2}\right]$, guarantees that the zeros of $f(T)$ are real (and positive).

References

[1] T. Andreescu and D. Andrica, *An Introduction to Diophantine Equations*, GIL Publishing House, Zalau, Romania, 2002.
[2] Euclid, *The Elements*, Sir Thomas L. Heath, editor, Dover Publications, Inc., New York, 1956.
[3] Euclid's Elements, aleph0.clarku.edu/˜djoyce/mathhist/alexandria.html
[4] M. Hajja, Problem 1767, *Math. Mag.*, 80 (2007), 145; solution, ibid., 81 (2008), 137.
[5] M. Hajja, Cyril F. Parry's variations on the Steiner-Lehmus theme, Preprint.
[6] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1–285.
[7] C. Kimberling, Encyclopaedia of Triangle Centers,
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html
[8] R. S. Millman and G. D. Parker, *Geometry – A Metric Approach with Models*, second edition, Springer-Verlag, New York, 1991.
[9] L. J. Mordell, *Diophantine Equations*, Academic Press, New York, 1969.
[10] B. M. Stewart, *Theory of Numbers*, second edition, The Macmillan Co., New York, 1964.

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