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Haruki's Lemma for Conics

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Abstract. We extend Haruki's lemma to conics.

1. Main results

In this paper we continue to explore Haruki's lemma introduced by Ross Honsberger in [2, 3]. In [1], we gave an extension of Haruki's lemma (Theorem 1 below) and studied a related locus problem, leading to certain interesting conics. [1]

Theorem 1 ([1, Lemma 2])**.** *Given two nonintersecting chords AB and CD in a circle and a variable point P on the arc AB remote from points C and D, let E and F be the intersections of chords PC, AB, and of PD, AB respectively. The following equalities hold:*

$$\frac{AE \cdot BF}{EF} = \frac{AC \cdot BD}{CD}, \quad (1)$$

$$\frac{AF \cdot BE}{EF} = \frac{AD \cdot BC}{CD}. \quad (2)$$

In this paper we generalize this result to conics.

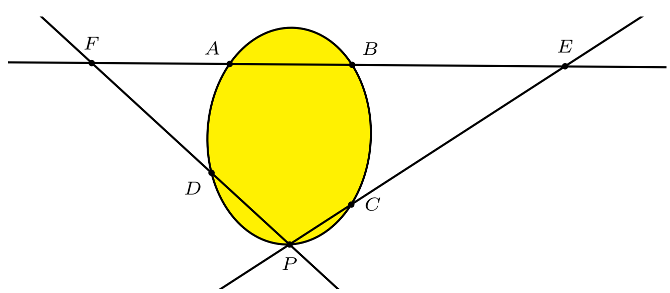

Figure 1.

Theorem 2. *Given a nondegenerate conic $\mathcal{C}$ with fixed points A, B, C, D on it, let P be a variable point distinct from A and B. Let E and F be the intersections of the lines PC, AB, and of PD, AB respectively. Then the ratios $\frac{AE \cdot BF}{EF}$ and $\frac{AF \cdot BE}{FE}$ are independent of the choice of P.*

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[1]See Remark following the proof of Theorem 2 below.

It turns out that this result still holds when the points A and B coincide. In this case, we replace the line AB by the tangent to the conic at A. With a minor change of notations, we have the following result.

Theorem 3. *Given a nondegenerate conic $\mathcal{C}$ with fixed points A, B, C on it, let P be a variable point distinct from A. Let E and F be the intersections of the lines PB, PC with the tangent to the conic at A. Then the ratio $\dfrac{AE \cdot AF}{EF}$ is independent of the choice of P.*

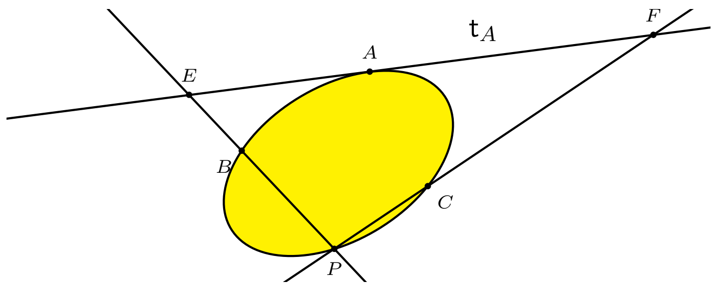

Figure 2

2. Proof of Theorem 2

We choose ABC as reference triangle. The nondegenerate conic $\mathcal{C}$ has equation of the form

$$fyz + gzx + hxy = 0 \tag{3}$$

for nonzero constants f, g, h. See Figure 1. Suppose D has homogeneous barycentric coordinates $(u : v : w)$, *i.e.*,

$$fvw + gwu + huv = 0. \tag{4}$$

Clearly, u, v, w are all nonzero. For an arbitrary point P with barycentric coordinates $(x : y : z)$, the coordinates of the intersections $E = AB \cap DC$ and $F = AB \cap PD$ can be easily determined:

$$E = (x : y : 0), \qquad F = (uz - wx : vz - wy : 0).$$

See [1, §6]. From these, we have the signed lengths of the various relevant segments:

$$AE = \tfrac{y}{x+y} \cdot c, \qquad EB = \tfrac{x}{x+y} \cdot c,$$
$$AF = \tfrac{vz-wy}{z(u+v)-w(x+y)} \cdot c, \qquad FB = \tfrac{uz-wx}{z(u+v)-w(x+y)} \cdot c,$$
$$EF = \tfrac{z(vx-uy)}{(x+y)(z(u+v)-w(x+y))} \cdot c,$$

where $c = AB$. It follows that $\dfrac{AE \cdot BF}{EF} = \dfrac{y(wx - uz)}{z(vx - uy)} \cdot c$. To calculate this fraction, note that from (4), we have $\frac{fw}{h} = -u(1 + k)$ for $k = \frac{gw}{hv}$. Now, from (3), we have

$$\begin{aligned}&\frac{fw}{h}\cdot yz+\frac{gw}{h}\cdot zx+w\cdot xy=0,\\&-u(1+k)yz+kvzx+wxy=0,\\&y(wx-uz)+kz(vx-uy)=0.\end{aligned}$$

Hence, $\dfrac{AE\cdot BF}{EF}=\dfrac{y(wx-uz)}{z(vx-uy)}\cdot c=-kc$, a constant.

A similar calculation gives $\dfrac{AF\cdot BE}{FE}=(1+k)c$, a constant. This completes the proof of the theorem.

Remark. Note that we have actually proved that

$$\frac{AE\cdot BF}{EF}=-\frac{gw}{hv}\cdot c\quad\text{and}\quad\frac{AF\cdot BE}{FE}=-\frac{fw}{hu}\cdot c.$$

In [1, Theorem 6], we have solved two loci problems in connection with Haruki's lemma. Denote, in Figure 1, $BC=a$, $CA=b$, $AB=c$, and $AD=a'$, $BD=b'$, $CD=c'$. The locus of points P satisfying (1) is the union of the two circumconics of $ABCD$

$$(cc'+\varepsilon bb')uyz-\varepsilon bb'vzx-cc'wxy=0,\quad\varepsilon=\pm1.$$

Now, with

$$f=(cc'+\varepsilon bb')u,\qquad g=-\varepsilon bb'v,\qquad h=-cc'w,$$

we have

$$\frac{AE\cdot BF}{EF}=-\frac{-\varepsilon bb'vw}{-cc'wv}\cdot c=-\varepsilon\cdot\frac{bb'}{c'}=\varepsilon\cdot\frac{AC\cdot BD}{CD}.$$

Similarly, the locus of points P satisfying (2) is the union of the two circumconics of $ABCD$

$$\varepsilon aa'uyz+(cc'-\varepsilon aa')vzx-cc'wxy=0,\quad\varepsilon=\pm1.$$

Now, with

$$f=\varepsilon aa'u,\qquad g=(cc'-\varepsilon aa')v,\qquad h=-cc'w,$$

we have

$$\frac{AF\cdot BE}{FE}=-\frac{fw}{hu}\cdot c=-\frac{\varepsilon aa'uw}{-cc'wu}\cdot c=\varepsilon\cdot\frac{aa'}{c'}=-\varepsilon\cdot\frac{AD\cdot BC}{DC}.$$

These confirm that Theorem 2 is consistent with Theorem 6 of [1].

3. Proof of Theorem 3

Again, we choose ABC as the reference triangle, and write the equation of the nondegenerate conic $\mathcal{C}$ in the form (3) with $fgh \neq 0$. The tangent at A is the line

$$\mathsf{t}_A : \qquad hy + gz = 0.$$

For an arbitrary point P with homogeneous barycentric coordinates $(x : y : z)$, the lines PB and PC intersect t_A respectively at

$$\begin{aligned} E =&(hx : -gz : hz), \\ F =&(gx : gy : -hy). \end{aligned}$$

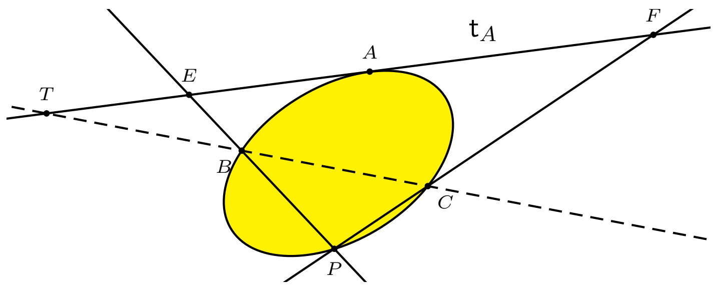

Figure 3

On the tangent line there is the point $T = (0 : -g : h)$, the intersection with the line BC. It is clearly possible to express the points E and F in terms of A and T. In fact, from

$$\begin{aligned} (hx, -gz, hz) =& hx(1, 0, 0) - z(0, g, -h), \\ (gx, gy, -hy) =& gx(1, 0, 0) + y(0, g, -h), \end{aligned}$$

we have, in absolute barycentric coordinates,

$$\begin{aligned} E =& \frac{hx}{hx - (g-h)z} \cdot A + \frac{-(g-h)z}{hx - (g-h)z} \cdot T, \\ F =& \frac{gx}{gx + (g-h)y} \cdot A + \frac{(g-h)y}{gx + (g-h)y} \cdot T. \end{aligned}$$

From these,

$$\frac{AE}{AT} = \frac{-(g-h)z}{hx - (g-h)z}, \qquad \frac{AF}{AT} = \frac{(g-h)y}{gx + (g-h)y}.$$

It follows that

$$\begin{aligned} \frac{EF}{AT} = \frac{AF - AE}{AT} &= \frac{(g-h)y}{gx + (g-h)y} + \frac{(g-h)z}{hx - (g-h)z} \\ &= \frac{(g-h)x(hy + gz)}{(gx + (g-h)y)(hx - (g-h)z)}. \end{aligned}$$

Therefore,

$$\frac{AE \cdot AF}{EF} = \frac{-(g-h)z \cdot (g-h)y}{(g-h)x(hy+gz)} \cdot AT = \frac{-(g-h)yz}{gzx+hxy} \cdot AT$$
$$= \frac{-(g-h)yz}{-fyz} \cdot AT = \frac{g-h}{f} \cdot AT.$$

This is independent of the choice of the point $P(x:y:z)$ on the conic. This completes the proof of Theorem 3.

References

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