

## Stronger Forms of the Steiner-Lehmus Theorem

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**Abstract.** We give a short proof based on Breusch's lemma of a stronger form of the Steiner-Lehmus theorem, and we discuss other possible stronger forms.

### 1. A stronger form of Steiner-Lehmus Theorem

Let  $a, b, c, A, B, C$  denote, in the standard manner, the side lengths and angles of a triangle  $ABC$ . An elegant lemma that was designed by Robert Breusch for solving an interesting 1961 MONTHLY problem [4] states that

$$\frac{p(ABC)}{a} = \frac{2}{1 - \tan(B/2) \tan(C/2)}, \quad (1)$$

where  $p(\dots)$  denotes the perimeter. Its simple proof is reproduced in [1], where it is used to give a very short proof of a theorem of Urquhart.

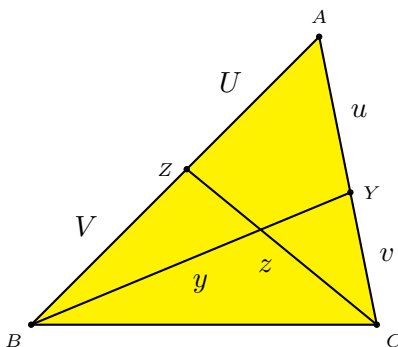


Figure 1

We now consider the Steiner-Lehmus configuration shown in Figure 1, where  $BY$  and  $CZ$  are the internal angle bisectors of angles  $B$  and  $C$ . Applying Breusch's lemma to triangles  $YBC$  and  $ZBC$ , we obtain

$$\frac{p(YBC)}{p(ZBC)} = \frac{1 - \tan(B/2) \tan(C/4)}{1 - \tan(B/4) \tan(C/2)}.$$

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The author would like to thank Yarmouk University for supporting this work, and Nikolaos Dergiades for his valuable comments and additions. Nikolaos is responsible for much of §2. Specifically, he is responsible for proving (8), which appeared as a conjecture in the first draft, and for strengthening the conjecture in (13) by testing more triangles and by introducing and evaluating the limit in (12).

If  $c > b$ , then

$$\tan \frac{C}{2} \tan \frac{B}{4} = \frac{2 \tan(B/4) \tan(C/4)}{1 - \tan^2(C/4)} > \frac{2 \tan(B/4) \tan(C/4)}{1 - \tan^2(B/4)} = \tan \frac{B}{2} \tan \frac{C}{4},$$

and therefore  $p(YBC) > p(ZBC)$ . Letting

$$|BY| = y, |CZ| = z, |AZ| = U, |ZB| = V, |AY| = u, |YC| = v,$$

we have proved the stronger form

$$c > b \iff y + v > z + V \quad (2)$$

of the traditional Steiner-Lehmus theorem

$$c > b \iff y > z. \quad (3)$$

To see that (2) is indeed stronger than (3), we need to show that  $V > v$ . By the angle bisector theorem, we have  $\frac{V}{U} = \frac{a}{b}$ . Therefore,  $\frac{V}{V+U} = \frac{a}{a+b}$ , and  $V = \frac{ac}{a+b}$ . A similar formula holds for  $v$ . Thus we have

$$V = \frac{ac}{a+b}, \quad v = \frac{ab}{a+c}, \quad (4)$$

and

$$V - v = \frac{ac}{a+b} - \frac{ab}{a+c} = \frac{a(c(a+c) - b(b+c))}{(a+b)(a+c)} = \frac{a(a+b+c)(c-b)}{(a+b)(a+c)}$$

Thus

$$c > b \implies V > v, \quad (5)$$

and (2) is stronger than (3).

It follows from (4) that

$$U = \frac{bc}{a+b}, \quad u = \frac{bc}{a+c}, \quad (6)$$

and therefore

$$c > b \implies U > u.$$

Thus the statement

$$c > b \implies y + b > z + c \quad (7)$$

would be stronger, and more pleasant, than (2). Unfortunately, (7) is not true. In fact, a recent MONTHLY problem [3] states that if  $a \geq c > b$ , then the reverse inequality  $y + b < z + c$  holds.

## 2. Additive stronger forms

2.1. One then wonders about the statement

$$c > b \implies y + u > z + U. \quad (8)$$

This is also stronger than the classical form (3). In order to prove (8), since  $c > b \implies y > z$  and  $U > u$ , it is sufficient to prove that

$$\frac{y^2 - z^2}{U - u} > y + z, \text{ or } \left( \frac{y^2 - z^2}{U - u} \right)^2 > (y + z)^2, \text{ or } \left( \frac{y^2 - z^2}{U - u} \right)^2 > 2(y^2 + z^2),$$

or

$$\frac{a(a+b+c)(a^8 + s_7a^7 + \cdots + s_1a + s_0)}{b^2c^2(a+c)^2(a+b)^2} > 0,$$

which is true because

$$\begin{aligned} s_7 &= 3(b+c), & s_6 &= 3(b^2 + 4bc + c^2), \\ s_5 &= b^2 + 16bc + 16c^2, & s_4 &= bc(2b + 5c)(5b + 2c), \\ s_3 &= bc(b+c)(2b^2 + 17bc + 2c^2), & s_2 &= 5b^2c^2(b+c)^2, \\ s_1 &= b^2c^2(b+c)(b^2 + c^2), & s_0 &= 2b^3c^3(b-c)^2. \end{aligned}$$

Combining (8) with (2) would yield the form

$$c > b \implies y + \frac{b}{2} > z + \frac{c}{2}$$

or

$$c > b \implies y - z > \frac{1}{2}(c - b). \quad (9)$$

2.2. In all cases, it is interesting to investigate the best constant  $\lambda$  for which

$$c > b \implies y - z \geq \lambda(c - b). \quad (10)$$

Similar questions can be asked about the best constants in

$$c > b \implies y - z \geq \lambda(U - u), \quad c > b \implies y - z \geq \lambda(V - v). \quad (11)$$

These may turn out to be quite easy given available computer packages such as BOTTEMA. For example, the stronger form  $c > b \implies y - z > 0.8568(c - b)$  of (9) was verified for all triangles whose side lengths are integers less than 51. The minimum value 0.8568 of the fraction  $\frac{y-z}{c-b}$ , verified for all triangles whose side lengths are integers less than 51, is attained for  $(a, b, c) = (48, 37, 38)$ , and the minimum value 0.856762, verified for all triangles whose side lengths are integers less than 501, is attained for  $(a, b, c) = (499, 388, 389)$ . Hence one may conjecture that the minimum value of  $\frac{y-z}{c-b}$  is attained when  $c$  tends to  $b$ . Note that

$$\lim_{c \rightarrow b} \frac{y-z}{c-b} = \frac{\sqrt{(a+2b)b(a^2+ab+2b^2)}}{2b(a+b)^2}. \quad (12)$$

Let  $f(x) := \frac{\sqrt{x+2}(x^2+x+2)}{2(x+1)^2}$ , so that the above limit is  $f(\frac{a}{b})$ . Since

$$f'(x) = \frac{x^3 + 4x^2 + x - 10}{4(x+1)^3\sqrt{x+2}},$$

we conclude that the minimum is  $f(q) = 0.856762$ , where  $q = 1.284277$  is the unique real zero of  $x^3 + 4x^2 + x - 10$ . Hence, we may conjecture that

$$c > b \iff y - z > f(q)(c - b). \quad (13)$$

### 3. Multiplicative stronger forms

3.1. One may also wonder about possibilities such as

$$c > b \implies yb > zc.$$

This is again false. In fact, it is proved in [5, Exercise 4, p. 15] that

$$y^2b^2 - z^2c^2 = \frac{abc(c-b)(a+b+c)^2(b^2 - bc + c^2 - a^2)}{(a+b)^2(a+c)^2}, \quad (14)$$

and therefore

$$c > b \implies (yb > zc \iff A < 60^\circ). \quad (15)$$

However, it is direct to check that

$$y^2b - z^2c = \frac{abc(c-b)(a+b+c)(b^2 + c^2 + ab + ac)}{(a+b)^2(a+c)^2}, \quad (16)$$

showing that

$$c > b \iff y^2b > z^2c, \quad (17)$$

yet another stronger form of the Steiner-Lehmus theorem (3).

3.2. Formulas (14) and (16) are derived from the formulas

$$y^2 = ac \left( 1 - \left( \frac{b}{a+c} \right)^2 \right), \quad z^2 = ab \left( 1 - \left( \frac{c}{a+b} \right)^2 \right). \quad (18)$$

These follow from Stewart's theorem using (6) and (4); see [5, Exercise 1, p. 15], where these are used to give a proof of the Steiner-Lehmus theorem via

$$y^2 - z^2 = \frac{(c-b)a(a+b+c)(a^2(a+b+c) + bc(b+c+3a))}{(a+b)^2(a+c)^2}.$$

Similarly one can prove the stronger forms

$$c > b \implies y^2u - z^2U > a(c-b), \quad (19)$$

$$c > b \implies y^2 - z^2 > V^2 - v^2 \quad (20)$$

using

$$\begin{aligned} y^2u - z^2U &= \frac{(c-b)abc(a+b+c)Q_1}{(a+b)^3(a+c)^3}, \\ (y^2 + v^2) - (z^2 + V^2) &= \frac{(c-b)a(a+b+c)Q_2}{(a+b)^2(a+c)^2}. \end{aligned}$$

where

$$\begin{aligned} Q_1 &= (a^3 + 2abc)(a+b+c) + bc(a^2 + b^2 + c^2), \\ Q_2 &= a^3 + b^2c + c^2b + 3abc - b^2a - c^2a. \end{aligned}$$

Here  $Q_2$  can be seen to be positive by substituting  $a = \beta + \gamma$ ,  $b = \alpha + \gamma$ ,  $c = \alpha + \beta$ .

*Remark.* It is well known that the Steiner-Lehmus theorem (3) is valid in neutral (or absolute) geometry; see [2, p. 119]. One wonders whether the same is true of the stronger forms (2), (8), and the other possible forms discussed in §2.

### References

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