

A New Proof of a Weighted Erdős-Mordell Type Inequality

Yu-Dong Wu

Dedicated to Miss Xiao-Ping Lü
on the occasion of the 24-th Teachers' Day

Abstract. In this short note, by making use of one of Liu's theorems and Cauchy-Schwarz Inequality, we solve a conjecture posed by Liu [3] and give a new proof of a weighted Erdős-Mordell type inequality. Some interesting corollaries are also given at the end.

1. Introduction and Main Results

Let P be an arbitrary point in the plane of triangle ABC . Denote by R_1 , R_2 , and R_3 the distances from P to the vertices A , B , and C , and r_1 , r_2 , and r_3 the signed distances from P to the sidelines BC , CA , and AB , respectively. The neat and famous inequality

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3), \quad (1)$$

conjectured by Paul Erdős in 1935, was first proved by L. J. Mordell and D. F. Barrow (see [2]). In 2005, Jian Liu [3] obtained a weighted Erdős-Mordell type inequality as follows.

Theorem 1. For $x, y, z \in \mathbb{R}$,

$$\begin{aligned} & x^2 \sqrt{R_2 + R_3} + y^2 \sqrt{R_3 + R_1} + z^2 \sqrt{R_1 + R_2} \\ & \geq \sqrt{2}(yz\sqrt{r_2 + r_3} + zx\sqrt{r_3 + r_1} + xy\sqrt{r_1 + r_2}). \end{aligned} \quad (2)$$

Liu's proof, however, is quite complicated. We give a simple proof of Theorem 1 as a corollary of a more general result, also conjectured by Liu in [3].

Publication Date: Month, 2008. Communicating Editor: Li Zhou.

The author would like to thank Professor Li Zhou for valuable comments which helped improve the presentation, and Professor Zhi-Hua Zhang for his careful reading of this paper.

Theorem 2. For $x, y, z \in \mathbb{R}$ and arbitrary positive real numbers u, v, w , we have

$$\begin{aligned} & x^2\sqrt{v+w} + y^2\sqrt{w+u} + z^2\sqrt{u+v} \\ & \geq 2 \left(yz\sqrt{u \sin \frac{A}{2}} + zx\sqrt{v \sin \frac{B}{2}} + xy\sqrt{w \sin \frac{C}{2}} \right). \end{aligned} \quad (3)$$

2. Preliminary Results

In order to prove our main results, we shall require the following two lemmas.

Lemma 3 ([4, 5]). For $x, y, z \in \mathbb{R}$, $p_i \in (-\infty, 0) \cup (0, +\infty)$, and $q_i \in \mathbb{R}$ for $i = 1, 2, 3$, the quadratic inequality of three variables

$$p_1x^2 + p_2y^2 + p_3z^2 \geq q_1yz + q_2zx + q_3xy$$

holds if and only if

$$\begin{cases} p_i > 0, \quad i = 1, 2, 3; \\ 4p_2p_3 > q_1^2, \quad 4p_3p_1 > q_2^2, \quad 4p_1p_2 > q_3^2, \\ 4p_1p_2p_3 \geq p_1q_1^2 + p_2q_2^2 + p_3q_3^2 + q_1q_2q_3. \end{cases}$$

Lemma 4. In $\triangle ABC$, we have

$$\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 1.$$

Proof. This follows from the formula $\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha)$ and the known identity

$$\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

□

3. Proof of Theorem 2

(1) For $u, v, w > 0$,

$$\begin{cases} \sqrt{v+w} > 0, \\ \sqrt{w+u} > 0, \\ \sqrt{u+v} > 0. \end{cases} \quad (4)$$

(2) From $\sin \frac{A}{2}, \sin \frac{B}{2}, \sin \frac{C}{2} \in (0, 1)$, we easily get

$$\begin{cases} 4\sqrt{(w+u)(u+v)} > 4u > 4u \sin \frac{A}{2}, \\ 4\sqrt{(u+v)(v+w)} > 4v > 4v \sin \frac{B}{2}, \\ 4\sqrt{(v+w)(w+u)} > 4w > 4w \sin \frac{C}{2}. \end{cases} \quad (5)$$

By the Cauchy-Schwarz inequality and Lemma 4, we have

$$\begin{aligned} & \left(u\sqrt{v+w}\sin\frac{A}{2} + v\sqrt{w+u}\sin\frac{B}{2} + w\sqrt{u+v}\sin\frac{C}{2} + \sqrt{2uvw}\sqrt{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \right)^2 \\ & \leq (u^2(v+w) + v^2(w+u) + w^2(u+v) + 2uvw) \\ & \quad \cdot \left(\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \right) \\ & = (u+v)(v+w)(w+u). \end{aligned} \tag{6}$$

From Lemma 3 and (4)–(6), we conclude that inequality (3) holds. The proof of Theorem 2 is complete.

4. Applications of Theorem 2

Proof of Theorem 1. If we take $u = R_1$, $v = R_2$, $w = R_3$ and with known inequalities (see [1])

$$2R_1 \sin\frac{A}{2} \geq r_2 + r_3, \quad 2R_2 \sin\frac{B}{2} \geq r_3 + r_1, \quad 2R_3 \sin\frac{C}{2} \geq r_1 + r_2,$$

we obtain Theorem 1 immediately. This completes the proof of Theorem 1.

Many further inequalities can be obtained from various substitutions for (u, v, w) . Here are two examples.

Corollary 5. For $\triangle ABC$ and real numbers x, y, z , we have

$$\begin{aligned} & x^2\sqrt{\sin\frac{B}{2} + \sin\frac{C}{2}} + y^2\sqrt{\sin\frac{C}{2} + \sin\frac{A}{2}} + z^2\sqrt{\sin\frac{A}{2} + \sin\frac{B}{2}} \\ & \geq 2\left(yz\sin\frac{A}{2} + zx\sin\frac{B}{2} + xy\sin\frac{C}{2}\right). \end{aligned}$$

Corollary 6. For $\triangle ABC$ and real numbers x, y, z , we have

$$\begin{aligned} & x^2\sqrt{\csc\frac{B}{2} + \csc\frac{C}{2}} + y^2\sqrt{\csc\frac{C}{2} + \csc\frac{A}{2}} + z^2\sqrt{\csc\frac{A}{2} + \csc\frac{B}{2}} \\ & \geq 2(yz + zx + xy). \end{aligned}$$

Further inequalities can also be obtained from substitutions of (x, y, z) by geometric elements of $\triangle ABC$. The reader is invited to experiment with the possibilities.

References

- [1] O. Bottema, R. Ž. Dordević, R. R. Janić and D. S. Mitrinović, *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969.
- [2] P. Erdős, L. J. Mordell, and D. F. Barrow. Problem 3740, *Amer. Math. Monthly*, 42 (1935) 396; solutions, *ibid.*, 44 (1937) 252–254.
- [3] J. Liu, A new extension of the Erdős-Mordell's type inequality, (in Chinese), *Jilin Normal University Journal* (Natural Science Edition), 26-4 (2005) 8–11.
- [4] J. Liu, Two theorems of three variables of quadratic type inequalities and applications, (in Chinese), *High School Mathematics* (Jiangsu), 5 (1996) 16–19.
- [5] J. Liu, Two results and several conjectures of a kind of geometry inequalities, (in Chinese), *Journal of East China Jiaotong University*, 3 (2002) 89–94.

Yu-Dong Wu: Department of Mathematics, Zhejiang Xinchang High School, Shaoxing, Zhejiang
312500, P. R. China

E-mail address: yudongwu@foxmail.com, 39387088@qq.com