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Some Identities Arising From Inversion of Pappus Chains in an Arbelos

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Abstract. We consider the inversive images, with respect to the incircle of an arbelos, of the three Pappus chains associated with the arbelos, and establish some identities connecting the radii of the circles involved.

In a previous work [1], we considered the three Pappus chains that can be drawn inside the arbelos and demonstrated some identities relating the radii of the circles in these chains. In Figure 1, the diameter AC of the left semicircle $\mathcal{C}_{\rm a}$ is $2a$, the diameter CB of the right semicircle $\mathcal{C}_{\rm b}$ is $2b$, and the diameter AB of the outer semicircle $\mathcal{C}_{\rm r}$ is $2r$, $r = a + b$. The first circle Γ_1 is common to all three chains and is the incircle of the arbelos.

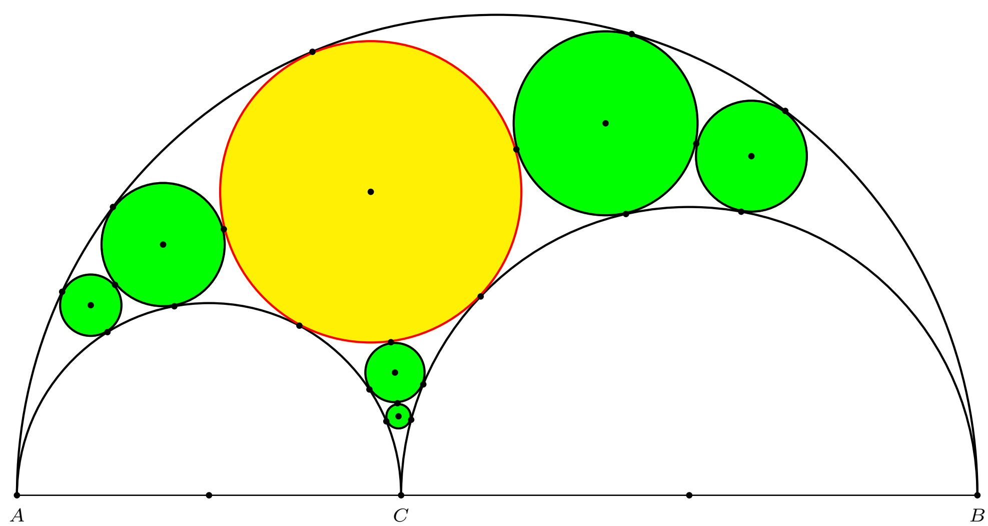

Figure 1. The Pappus chains in an arbelos

With reference to Figure 1, we denote by $\Gamma_{\rm r}$, $\Gamma_{\rm a}$ and $\Gamma_{\rm b}$ the chains converging to C, A, B respectively. Table 1 gives the coordinates of the centers and the radii of the circles in the chains, referring to a Cartesian reference system with origin at C and x-axis along AB.

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Table 1: Center coordinates and radii of the circles in the Pappus chains

Chain	$\Gamma_{\rm r}$	$\Gamma_{\rm a}$	$\Gamma_{\rm b}$
Abscissa of n-th circle	$x_{{\rm r}n}=\frac{ab(a-b)}{n^2r^2-ab}$	$x_{{\rm a}n}=2b-\frac{rb(r+b)}{n^2a^2+rb}$	$x_{{\rm b}n}=-2a+\frac{ra(r+a)}{n^2b^2+ra}$
Ordinate of n-th circle	$y_{{\rm r}n}=\frac{2nrab}{n^2r^2-ab}$	$y_{{\rm a}n}=\frac{2nrab}{n^2a^2+rb}$	$y_{{\rm b}n}=\frac{2nrab}{n^2b^2+ra}$
Radius of n-th circle	$\rho_{{\rm r}n}=\frac{rab}{n^2r^2-ab}$	$\rho_{{\rm a}n}=\frac{rab}{n^2a^2+rb}$	$\rho_{{\rm b}n}=\frac{rab}{n^2b^2+ra}$

The following proposition was established in [1].

Proposition 1. *Given a generic arbelos with its three Pappus chains, the following identities hold for each integer* n*:*

$$\rho_{\rm inc}\left(\frac{1}{\rho_{{\rm r}n}}+\frac{1}{\rho_{{\rm a}n}}+\frac{1}{\rho_{{\rm b}n}}\right)=2n^2+1, \tag{1}$$

$$\rho_{\rm inc}^2\left(\frac{1}{\rho_{{\rm r}n}^2}+\frac{1}{\rho_{{\rm a}n}^2}+\frac{1}{\rho_{{\rm b}n}^2}\right)=2n^4+1, \tag{2}$$

$$\rho_{\rm inc}^2\left(\frac{1}{\rho_{{\rm r}n}}\cdot\frac{1}{\rho_{{\rm a}n}}+\frac{1}{\rho_{{\rm a}n}}\cdot\frac{1}{\rho_{{\rm b}n}}+\frac{1}{\rho_{{\rm b}n}}\cdot\frac{1}{\rho_{{\rm r}n}}\right)=n^4+2n^2. \tag{3}$$

In particular, the center of the incircle of the arbelos is the point

$$(x_{\rm inc},\ y_{\rm inc})=\left(\frac{ab(a-b)}{a^2+ab+b^2},\ \frac{2ab(a+b)}{a^2+ab+b^2}\right).$$

Its radius is

$$\rho_{\rm inc}=\frac{ab(a+b)}{a^2+ab+b^2}.$$

We now consider the inversion of these three Pappus chains with respect to the incircle of arbelos. See Figure 2. For convenience, we record a useful formula, which can be found in [2], we use for the computation of the centers and radii of the inversive images of the circles in the Pappus chains.

Lemma 2. *With respect the circle of center* (x_0, y_0) *and radius* R_0*, the inversive image of the circle with center* $(x_C,\ y_C)$ *and radius* R *is the circle with center* $(x_C^{\rm i},\ y_C^{\rm i})$ *and radius* $R^{\rm i}$ *given by*

$$x_C^{\rm i}=x_0+\frac{R_0^2}{(x_C-x_0)^2+(y_C-y_0)^2-R^2}(x_C-x_0),$$

$$y_C^{\rm i}=y_0+\frac{R_0^2}{(x_C-x_0)^2+(y_C-y_0)^2-R^2}(y_C-y_0),$$

$$R^{\rm i}=\left|\frac{R_0^2}{(x_C-x_0)^2+(y_C-y_0)^2-R^2}\right|R.$$

We give in Table 2 the coordinates of the centers of the inversive images of the circles in the Pappus chains, and their radii.

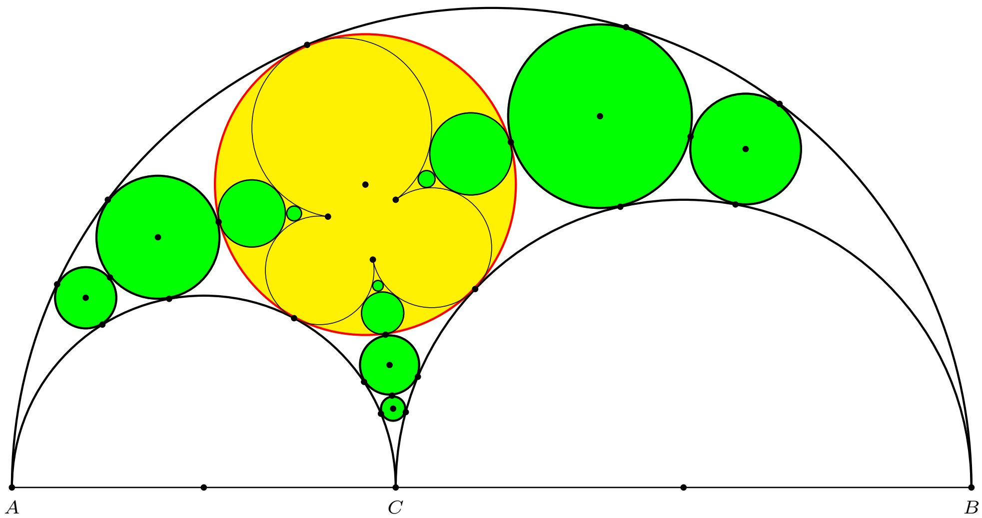

Figure 2. Inversive images of the Pappus chains

Table 2: Center coordinates and radii of inversive images of circles in the Pappus chains

	Inverted chain $\Gamma_{\mathrm{r}}^{\mathrm{i}}$
Abscissa of n-th circle	$x_{\mathrm{r}n}^{\mathrm{i}} = x_{\mathrm{inc}} + \frac{\rho_{\mathrm{inc}}^2(x_{\mathrm{r}n}-x_{\mathrm{inc}})}{(x_{\mathrm{r}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{r}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Ordinate of n-th circle	$y_{\mathrm{r}n}^{\mathrm{i}} = y_{\mathrm{inc}} + \frac{\rho_{\mathrm{inc}}^2(y_{\mathrm{r}n}-y_{\mathrm{inc}})}{(x_{\mathrm{r}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{r}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Radius of n-th circle	$\rho_{\mathrm{r}n}^{\mathrm{i}} = \left\lvert\frac{\rho_{\mathrm{inc}}^2}{(x_{\mathrm{r}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{r}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}\right\rvert \rho_{\mathrm{r}n}$
	Inverted chain $\Gamma_{\mathrm{a}}^{\mathrm{i}}$
Abscissa of n-th circle	$x_{\mathrm{a}n}^{\mathrm{i}} = x_{\mathrm{inc}} + \frac{\rho_{\mathrm{inc}}^2(x_{\mathrm{a}n}-x_{\mathrm{inc}})}{(x_{\mathrm{a}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{a}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Ordinate of n-th circle	$y_{\mathrm{a}n}^{\mathrm{i}} = y_{\mathrm{inc}} + \frac{\rho_{\mathrm{inc}}^2(y_{\mathrm{a}n}-y_{\mathrm{inc}})}{(x_{\mathrm{a}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{a}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Radius of n-th circle	$\rho_{\mathrm{a}n}^{\mathrm{i}} = \left\lvert\frac{\rho_{\mathrm{inc}}^2}{(x_{\mathrm{a}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{a}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}\right\rvert \rho_{\mathrm{a}n}$
	Inverted chain $\Gamma_{\mathrm{b}}^{\mathrm{i}}$
Abscissa of n-th circle	$x_{\mathrm{b}n}^{\mathrm{i}} = x_{\mathrm{inc}} + \frac{\rho_{\mathrm{inc}}^2(x_{\mathrm{b}n}-x_{\mathrm{inc}})}{(x_{\mathrm{b}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{b}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Ordinate of n-th circle	$y_{\mathrm{b}n}^{\mathrm{i}} = y_{\mathrm{inc}} + \frac{\rho_{\mathrm{inc}}^2(y_{\mathrm{b}n}-y_{\mathrm{inc}})}{(x_{\mathrm{b}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{b}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}$
Radius of n-th circle	$\rho_{\mathrm{b}n}^{\mathrm{i}} = \left\lvert\frac{\rho_{\mathrm{inc}}^2}{(x_{\mathrm{b}n}-x_{\mathrm{inc}})^2+(y_{\mathrm{b}n}-y_{\mathrm{inc}})^2-\rho_{\mathrm{inc}}^2}\right\rvert \rho_{\mathrm{b}n}$

From these data, we can deduce some identities connecting the radii of these circles.

Theorem 3. *For the circles in the Pappus chains and their inversive images in the incircle, the following identities hold. For $n \geq 2$,*

$$\frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{r}n}} - \frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{r}n}} = \frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{a}n}} - \frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{a}n}} = \frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{b}n}} - \frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{b}n}} = 4n^2 - 8n + 2, \tag{4}$$

$$\frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{r}n}} + \frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{a}n}} + \frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{b}n}} = 14n^2 - 24n + 7, \tag{5}$$

$$\frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{r}n}} + \frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{a}n}} + \frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{b}n}} = \frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{r}n}} + \frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{a}n}} + \frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{b}n}} = \frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{r}n}} + \frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{a}n}} + \frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{b}n}} = 6n^2 - 8n + 3, \tag{6}$$

$$\frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{r}n}} + \frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{a}n}} + \frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{b}n}} = \frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{r}n}} + \frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{a}n}} + \frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{b}n}} = \frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{r}n}} + \frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{a}n}} + \frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{b}n}} = 10n^2 - 16n + 5, \tag{7}$$

$$\frac{\rho^2_{\mathrm{inc}}}{\rho_{\mathrm{r}n}\rho^{\mathrm{i}}_{\mathrm{r}n}} + \frac{\rho^2_{\mathrm{inc}}}{\rho_{\mathrm{a}n}\rho^{\mathrm{i}}_{\mathrm{a}n}} + \frac{\rho^2_{\mathrm{inc}}}{\rho_{\mathrm{b}n}\rho^{\mathrm{i}}_{\mathrm{b}n}} = 10n^4 - 16n^3 + 8n^2 - 8n + 3, \tag{8}$$

$$\frac{\rho^2_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{a}n}\rho^{\mathrm{i}}_{\mathrm{b}n}} + \frac{\rho^2_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{a}n}\rho^{\mathrm{i}}_{\mathrm{r}n}} + \frac{\rho^2_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{b}n}\rho^{\mathrm{i}}_{\mathrm{r}n}} = 65n^4 - 224n^3 + 258n^2 - 112n + 16, \tag{9}$$

$$\left(\frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{r}n}}\right)^2 + \left(\frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{a}n}}\right)^2 + \left(\frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{b}n}}\right)^2 = 66n^4 - 224n^3 + 256n^2 - 112n + 17. \tag{10}$$

From (9), (10) above, and also (2), (3) in Proposition 1, we have

$$\begin{aligned}
&\left(\frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{r}n}}\right)^2 + \left(\frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{a}n}}\right)^2 + \left(\frac{\rho_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{b}n}}\right)^2 - \left(\frac{\rho^2_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{a}n}\rho^{\mathrm{i}}_{\mathrm{b}n}} + \frac{\rho^2_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{a}n}\rho^{\mathrm{i}}_{\mathrm{r}n}} + \frac{\rho^2_{\mathrm{inc}}}{\rho^{\mathrm{i}}_{\mathrm{b}n}\rho^{\mathrm{i}}_{\mathrm{r}n}}\right) \\
=& \left(\frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{r}n}}\right)^2 + \left(\frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{a}n}}\right)^2 + \left(\frac{\rho_{\mathrm{inc}}}{\rho_{\mathrm{b}n}}\right)^2 - \left(\frac{\rho^2_{\mathrm{inc}}}{\rho_{\mathrm{a}n}\rho_{\mathrm{b}n}} + \frac{\rho^2_{\mathrm{inc}}}{\rho_{\mathrm{a}n}\rho_{\mathrm{r}n}} + \frac{\rho^2_{\mathrm{inc}}}{\rho_{\mathrm{b}n}\rho_{\mathrm{r}n}}\right) \\
=& (n^2 - 1)^2.
\end{aligned}$$

References

[1] G. Lucca, Three Pappus chains inside the arbelos: some identities, *Forum Geom.*, 7 (2008) 107–109.

[2] E. W. Weisstein, Inversion from MathWorld – A Wolfram Web Resource, `http://mathworld.wolfram.com/Inversion.html`.

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