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On the Nagel Line and a Prolific Polar Triangle

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Abstract. For a given triangle ABC, the polar triangle of the medial triangle with respect to the incircle is shown to have as its vertices the orthocenters of triangles AIB, BIC and AIC. We prove results which relate this polar triangle to the Nagel line and, eventually, to the Feuerbach point.

1. A prolific triangle

In a triangle ABC we construct a triad of circles $\mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c$ that are orthogonal to the incircle Γ of the triangle, with their centers at the midpoints D, E, F of the sides BC, AC, AB. These circles pass through the points of tangency X, Y, Z of the incircle with the respective sides. We denote by ℓ_a (respectively ℓ_b, ℓ_c) the radical axis of Γ and $\mathcal{C}_a$ (respectively $\mathcal{C}_b$, $\mathcal{C}_c$), and examine the triangle $A^*B^*C^*$ bounded by these lines (see Figure 1). J.-P. Ehrmann [1] has shown that this triangle has the same area as triangle ABC.

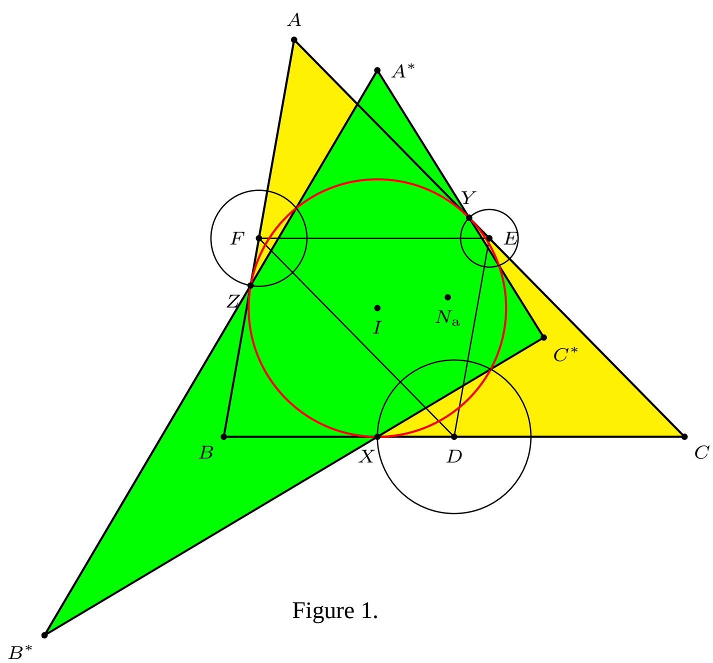

Figure 1.

Lemma 1. *The triangle $A^*B^*C^*$ is the polar triangle of the medial triangle DEF of triangle ABC with respect to Γ.*

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Proof. Because $\mathcal{C}_a$ is orthogonal to Γ, the line ℓ_a is the polar of D with respect to Γ. Similarly, ℓ_b and ℓ_c are the polars of E and F with respect to the same circle. $\square$

Note that Lemma 1 implies that triangles $A^*B^*C^*$ and XYZ are perspective with center I: $A^*I \perp EF$ because EF is the polar line of A^* with respect to Γ. Because $EF \parallel BC$ and $BC \perp XI$, the assertion follows.

Lemma 2. *The lines XY, BI, EF, and AC^* are concurrent at a point of $\mathcal{C}_b$, as are the lines YZ, BI, DE, and AB^** (see Figure 2).

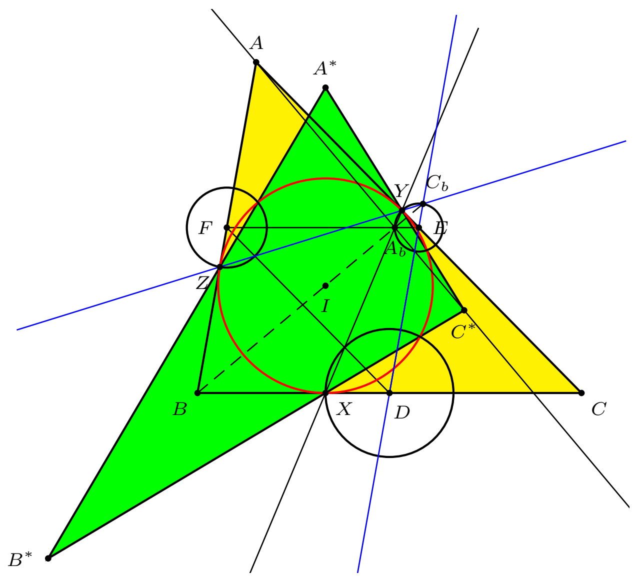

Figure 2.

Proof. Let A_b as the point on EF, on the same side of F as E, so that $FA_b = FA$.

(i) Because $FA = FA_b = FB$, the points A, A_b and B all lie on a circle with center F. This implies that $\angle ABC = \angle AFA_b = 2\angle ABA_b$, yielding $\angle ABI = \angle ABA_b$. This shows that A_b lies on BI.

(ii) Because $YC = \frac{1}{2}(AC + CB - BA) = EC + EF - FA$, we have

$$EY = YC - EC = EF - FA = FE - FA_b = EA_b,$$

showing that A_b lies on $\mathcal{C}_b$. Also, noting that $CX = CY$, we have $\dfrac{EY}{CY} = \dfrac{EA_b}{CX}$. This implies that triangles EYA_b and CYX are isosceles and similar. From this we deduce that A_b lies on XY.

A similar argument shows that DE, BI, YZ are concurrent at a point C_b on the circle $\mathcal{C}_b$. We will use this to prove the last part of this lemma.

(iii) Because YZ and DE are the polar lines of A and C^* with respect to Γ, AC^* is the polar line of C_b, which also lies on BI. This implies that $AC^* \perp BI$, so the intersection of AC^* and BI lies on the circle with diameter AB. We have shown that A_b lies on this circle, and on BI, so A_b also lies on AC^*.

Similarly, C_b also lies on the line AB^*. □

Note that the points A_b and C_b are the orthogonal projections of A and C on BI. Analogous statements can be made of quadruples of lines intersecting on the circles $\mathcal{C}_a$ and $\mathcal{C}_c$. Reference to this configuration can be found, for example, in a problem on the $2002 - 2003$ Hungarian Mathematical Olympiad. A solution and further references can be found in *Crux Mathematicorum with Mathematical Mayhem*, 33 (2007) 415–416.

We are now ready for our first theorem, conjectured in 2002 by D. Grinberg [2].

Theorem 3. *The points A^*, B^*, and C^* are the respective orthocenters of triangles BIC, CIA, and AIB.*

Proof. Because the point A_b lies on the polar lines of A^* and C with respect to Γ, we know that $A^*C \perp BI$. Combining this with the fact that $A^*I \perp BC$ we conclude that A^* is indeed the orthocenter of triangle BIC. □

Theorem 4. *The medial triangle DEF is perspective with triangle $A^*B^*C^*$, at the Mittenpunkt M_t* [1] *of triangle ABC* (see Figure 3).

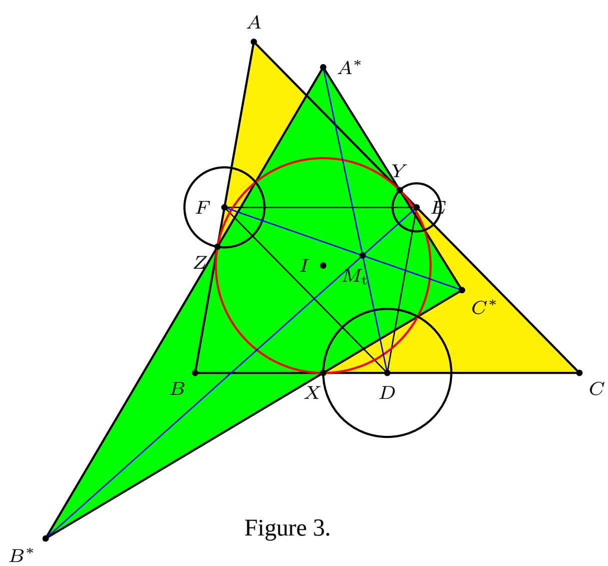

Figure 3.

Proof. Because A^*C is perpendicular to BI, it is parallel to the external bisector of angle B. A similar argument holds for BA^*, so we conclude that A^*BI_aC is a parallelogram. It follows that A^*, D, and I_a are collinear. This shows that M_t lies on I_aD, and similar arguments show that M_t lies on the lines I_bE and I_cF. □

We already know that triangle $A^*B^*C^*$ and triangle XYZ are perspective at the incenter I. By proving Theorem 4, we have in fact found two additional triangles that are perspective with triangle $A^*B^*C^*$: the medial triangle DEF and the

[1]The Mittenpunkt (called $X(9)$ in [4]) is the point of concurrency of the lines joining D to the excenter I_a, E to the excenter I_b, and C to the excenter I_c. It is also the symmedian point of the excentral triangle $I_aI_bI_c$.

excentral triangle $I_aI_bI_c$, both with center $M_{\rm t}$. This is however just a taste of the many special properties of triangle $A^*B^*C^*$, which will be treated throughout the rest of this paper.

Theorem 3 shows that B, C, A^*, I are four points that form an orthocentric system. A consequence of this is that I is the orthocenter of triangles A^*BC, AB^*C, ABC^*. In the following theorem we prove a similar result that will produce an unexpected point.

Theorem 5. *The Nagel point $N_{\rm a}$ of triangle ABC is the common orthocenter of triangles AB^*C^*, A^*BC^*, A^*B^*C.*

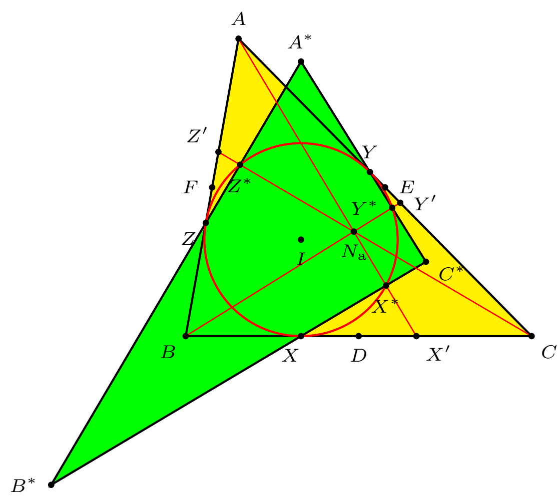

Figure 4.

Proof. Consider the homothety $\zeta := \mathsf{h}(D, -1)$. [2] This carries A into the vertex A' of the anticomplementary triangle $A'B'C'$ of ABC. It follows from Theorem 4 that $\zeta(A^*) = I_a$. This implies that $A'A^*$ is the bisector of $\angle BA'C$.

The Nagel line is the line IG joining the incenter and the centroid. It is so named because it also contains the Nagel point $N_{\rm a}$. Since $2IG = GN_{\rm a}$, the Nagel point $N_{\rm a}$ is the incenter of the anticomplementary triangle. This implies that $A'N_{\rm a}$ is the bisector of $\angle BA'C$. Hence, ζ carries $A^*N_{\rm a}$ into AI, so $A^*N_{\rm a}$ and AI are parallel. From this, $A^*N_{\rm a} \perp CB^*$.

Similarly, we deduce that $B^*N_{\rm a} \perp CA^*$, so $N_{\rm a}$ is the orthocenter of triangle A^*B^*C. □

The next theorem was proved by J.-P. Ehrmann in [1] using barycentric coordinates. We present a synthetic proof here.

Theorem 6 (Ehrmann)**.** *The centroid G^* of triangle $A^*B^*C^*$ is the point dividing IH in the ratio $IG^* : G^*H = 2 : 1$.*

[2] A homothety with center P and factor k is denoted by $\mathsf{h}(P, k)$.

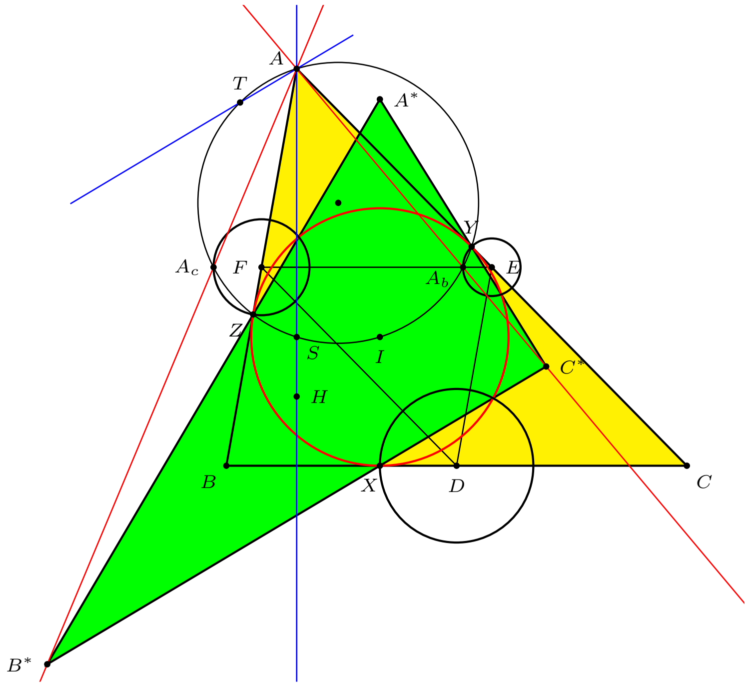

Figure 5.

Proof. The four points A, A_b, I, A_c all lie on a circle with diameter IA, which we will call $\mathcal{C}'_a$. Let H be the orthocenter of triangle ABC, and S the (second) intersection of $\mathcal{C}'_a$ with the altitude AH. Construct also the parallel AT to B^*C^* through A to intersect the circle at T (see Figure 5).

Denote by R_b and R_c the circumradii of triangles AIC and AIB respectively. Because C^* is the orthocenter of triangle AIB, we can write $AC^* = R_c \cdot \cos \frac{A}{2}$, and similarly for AB^*. Using this and the property $B^*C^* \parallel AT$, we have

$$\frac{\sin TAA_b}{\sin TAA_c} = \frac{\sin AC^*B^*}{\sin AB^*C^*} = \frac{AB^*}{AC^*} = \frac{R_b}{R_c} = \frac{\sin\frac{B}{2}}{\sin\frac{C}{2}} = \frac{IC}{IB}.$$

The points A_b, A_c are on EF according to Lemma 2, so triangle IA_bA_c and triangle IBC are similar. This implies $\dfrac{IC}{IB} = \dfrac{IA_c}{IA_b}$.

In any triangle, the orthocenter and circumcenter are known to be each other's isogonal conjugates. Applying this to triangle AA_bA_c, we find that $\angle SAA_b = \angle A_cAI$. We can now see that $\dfrac{SA_b}{SA_c} = \dfrac{IA_c}{IA_b}$.

Combining these results, we obtain

$$\frac{SA_b}{SA_c} = \frac{IA_c}{IA_b} = \frac{IC}{IB} = \frac{\sin TAA_b}{\sin TAA_c} = \frac{TA_b}{TA_c}.$$

This proves that $TA_c \cdot SA_b = SA_c \cdot TA_b$, so TA_cSA_b is a harmonic quadrilateral. It follows that AC^*, AB^* divide AH, AT harmonically. Because $AT \parallel B^*C^*$, we know that AH must pass through the midpoint of B^*C^*.

Let us call D^* the midpoint of B^*C^*, and consider the homothety $\xi = \mathsf{h}(G^*, -2)$. Because ξ takes D^* to B^* while $AH \parallel A^*X$, we know that ξ takes AH to A^*X. Similar arguments applied to B and B^* establish that ξ takes H to I. □

2. Two more triads of circles

Consider again the orthogonal projections A_b, A_c of A on the bisectors BI and CI. It is clear that the circle $\mathcal{C}'_a$ with diameter IA in Theorem 6 contains the points Y and Z as well. Similarly, we consider the circles $\mathcal{C}'_b$ and $\mathcal{C}'_c$ with diameters IB and IC (see Figure 6). It is easy to determine the intersections of the circles from the two triads $\mathcal{C}_a$, $\mathcal{C}_b$, $\mathcal{C}_c$, and $\mathcal{C}'_a$, $\mathcal{C}'_b$, $\mathcal{C}'_c$, which we summarize in the following table.

Table 1. Intersections of circles

	$\mathcal{C}'_a$	$\mathcal{C}'_b$	$\mathcal{C}'_c$
$\mathcal{C}_a$		$X,\ B_a$	$X,\ C_a$
$\mathcal{C}_b$	$Y,\ A_b$		$Y,\ X_b$
$\mathcal{C}_c$	$Z,\ A_c$	$Z,\ B_c$	

Now we introduce another triad of circles.

Let X^* (respectively Y^*, Z^*) be the intersection of Γ with $\mathcal{C}_a$ (respectively $\mathcal{C}_b$, $\mathcal{C}_c$) different from X (respectively Y, Z). Consider also the orthogonal projections A^*_b and A^*_c of A^* onto B^*N_a and C^*N_a, and similarly defined B^*_a, B^*_c, C^*_a, C^*_b.

Lemma 7. *The six points A^*, A^*_b, A^*_c, Y^*, Z^*, and N_a all lie on the circle with diameter A^*N_a* (see Figure 6).

Proof. The points A^*_b and A^*_c lie on the circle with diameter A^*N_a by definition.

We know that the Nagel point and the Gergonne point are isotomic conjugates, so if we call Y' the intersection of BN_a and AC, it follows that $YE = Y'E$. Therefore, Y' lies on $\mathcal{C}_b$.

Clearly YY' is a diameter of $\mathcal{C}_b$. It follows from Theorem 5 that BN_a is perpendicular to A^*C^*, so their intersection point must lie on $\mathcal{C}_b$. Since Y^* is the intersection point of A^*C^* and $\mathcal{C}_b$ different from Y, it follows that Y^* lies on BN_a.

Combining the above results, we obtain that $N_\mathrm{a}Y^* \perp A^*Y^*$, so Y^* lies on the circle with diameter A^*N_a. A similar proof holds for Z^*. □

We will call this circle $\mathcal{C}^*_a$. Likewise, $\mathcal{C}^*_b$ and $\mathcal{C}^*_c$ are the ones with diameters B^*N_a and C^*N_a. Here are the intersections of the circles in the two triads $\mathcal{C}_a$, $\mathcal{C}_b$, $\mathcal{C}_c$, and $\mathcal{C}^*_a$, $\mathcal{C}^*_b$, $\mathcal{C}^*_c$.

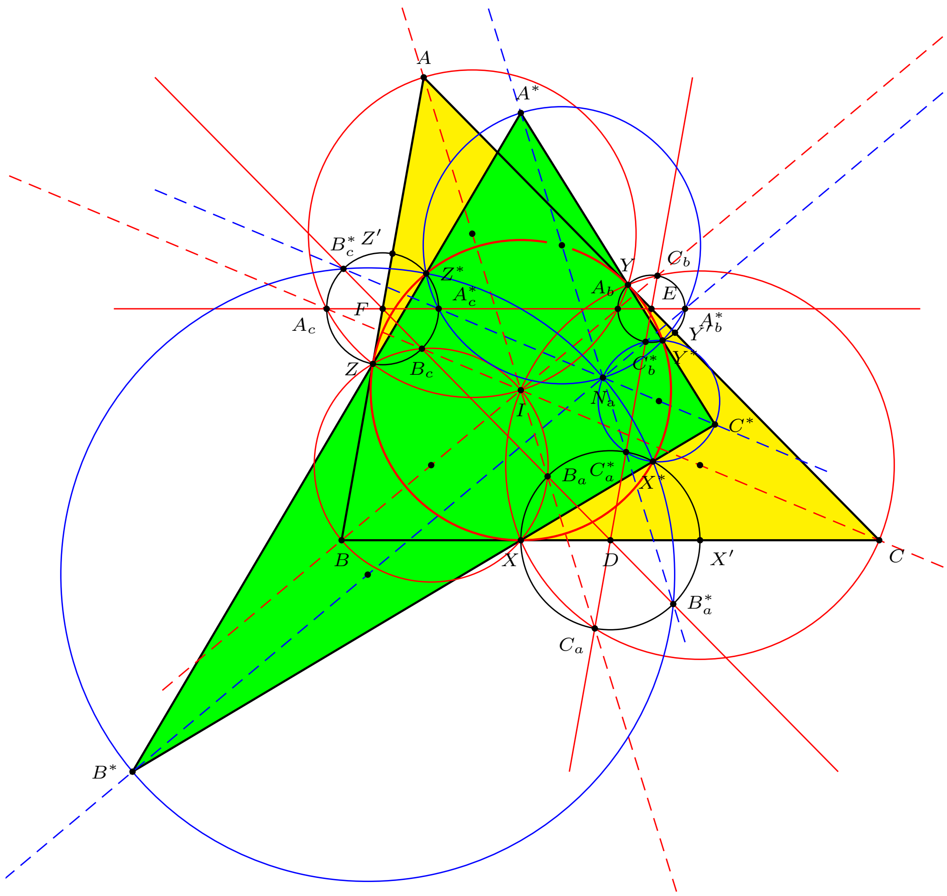

Figure 6.

Table 2. Intersections of circles

	$\mathcal{C}_a^*$	$\mathcal{C}_b^*$	$\mathcal{C}_c^*$
$\mathcal{C}_a$		$X^*,\ B_a^*$	$X^*,\ C_a^*$
$\mathcal{C}_b$	$Y^*,\ A_b^*$		$Y^*,\ X_b^*$
$\mathcal{C}_c$	$Z^*,\ A_c^*$	$Z^*,\ B_c^*$	

Lemma 8. *The circle $\mathcal{C}_a^*$ intersects $\mathcal{C}_b$ in the points Y^* and A_b^*. The point A_b^* lies on EF* (see Figure 7).

Proof. The point Y^* lies on $\mathcal{C}_b$ by definition, and on $\mathcal{C}_a^*$ by Lemma 7.

Consider the homothety $\phi := \mathsf{h}(E, -1)$. We already know that $\phi(AC^*) = CA^*$ and $\phi(BI) = B^*N_\text{a}$. This shows that the intersection points are mapped onto each other, or $\phi(A_b) = A_b^*$. It follows that A_b^* lies on $\mathcal{C}_b$ and EF. □

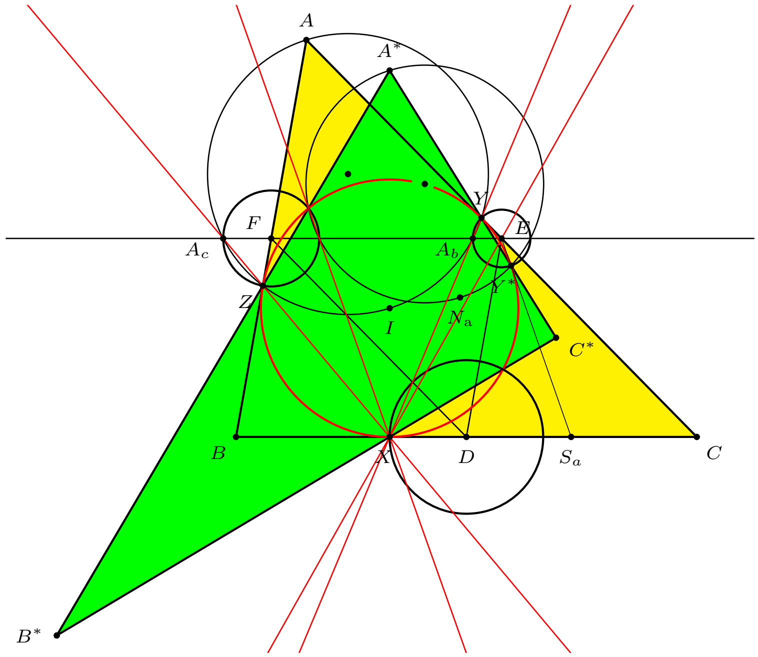

Figure 7.

The two triads of circles have some remarkable properties, strongly related to the Nagel line and eventually to the Feuerbach point. We will start with a property that may be helpful later on.

Theorem 9. *The point X has equal powers with respect to the circles $\mathcal{C}_b$, $\mathcal{C}_c$, $\mathcal{C}_a^*$, and $\mathcal{C}_a'$* (see Figure 7).

Proof. Let us call S_a the intersection of EY^* and BC, and S_b the intersection of XY^* and EF. Because EY^* is tangent to Γ, we have $S_aY^* = S_aX$. Because triangles XS_aY^* and S_bEY^* are similar, it follows that $EY^* = ES_b$. This implies that S_b lies on $\mathcal{C}_b$ so in fact S_b and A_b^* coincide. This shows that X lies on $Y^*A_b^*$. Similar arguments can be used to prove that X lies on $Z^*A_c^*$.

From Table 1, it follows that A_bY (respectively A_cZ) is the radical axis of the circles $\mathcal{C}_a'$ and $\mathcal{C}_b$ (respectively $\mathcal{C}_c$). Lemma 2 implies that X lies on both A_bY and A_cZ, so it is the radical center of $\mathcal{C}_a'$, $\mathcal{C}_b$ and $\mathcal{C}_c$.

From Lemma 8, it follows that $Y^*A_b^*$ (respectively $Z^*A_c^*$) is the radical axis of the circles $\mathcal{C}_b$ and $\mathcal{C}_a^*$ (respectively $\mathcal{C}_c$ and $\mathcal{C}_b^*$). We have just proved that X lies on both $Y^*A_b^*$ and $Z^*A_c^*$, so it is the radical center of $\mathcal{C}_a^*$, $\mathcal{C}_b$, and $\mathcal{C}_c$. The conclusion follows. □

3. Some similitude centers and the Nagel line

Denote by U, V, W the intersections of the Nagel line IG with the lines EF, DF and DE respectively (see Figure 8).

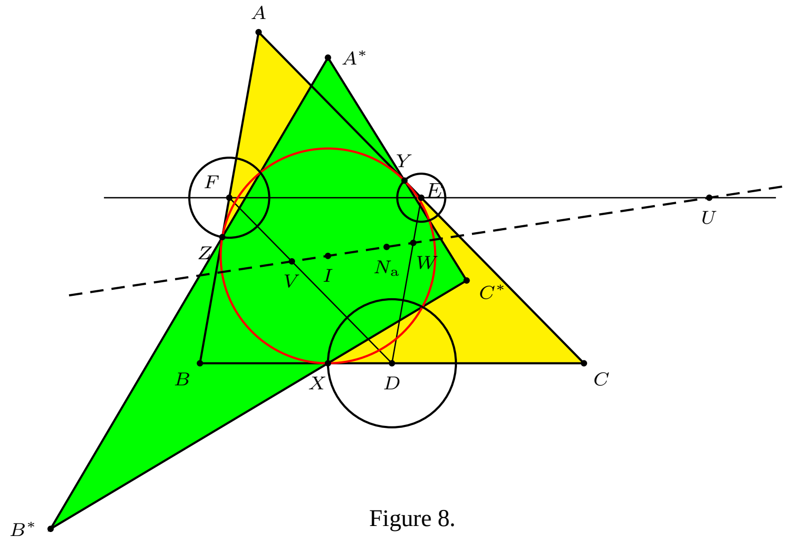

Figure 8.

Theorem 10. *The point U is a center of similitude of circles $\mathcal{C}'_a$ and $\mathcal{C}^*_a$. Likewise, V is a center of similitude of circles $\mathcal{C}'_b$ and $\mathcal{C}^*_b$, and W of $\mathcal{C}'_c$ and $\mathcal{C}^*_c$.*

Proof. We know from Lemma 2 and Theorem 5 that $A^*A^*_b \parallel AA_b$, and $AI \parallel A^*N_\mathrm{a}$, as well as $A^*_bN_\mathrm{a} \parallel A_bI$. Hence triangles triangle $A^*N_\mathrm{a}A^*_b$ and triangle AIA_b have parallel sides. It follows from Desargues' theorem that AA^*, $A_bA^*_b$, IN_a are concurrent. Clearly, the point of concurrency is a center of similitude of both triangles, and therefore also of their circumcircles, $\mathcal{C}^*_a$ and $\mathcal{C}_a$. This point of concurrency is the intersection point of EF and the Nagel line as shown above, so the theorem is proved. □

Theorem 11. *The point U is a center of similitude of circles $\mathcal{C}_b$ and $\mathcal{C}_c$. Likewise, V is a center of similitude of circles $\mathcal{C}_c$ and $\mathcal{C}_a$, and W of $\mathcal{C}_a$ and $\mathcal{C}_b$.*

Proof. By Theorem 10, we know that

$$\frac{A_bU}{A_cU} = \frac{A^*_bU}{A^*_cU}. \quad (1)$$

By Table 1 and Theorem 8, we know that A_b, A^*_c lie on $\mathcal{C}_c$ and A_b, A^*_b lie on $\mathcal{C}_b$. Knowing that U lies on EF, the line connecting the centers of $\mathcal{C}_b$ and $\mathcal{C}_c$, relation (1) now directly expresses that U is a center of similitude of $\mathcal{C}_b$ and $\mathcal{C}_c$. □

Depending on the shape of triangle ABC, the center of similitude of $\mathcal{C}_b$ and $\mathcal{C}_c$ which occurs in the above theorem could be either external or internal. Whichever it is, we will meet the other in the next theorem.

Theorem 12. *The lines BV and CW intersect at a point on EF. This point is the center of similitude different from U of $\mathcal{C}_b$ and $\mathcal{C}_c$* (see Figure 9).

Proof. Let us call U' the point of intersection of BV and EF. We have that $G = BE \cap CF$ and $V = DF \cap BU'$. By the theorem of Pappus-Pascal applied to the collinear triples E, U', F and C, D, B, the intersection of $U'C$ and DE must lie

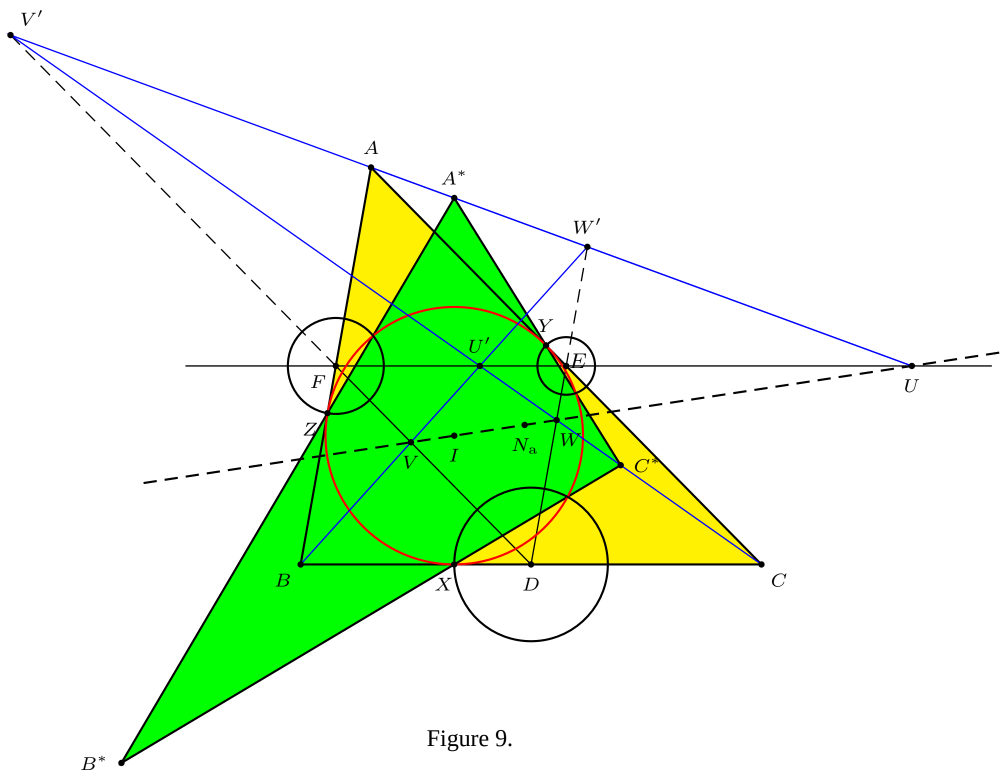

Figure 9.

on GV, and therefore, it must be W. It follows that BV and CW are concurrent in the point U' on EF.

By similarity of triangles, we have $\frac{DB}{DV} = \frac{FU'}{FV}$ and $\frac{DC}{DW} = \frac{EU'}{EW}$. This gives us:

$$\frac{WE}{WD} \cdot \frac{VD}{VF} \cdot \frac{U'F}{U'E} = \frac{EU'}{DC} \cdot \frac{DB}{FU'} \cdot \frac{U'F}{U'E} = \frac{DB}{DC} = -1.$$

Hence DU', EV, FW are concurrent by Ceva's theorem applied to triangle DEF. By Menelaus's theorem applied to the transversal WVU we obtain that U' is the harmonic conjugate of U with respect to E and F. Therefore, it is a center of similitude of $\mathcal{C}_b$ and $\mathcal{C}_c$. □

Let us call X'', Y'', Z'' the antipodes of X, Y, Z respectively on the incircle Γ.

Theorem 13. *The point X'' is the center of similitude different from U of circles $\mathcal{C}'_a$ and $\mathcal{C}^*_a$. Likewise, Y'' is a center of similitude of $\mathcal{C}'_b$ and $\mathcal{C}^*_b$, and Z'' one of $\mathcal{C}'_c$ and $\mathcal{C}^*_c$.*

Proof. We construct the line $l_{X''}$ which passes through X'' and is parallel to BC. The triangle bounded by $AC, AB, l_{X''}$ has Γ as its excircle opposite A. This implies that its Nagel point lies on AX'', and because it is homothetic to triangle ABC from center A, we have that X'' lies on AN_{a}. We have also proved that A^*,

I, X are collinear, so it follows that X'' lies on A^*I. Hence the intersection point of AN_{a} and A^*I is X'', a center of similitude of $\mathcal{C}_a$ and $\mathcal{C}_a^*$, different from U. □

Having classified all similitude centers of the pairs of circles $\mathcal{C}'_a, \mathcal{C}^*_a$ and $\mathcal{C}_b, \mathcal{C}_c$ (and we obtain similar results for the other pairs of circles), we now establish a surprising concurrency. Not only does this involve hitherto inconspicuous points introduced at the beginning of §2, it also strongly relates the triangle $A^*B^*C^*$ to the Nagel line of ABC.

Theorem 14. *The triangles $A^*B^*C^*$ and $X^*Y^*Z^*$ are perspective at a point on the Nagel line* (see Figure 10).

Proof. Considering the powers of A^*, B^*, C^* with respect to the incircle Γ of triangle ABC, we have

$$A^*Z{\cdot}A^*Z^* = A^*Y^*{\cdot}A^*Y, \quad B^*X^*{\cdot}B^*X = B^*Z^*{\cdot}B^*Z, \quad C^*X{\cdot}C^*X^* = C^*Y{\cdot}C^*Y^*.$$

From these,

$$\begin{aligned}\frac{B^*X^*}{X^*C^*}\cdot\frac{C^*Y^*}{Y^*A^*}\cdot\frac{A^*Z^*}{Z^*B^*} &= \frac{B^*X^*}{Z^*B^*}\cdot\frac{C^*Y^*}{X^*C^*}\cdot\frac{A^*Z^*}{Y^*A^*}\\ &= \frac{B^*Z}{XB^*}\cdot\frac{C^*X}{YC^*}\cdot\frac{A^*Y}{ZA^*} = \frac{B^*Z}{ZA^*}\cdot\frac{C^*X}{XB^*}\cdot\frac{A^*Y}{YC^*} = 1\end{aligned}$$

since $A^*B^*C^*$ and XYZ are perspective. By Ceva's theorem, we conclude that $A^*B^*C^*$ and $X^*Y^*Z^*$ are perspective, *i.e.*, A^*X^*, B^*Y^*, C^*Z^* intersect at a point Q.

To prove that Q lies on the Nagel line, however, we have to go a considerable step further. First, note that $A_b^*Y^*ZA_c$ is a cyclic quadrilateral, because $XA_b^* \cdot XY^* = XA_c \cdot XZ$ using Theorem 9. We call N_c the point where DE meets ZY^* and working with directed angles we deduce that

$$\measuredangle ZY^*A_b^* = \measuredangle ZA_cU = \measuredangle N_cA_bU = \measuredangle N_cA_bA_b^* = \measuredangle N_cY^*A_b^*$$

We conclude that N_c, Y^*, Z and therefore also Z, Y^*, U are collinear. Similar proofs show that

$$U \in YZ^*,\ V \in XZ^*,\ V \in ZX^*, W \in XY^*,\ W \in YX^*.$$

If we construct the intersection points

$$J = FZ^* \cap BC \quad \text{and} \quad K = DX^* \cap AB,$$

we know that the pole of JK with respect to Γ is the intersection of XZ^* with X^*Z, which is V. The fact that JK is the polar line of V shows that B^* lies on JK, and that JK is perpendicular to the Nagel line.

Now we construct the points

$$O = EF \cap DX^*, \quad P = DE \cap FZ^*, \quad R = OD \cap FZ^*.$$

Recalling Lemma 1 and the definitions of X^* and Z^* following Lemma 3, we see that OP is the polar line of Q with respect to Γ. We also know by similarity of the triangles ORF and DRJ that $OR \cdot RJ = DR \cdot RF$. Likewise, we find by similarity of the triangles KFR and DPR that $RF \cdot DR = KR \cdot RP$. Combining these identities we get $OR \cdot RJ = KR \cdot RP$, and this proves that OP and JK are

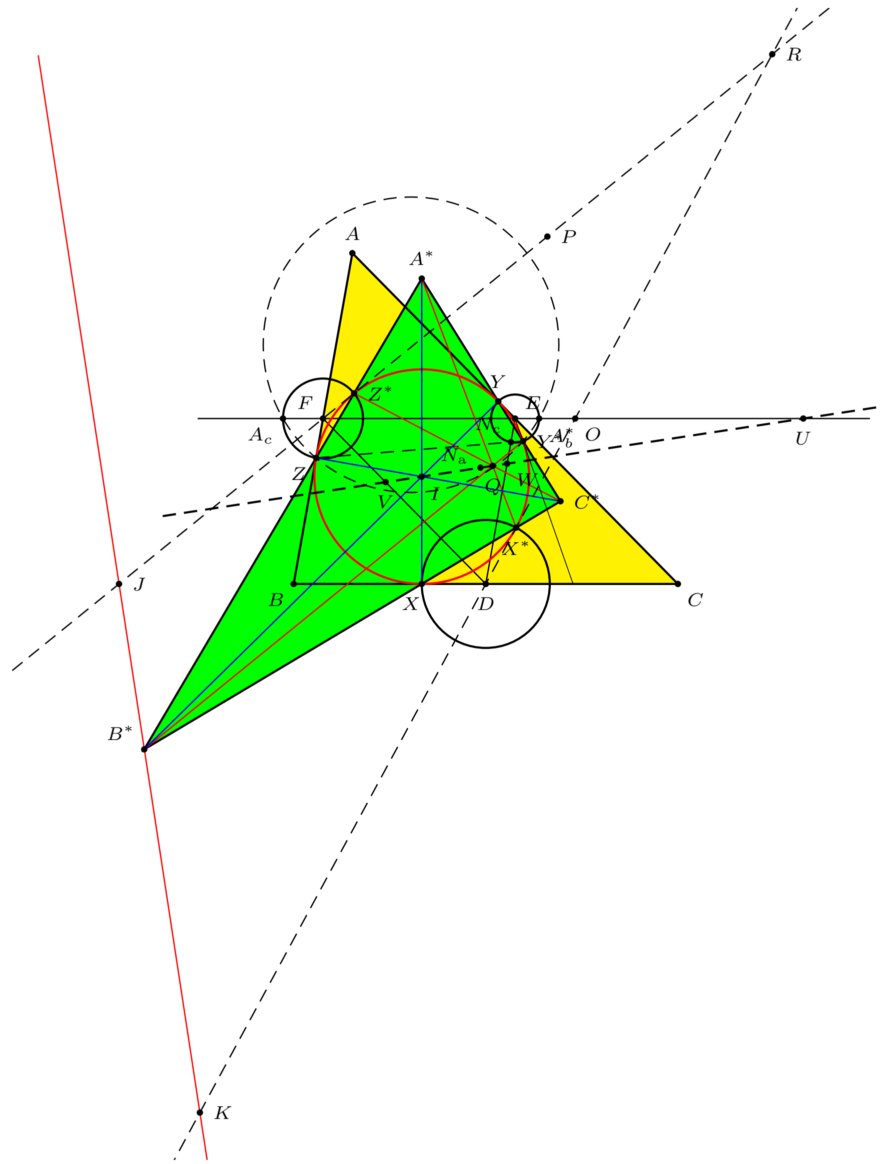

Figure 10.

parallel. Thus, OP is perpendicular to the Nagel line, whence its pole Q lies on the Nagel line. □

4. The Feuerbach point

Theorem 15. *The line connecting the centers of $\mathcal{C}'_a$ and $\mathcal{C}^*_a$ passes through the Feuerbach point of triangle ABC; so do the lines joining the centers of $\mathcal{C}'_b$, $\mathcal{C}^*_b$ and those of $\mathcal{C}'_c$, $\mathcal{C}^*_c$* (see Figure 11).

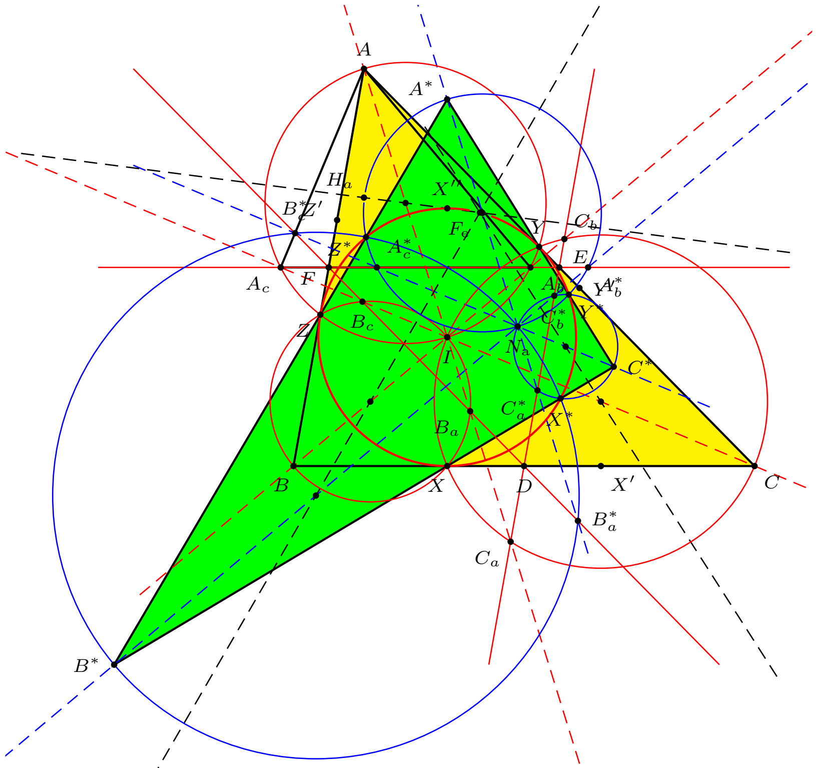

Figure 11.

Proof. Let us call H_a the orthocenter of triangle AA_bA_c. Since AI is the diameter of $\mathcal{C}'_a$ (as in the proof of Theorem 6), we have $AH_a = AI \cdot \cos A_bAA_c = AI \cdot \sin \frac{A}{2}$, where the last equality follows from $\frac{\pi}{2} - \frac{A}{2} = \angle BIC = \angle A_bIA_c = \pi - \angle A_bAA_c$. By observing triangle AIZ, for instance, and writing r for the inradius of triangle ABC we find that

$$AH_a = AI \cdot \sin \frac{A}{2} = r.$$

Now consider the homothety χ with factor -1 centered at the midpoint of AI (which is also the center of $\mathcal{C}'_a$). We have that $\chi(A) = I$ and $\chi(AH_a) = A^*I$. But we just proved that $AH_a = r = IX''$, so it follows that $\chi(H_a) = X''$. This shows that X'' lies on the Euler line of triangle AA_bA_c, so the line joining the centers of $\mathcal{C}'_a$ and $\mathcal{C}^*_a$ is exactly the Euler line of triangle AA_aA_b.

According to A. Hatzipolakis ([3]; see also [5]), the Euler line of triangle AA_bA_c passes through the Feuerbach point of triangle ABC. From this our conclusion follows immediately. □

In summary, the Euler line of triangle AA_bA_c and the Nagel line of triangle ABC intersect on EF. We will show that the circles $\mathcal{C}_a$, $\mathcal{C}^*_a$ have another amazing connection to the Feuerbach point.

Theorem 16. *The radical axis of $\mathcal{C}'_a$ and $\mathcal{C}^*_a$ passes through the Feuerbach point of triangle ABC; so do the radical axes of $\mathcal{C}'_b$, $\mathcal{C}^*_b$, and of $\mathcal{C}'_c$, $\mathcal{C}^*_c$* (see Figure 12).

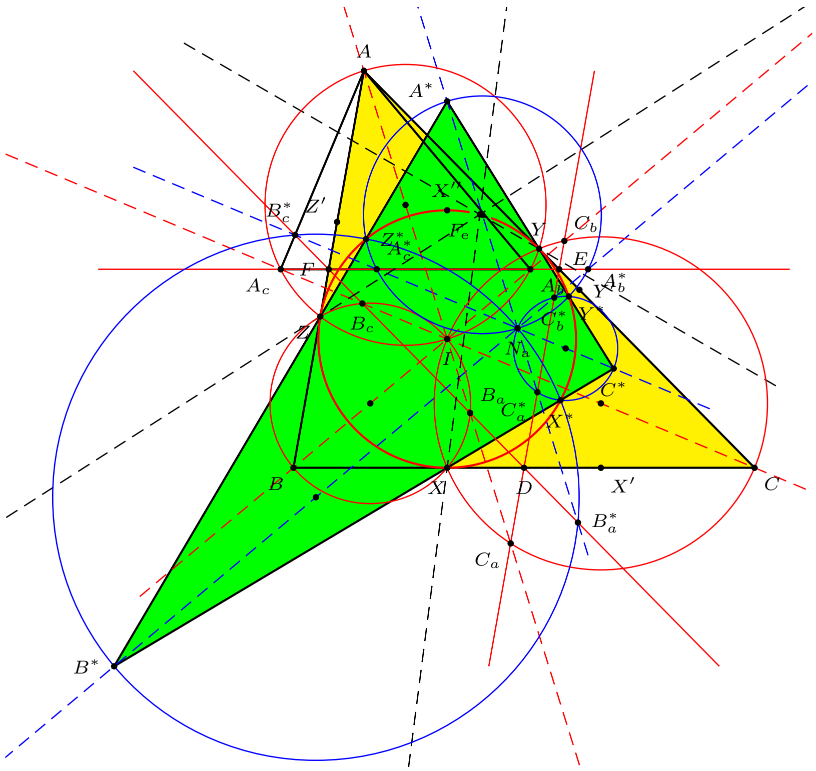

Figure 12.

Proof. Because the radical axis of two circles is perpendicular to the line joining the centers of the circles, the radical axis $\mathcal{R}_a$ of $\mathcal{C}'_a$ and $\mathcal{C}^*_a$ is perpendicular to the Euler line of triangle AA_bA_c. Since this Euler line contains X'', and $\mathcal{R}_a$ contains X (see Theorem 9), their intersection lies on Γ. This point is also the intersection point of the Euler line with Γ, different from X''. It is the Feuerbach point of ABC. □

References

[1] J. P. Ehrmann, Hyacinthos message 6130, December 10, 2002.
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[3] A. Hatzipolakis, Hyacinthos message 10485, September 18, 2004.
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