On the Nagel Line and a Prolific Polar Triangle

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Abstract. For a given triangle $ABC$, the polar triangle of the medial triangle with respect to the incircle is shown to have as its vertices the orthocenters of triangles $AIB$, $BIC$ and $AIC$. We prove results which relate this polar triangle to the Nagel line and, eventually, to the Feuerbach point.

1. A prolific triangle

In a triangle $ABC$ we construct a triad of circles $C_a, C_b, C_c$ that are orthogonal to the incircle $\Gamma$ of the triangle, with their centers at the midpoints $D, E, F$ of the sides $BC, AC, AB$. These circles pass through the points of tangency $X, Y, Z$ of the incircle with the respective sides. We denote by $\ell_a$ (respectively $\ell_b, \ell_c$) the radical axis of $\Gamma$ and $C_a$ (respectively $C_b, C_c$), and examine the triangle $A^*B^*C^*$ bounded by these lines (see Figure 1). J.-P. Ehrmann [1] has shown that this triangle has the same area as triangle $ABC$.

Lemma 1. The triangle $A^*B^*C^*$ is the polar triangle of the medial triangle $DEF$ of triangle $ABC$ with respect to $\Gamma$.


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Proof. Because $C_a$ is orthogonal to $\Gamma$, the line $\ell_a$ is the polar of $D$ with respect to $\Gamma$. Similarly, $\ell_b$ and $\ell_c$ are the polars of $E$ and $F$ with respect to the same circle. \qed

Note that Lemma 1 implies that triangles $A^*B^*C^*$ and $XYZ$ are perspective with center $I$: $A^*I \perp EF$ because $EF$ is the polar line of $A^*$ with respect to $\Gamma$. Because $EF \parallel BC$ and $BC \perp XI$, the assertion follows.

Lemma 2. The lines $XY$, $BI$, $EF$, and $AC^*$ are concurrent at a point of $C_b$, as are the lines $YZ$, $BI$, $DE$, and $AB^*$ (see Figure 2).

Proof. Let $A_b$ as the point on $EF$, on the same side of $F$ as $E$, so that $FA_b = FA$.

(i) Because $FA = FA_b = FB$, the points $A$, $A_b$, and $B$ all lie on a circle with center $F$. This implies that $\angle ABC = \angle AFA_b = 2\angle ABA_b$, yielding $\angle ABI = \angle ABA_b$. This shows that $A_b$ lies on $BI$.

(ii) Because $YC = \frac{1}{2}(AC + CB - BA) = EC + EF - FA$, we have

$$EY = YC - EC = EF - FA = FE - FA_b = EA_b,$$

showing that $A_b$ lies on $C_b$. Also, noting that $CX = CY$, we have $\frac{EY}{CY} = \frac{EA_b}{CX}$. This implies that triangles $EYA_b$ and $CYX$ are isosceles and similar. From this we deduce that $A_b$ lies on $XY$.

A similar argument shows that $DE$, $BI$, $YZ$ are concurrent at a point $C_b$ on the circle $C_b$. We will use this to prove the last part of this lemma.

(iii) Because $YZ$ and $DE$ are the polar lines of $A$ and $C^*$ with respect to $\Gamma$, $AC^*$ is the polar line of $C_b$, which also lies on $BI$. This implies that $AC^* \perp BI$, so the intersection of $AC^*$ and $BI$ lies on the circle with diameter $AB$. We have shown that $A_b$ lies on this circle, and on $BI$, so $A_b$ also lies on $AC^*$.
Similarly, $C_b$ also lies on the line $AB^*$.

Note that the points $A_b$ and $C_b$ are the orthogonal projections of $A$ and $C$ on $BI$. Analogous statements can be made of quadruples of lines intersecting on the circles $C_a$ and $C_c$. Reference to this configuration can be found, for example, in a problem on the 2002−2003 Hungarian Mathematical Olympiad. A solution and further references can be found in *Crux Mathematicorum with Mathematical Mayhem*, 33 (2007) 415–416.

We are now ready for our first theorem, conjectured in 2002 by D. Grinberg [2].

**Theorem 3.** The points $A^*$, $B^*$, and $C^*$ are the respective orthocenters of triangles $BIC$, $CIA$, and $AIB$.

**Proof.** Because the point $A_b$ lies on the polar lines of $A^*$ and $C$ with respect to $\Gamma$, we know that $A^*C \perp BI$. Combining this with the fact that $A^*I \perp BC$ we conclude that $A^*$ is indeed the orthocenter of triangle $BIC$. □

**Theorem 4.** The medial triangle $DEF$ is perspective with triangle $A^*B^*C^*$, at the Mittenpunkt $M_t$ of triangle $ABC$ (see Figure 3).

**Proof.** Because $A^*C$ is perpendicular to $BI$, it is parallel to the external bisector of angle $B$. A similar argument holds for $BA^*$, so we conclude that $A^*BI_aC$ is a parallelogram. It follows that $A^*$, $D$, and $I_a$ are collinear. This shows that $M_t$ lies on $I_aD$, and similar arguments show that $M_t$ lies on the lines $I_bE$ and $I_cF$. □

We already know that triangle $A^*B^*C^*$ and triangle $XYZ$ are perspective at the incenter $I$. By proving Theorem 4, we have in fact found two additional triangles that are perspective with triangle $A^*B^*C^*$: the medial triangle $DEF$ and the

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1The Mittenpunkt (called $X(9)$ in [4]) is the point of concurrency of the lines joining $D$ to the excenter $I_a$, $E$ to the excenter $I_b$, and $C$ to the excenter $I_c$. It is also the symmedian point of the excentral triangle $I_aI_bI_c$. 
excentral triangle $I_aI_bI_c$, both with center $M_t$. This is however just a taste of the many special properties of triangle $A^*B^*C^*$, which will be treated throughout the rest of this paper.

Theorem 3 shows that $B, C, A^*, I$ are four points that form an orthocentric system. A consequence of this is that $I$ is the orthocenter of triangles $A^*BC$, $AB^*C$, $ABC^*$. In the following theorem we prove a similar result that will produce an unexpected point.

**Theorem 5.** The Nagel point $N_a$ of triangle $ABC$ is the common orthocenter of triangles $AB^*C^*$, $A^*BC^*$, $A^*B^*C$.

**Proof.** Consider the homothety $\zeta := h(D, -1)$. $^2$ This carries $A$ into the vertex $A'$ of the anticomplementary triangle $A'B'C'$ of $ABC$. It follows from Theorem 4 that $\zeta(A') = I_o$. This implies that $A'A^*$ is the bisector of $\angle BA'C$.

The Nagel line is the line $IG$ joining the incenter and the centroid. It is so named because it also contains the Nagel point $N_a$. Since $2IG = GN_a$, the Nagel point $N_a$ is the incenter of the anticomplementary triangle. This implies that $A'N_a$ is the bisector of $\angle BA'C$. Hence, $\zeta$ carries $A^*N_a$ into $AI$, so $A^*N_a$ and $AI$ are parallel. From this, $A^*N_a \perp CB^*$.

Similarly, we deduce that $B^*N_a \perp CA^*$, so $N_a$ is the orthocenter of triangle $A^*B^*C$.

The next theorem was proved by J.-P. Ehrmann in [1] using barycentric coordinates. We present a synthetic proof here.

**Theorem 6** (Ehrmann). The centroid $G^*$ of triangle $A^*B^*C^*$ is the point dividing $IH$ in the ratio $IG^*: G^*H = 2:1$.

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$^2$A homothety with center $P$ and factor $k$ is denoted by $h(P, k)$. 
**Proof.** The four points $A, A_b, I, A_c$ all lie on a circle with diameter $IA$, which we will call $C_a'$. Let $H$ be the orthocenter of triangle $ABC$, and $S$ the (second) intersection of $C_a'$ with the altitude $AH$. Construct also the parallel $AT$ to $B^*C^*$ through $A$ to intersect the circle at $T$ (see Figure 5).

Denote by $R_b$ and $R_c$ the circumradii of triangles $AIC$ and $AIB$ respectively. Because $C^*$ is the orthocenter of triangle $AIB$, we can write $AC^* = R_c \cdot \cos \frac{\angle B}{2}$, and similarly for $AB^*$. Using this and the property $B^*C^* \parallel AT$, we have

$$\frac{\sin TAA_b}{\sin TAA_c} = \frac{\sin AC^*B^*}{\sin AB^*C^*} = \frac{AB^*}{AC^*} = \frac{R_b}{R_c} = \frac{\sin \frac{B}{2}}{\sin \frac{C}{2}} = \frac{IC}{IB}$$

The points $A_b, A_c$ are on $EF$ according to Lemma 2, so triangle $IA_bA_c$ and triangle $IBC$ are similar. This implies $\frac{IC}{IB} = \frac{IA_c}{IA_b}$.

In any triangle, the orthocenter and circumcenter are known to be each other’s isogonal conjugates. Applying this to triangle $AA_bA_c$, we find that $\angle SAA_b = \angle A_cAI$. We can now see that $\frac{SA_b}{SA_c} = \frac{IA_c}{IA_b}$.

Combining these results, we obtain

$$\frac{SA_b}{SA_c} = \frac{IA_c}{IA_b} = \frac{IC}{IB} = \frac{\sin TAA_b}{\sin TAA_c} = \frac{T_A B}{T_A C}.$$
This proves that $TA_c \cdot SA_b = SA_c \cdot TA_b$, so $TA_cSA_b$ is a harmonic quadrilateral. It follows that $AC^*, AB^*$ divide $AH, AT$ harmonically. Because $AT \parallel B^*C^*$, we know that $AH$ must pass through the midpoint of $B^*C^*$.

Let us call $D^*$ the midpoint of $B^*C^*$, and consider the homothety $\xi = h(G^*, -2)$. Because $\xi$ takes $D^*$ to $B^*$ while $AH \parallel A^*X$, we know that $\xi$ takes $AH$ to $A^*X$. Similar arguments applied to $B$ and $B^*$ establish that $\xi$ takes $H$ to $I$. $\blacksquare$

2. Two more triads of circles

Consider again the orthogonal projections $A_b, A_c$ of $A$ on the bisectors $BI$ and $CI$. It is clear that the circle $C'_a$ with diameter $IA$ in Theorem 6 contains the points $Y$ and $Z$ as well. Similarly, we consider the circles $C'_b$ and $C'_c$ with diameters $IB$ and $IC$ (see Figure 6). It is easy to determine the intersections of the circles from the two triads $C_a, C_b, C_c$, and $C'_a, C'_b, C'_c$, which we summarize in the following table.

<table>
<thead>
<tr>
<th></th>
<th>$C'_a$</th>
<th>$C'_b$</th>
<th>$C'_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_a$</td>
<td>$X, B_a$</td>
<td>$X, C_a$</td>
<td></td>
</tr>
<tr>
<td>$C_b$</td>
<td>$Y, A_b$</td>
<td>$Y, X_b$</td>
<td></td>
</tr>
<tr>
<td>$C_c$</td>
<td>$Z, A_c$</td>
<td>$Z, B_c$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Intersections of circles

Now we introduce another triad of circles.

Let $X^*$ (respectively $Y^*, Z^*$) be the intersection of $\Gamma$ with $C_a$ (respectively $C_b$, $C_c$) different from $X$ (respectively $Y$, $Z$). Consider also the orthogonal projections $A_b^*$ and $A_c^*$ of $A^*$ onto $B^*N_a$ and $C^*N_a$, and similarly defined $B_a^*, B_c^*, C_a^*, C_b^*$.

Lemma 7. The six points $A^*, A_b^*, A_c^*, Y^*, Z^*$, and $N_a$ all lie on the circle with diameter $A^*N_a$ (see Figure 6).

Proof: The points $A_b^*$ and $A_c^*$ lie on the circle with diameter $A^*N_a$ by definition.

We know that the Nagel point and the Gergonne point are isotomic conjugates, so if we call $Y'$ the intersection of $BN_a$ and $AC$, it follows that $YE = Y'E$. Therefore, $Y'$ lies on $C_b$.

Clearly $YY'$ is a diameter of $C_b$. It follows from Theorem 5 that $BN_a$ is perpendicular to $A^*C^*$, so their intersection point must lie on $C_b$. Since $Y^*$ is the intersection point of $A^*C^*$ and $C_b$ different from $Y$, it follows that $Y^*$ lies on $BN_a$.

Combining the above results, we obtain that $N_aY^* \perp A^*Y^*$, so $Y^*$ lies on the circle with diameter $A^*N_a$. A similar proof holds for $Z^*$. $\blacksquare$

We will call this circle $C^*_a$. Likewise, $C^*_b$ and $C^*_c$ are the ones with diameters $B^*N_a$ and $C^*N_a$. Here are the intersections of the circles in the two triads $C_a, C_b, C_c$, and $C^*_a, C^*_b, C^*_c$.  


Table 2. Intersections of circles

<table>
<thead>
<tr>
<th></th>
<th>$C_a^*$</th>
<th>$C_b^*$</th>
<th>$C_c^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_a$</td>
<td>$X^<em>$, $B_a^</em>$</td>
<td>$X^<em>$, $C_a^</em>$</td>
<td></td>
</tr>
<tr>
<td>$C_b$</td>
<td>$Y^<em>$, $A_b^</em>$</td>
<td>$Y^<em>$, $X_b^</em>$</td>
<td></td>
</tr>
<tr>
<td>$C_c$</td>
<td>$Z^<em>$, $A_c^</em>$</td>
<td>$Z^<em>$, $B_c^</em>$</td>
<td></td>
</tr>
</tbody>
</table>

Lemma 8. The circle $C_a^*$ intersects $C_b$ in the points $Y^*$ and $A_b^*$. The point $A_b^*$ lies on $EF$ (see Figure 7).

Proof. The point $Y^*$ lies on $C_b$ by definition, and on $C_a^*$ by Lemma 7.

Consider the homothety $\phi := h(E, -1)$. We already know that $\phi(AC^*) = CA^*$ and $\phi(BI) = B^*N_a$. This shows that the intersection points are mapped onto each other, or $\phi(A_b) = A_b^*$. It follows that $A_b^*$ lies on $C_b$ and $EF$. 

□
The two triads of circles have some remarkable properties, strongly related to the Nagel line and eventually to the Feuerbach point. We will start with a property that may be helpful later on.

**Theorem 9.** The point $X$ has equal powers with respect to the circles $C_b$, $C_c$, $C^*_a$, and $C'_a$ (see Figure 7).

*Proof.* Let us call $S_a$ the intersection of $EY^*$ and $BC$, and $S_b$ the intersection of $XY^*$ and $EF$. Because $EY^*$ is tangent to $\Gamma$, we have $S_aY^* = S_bX$. Because triangles $XS_aY^* = S_bEY^*$ are similar, it follows that $EY^* = ES_b$. This implies that $S_b$ lies on $\Gamma$ so in fact $S_b$ and $A^*_b$ coincide. This shows that $X$ lies on $Y^*A^*_b$.

Similar arguments can be used to prove that $X$ lies on $Z^*A^*_c$.

From Table 1, it follows that $A_bY$ (respectively $A_cZ$) is the radical axis of the circles $C'_a$ and $C_b$ (respectively $C_c$). Lemma 2 implies that $X$ lies on both $A_bY$ and $A_cZ$, so it is the radical center of $C'_a$, $C_b$, and $C_c$.

From Lemma 8, it follows that $Y^*A^*_b$ (respectively $Z^*A^*_c$) is the radical axis of the circles $C_b$ and $C^*_a$ (respectively $C_c$ and $C'_b$). We have just proved that $X$ lies on both $Y^*A^*_b$ and $Z^*A^*_c$, so it is the radical center of $C'_a$, $C_b$, and $C_c$. The conclusion follows. \[\Box\]

### 3. Some similitude centers and the Nagel line

Denote by $U$, $V$, $W$ the intersections of the Nagel line $IG$ with the lines $EF$, $DF$ and $DE$ respectively (see Figure 8).
Theorem 10. The point $U$ is a center of similitude of circles $C'_a$ and $C'^*_a$. Likewise, $V$ is a center of similitude of circles $C'_b$ and $C'^*_b$, and $W$ of $C'_c$ and $C'^*_c$.

Proof. We know from Lemma 2 and Theorem 5 that $A^*A^*_b \parallel AA_b$ and $AI \parallel A^*N_a$, as well as $A^*_bN_a \parallel A_bI$. Hence triangles triangle $A^*N_aA^*_b$ and triangle $AIA_b$ have parallel sides. It follows from Desargues’ theorem that $AA^*_c$, $A^*_bA^*_c$, $IN_a$ are concurrent. Clearly, the point of concurrency is a center of similitude of both triangles, and therefore also of their circumcircles, $C'^*_a$ and $C'^*_a$. This point of concurrency is the intersection point of $EF$ and the Nagel line as shown above, so the theorem is proved.

Theorem 11. The point $U$ is a center of similitude of circles $C_b$ and $C_c$. Likewise, $V$ is a center of similitude of circles $C_c$ and $C_a$, and $W$ of $C_a$ and $C_b$.

Proof. By Theorem 10, we know that

$$\frac{A_bU}{A_cU} = \frac{A^*_bU}{A^*_cU}.$$ (1)

By Table 1 and Theorem 8, we know that $A_b, A^*_c$ lie on $C_c$ and $A^*_b, A^*_b$ lie on $C_b$. Knowing that $U$ lies on $EF$, the line connecting the centers of $C_b$ and $C_c$, relation (1) now directly expresses that $U$ is a center of similitude of $C_b$ and $C_c$.

Depending on the shape of triangle $ABC$, the center of similitude of $C_b$ and $C_c$ which occurs in the above theorem could be either external or internal. Whichever it is, we will meet the other in the next theorem.

Theorem 12. The lines $BV$ and $CW$ intersect at a point on $EF$. This point is the center of similitude different from $U$ of $C_b$ and $C_c$ (see Figure 9).

Proof. Let us call $U'$ the point of intersection of $BV$ and $EF$. We have that $G = BE \cap CF$ and $V = DF \cap BU'$. By the theorem of Pappus-Pascal applied to the collinear triples $E, U', F$ and $C, D, B$, the intersection of $U'C$ and $DE$ must lie
on $GV$, and therefore, it must be $W$. It follows that $BV$ and $CW$ are concurrent in the point $U'$ on $EF$.

By similarity of triangles, we have $\frac{DB}{DV} = \frac{FU'}{FV}$ and $\frac{DC}{DW} = \frac{EU'}{EW}$.

This gives us:

\[
\frac{WE}{WD} \cdot \frac{VD}{VF} \cdot \frac{U'F}{U'E} = \frac{EU'}{DC} \cdot \frac{DB}{FU'} \cdot \frac{U'F}{U'E} = \frac{DB}{DC} = -1.
\]

Hence $DU'$, $EV$, $FW$ are concurrent by Ceva’s theorem applied to triangle $DEF$. By Menelaus’s theorem applied to the transversal $WVU$ we obtain that $U'$ is the harmonic conjugate of $U$ with respect to $E$ and $F$. Therefore, it is a center of similitude of $C_b$ and $C_c$. □

Let us call $X''$, $Y''$, $Z''$ the antipodes of $X, Y, Z$ respectively on the incircle $\Gamma$.

**Theorem 13.** The point $X''$ is the center of similitude different from $U$ of circles $C'_a$ and $C'_a$. Likewise, $Y''$ is a center of similitude of $C'_b$ and $C'_b$, and $Z''$ one of $C'_c$ and $C'_c$.

**Proof.** We construct the line $l_{X''}$ which passes through $X''$ and is parallel to $BC$. The triangle bounded by $AC, AB, l_{X''}$ has $\Gamma$ as its excircle opposite $A$. This implies that its Nagel point lies on $AX''$, and because it is homothetic to triangle $ABC$ from center $A$, we have that $X''$ lies on $AN_a$. We have also proved that $A''$, ...
I, X are collinear, so it follows that \(XX'\) lies on \(A*I\). Hence the intersection point of \(AN_a\) and \(A*I\) is \(XX'\), a center of similitude of \(C_a\) and \(C'_a\), different from \(U\).  

Having classified all similitude centers of the pairs of circles \(C'_a, C'_a\) and \(C_b, C_c\) (and we obtain similar results for the other pairs of circles), we now establish a surprising concurrency. Not only does this involve hitherto inconspicuous points introduced at the beginning of §2, it also strongly relates the triangle \(A*B*C*\) to the Nagel line of \(ABC\).

**Theorem 14.** The triangles \(A*B*C*\) and \(X*Y*Z*\) are perspective at a point on the Nagel line (see Figure 10).

**Proof.** Considering the powers of \(*A*, \(*B*, \(*C*\) with respect to the incircle \(\Gamma\) of triangle \(ABC\), we have
\[
A*Z * A*Z* = A*Y* \cdot A*Y, \quad B*Z * B*Z = B*Z* \cdot B*Z, \quad C*X* \cdot C*X* = C*Y* \cdot C*Y*.
\]
From these,
\[
\frac{B*X*}{X*C*} \cdot \frac{C*Y*}{Y*A*} \cdot \frac{A*Z*}{Z*B*} = \frac{B*X*}{Z*B*} \cdot \frac{C*Y*}{X*C*} \cdot \frac{A*Z*}{Y*A*} = \frac{B*Z}{X*B*} \cdot \frac{C*X}{Y*C*} \cdot \frac{A*Y}{Z*A*} = \frac{B*Z*}{Z*A*} \cdot \frac{C*Y*}{X*B*} \cdot \frac{A*Y*}{Y*C*} = 1
\]
since \(A*B*C*\) and \(XYZ\) are perspective. By Ceva’s theorem, we conclude that \(A*B*C*\) and \(X*Y*Z*\) are perspective, i.e., \(A*X*, B*Y*, C*Z*\) intersect at a point \(Q\).

To prove that \(Q\) lies on the Nagel line, however, we have to go a considerable step further. First, note that \(A*Y*ZA_c\) is a cyclic quadrilateral, because \(X \cdot B*Y* = X \cdot A_c \cdot XZ\) using Theorem 9. We call \(N_e\) the point where \(DE\) meets \(ZY*\) and working with directed angles we deduce that
\[
\angle ZY* \cdot A_b^* = \angle ZA_eU = \angle N_eA_bU = \angle N_eA_cA_b^* = \angle N_cY* \cdot A_b^*
\]

We conclude that \(N_e, Y*, Z\) and therefore also \(Z, Y*, U\) are collinear. Similar proofs show that
\[
U \in YZ*, \quad V \in XZ*, \quad V \in ZX*, \quad W \in XY*, \quad W \in YX*.
\]
If we construct the intersection points
\[
J = FZ* \cap BC \quad \text{and} \quad K = DX* \cap AB,
\]
we know that the pole of \(JK\) with respect to \(\Gamma\) is the intersection of \(XZ*\) with \(X*Z\), which is \(V\). The fact that \(JK\) is the polar line of \(V\) shows that \(B*\) lies on \(JK\), and that \(JK\) is perpendicular to the Nagel line.

Now we construct the points
\[
O = EF \cap DX*, \quad P = DE \cap FZ*, \quad R = OD \cap FZ*.
\]
Recalling Lemma 1 and the definitions of \(X*\) and \(Z*\) following Lemma 3, we see that \(OP\) is the polar line of \(Q\) with respect to \(\Gamma\). We also know by similarity of the triangles \(ORF\) and \(DRJ\) that \(OR \cdot RJ = DR \cdot RF\). Likewise, we find by similarity of the triangles \(KFR\) and \(DPR\) that \(RF \cdot DR = KR \cdot RP\). Combining these identities we get \(OR \cdot RJ = KR \cdot RP\), and this proves that \(OP\) and \(JK\) are
parallel. Thus, $OP$ is perpendicular to the Nagel line, whence its pole $Q$ lies on the Nagel line. □

4. The Feuerbach point

**Theorem 15.** The line connecting the centers of $C_a^\prime$ and $C_a^\ast$ passes through the Feuerbach point of triangle $ABC$; so do the lines joining the centers of $C_b^\prime$, $C_b^\ast$ and those of $C_c^\prime$, $C_c^\ast$ (see Figure 11).
Proof. Let us call $H_a$ the orthocenter of triangle $AA_bA_c$. Since $AI$ is the diameter of $C'_a$ (as in the proof of Theorem 6), we have $AH_a = AI \cdot \cos \angle A_bAA_c = AI \cdot \sin \frac{\pi}{2}$, where the last equality follows from $\pi - \frac{A}{2} = \angle BIC = \angle A_bIA_c = \pi - \angle A_bAA_c$.

By observing triangle $AIZ$, for instance, and writing $r$ for the inradius of triangle $ABC$, we find that $AH_a = AI \cdot \sin \frac{A}{2} = r$.

Now consider the homothety $\chi$ with factor $-1$ centered at the midpoint of $AI$ (which is also the center of $C'_a$). We have that $\chi(A) = I$ and $\chi(AH_a) = A^* I$. But we just proved that $AH_a = r = IX''$, so it follows that $\chi(H_a) = X''$. This shows that $X''$ lies on the Euler line of triangle $AA_bA_c$, so the line joining the centers of $C'_a$ and $C''_a$ is exactly the Euler line of triangle $AA_bA_c$.

According to A. Hatzipolakis ([3]; see also [5]), the Euler line of triangle $AA_bA_c$ passes through the Feuerbach point of triangle $ABC$. From this our conclusion follows immediately.

In summary, the Euler line of triangle $AA_bA_c$ and the Nagel line of triangle $ABC$ intersect on $EF$. We will show that the circles $C_a, C''_a$ have another amazing connection to the Feuerbach point.
Theorem 16. The radical axis of \( C'_a \) and \( C^*_a \) passes through the Feuerbach point of triangle \( ABC \); so do the radical axes of \( C'_b \), \( C^*_b \), and of \( C'_c \), \( C^*_c \) (see Figure 12).

Proof. Because the radical axis of two circles is perpendicular to the line joining the centers of the circles, the radical axis \( R_a \) of \( C'_a \) and \( C^*_a \) is perpendicular to the Euler line of triangle \( AA_bA_c \). Since this Euler line contains \( X'' \), and \( R_a \) contains \( X \) (see Theorem 9), their intersection lies on \( \Gamma \). This point is also the intersection point of the Euler line with \( \Gamma \), different from \( X'' \). It is the Feuerbach point of \( ABC \). □

References


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