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An Elementary Proof of a Theorem by Emelyanov

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Abstract. In this note, we provide an alternative proof of a theorem by Lev Emelyanov stating that the Miquel point of any complete quadrilateral (in general position) lies on the nine-point circle of the triangle formed by the diagonals of that same complete quadrilateral.

1. Introduction and terminology

In their recent book on the geometry of conics, Akopyan and Zaslavsky prove a curious theorem by Lev Emelyanov on complete quadrilaterals. Their proof is very concise, but it does rely on the theory of conic sections, as presumably does Emelyanov's original proof. Indeed, it is the authors' contention that the theorem does not seem to allow for a "short and simple" proof without using the so-called inscribed parabola of the complete quadrilateral.[1] In this note, we will show that actually it is possible to avoid the use of conic sections and to give a proof that uses elementary means only. It is left to the reader to decide whether our proof is reasonably short and simple.

Recall that a *complete* quadrilateral is usually defined as the configuration of four given lines, no three of which are concurrent, and the six points at which they intersect each other. For this paper, we will also assume that no two of the lines are parallel. Without loss of generality, we can think of a complete quadrilateral as the configuration associated with a quadrilateral $ABCD$ in the traditional sense with no two sides parallel and no two vertices coinciding, together with the points $F = AD \cap BC$ and $G = AB \cap CD$. By abuse of notation, we will refer to a generic complete quadrilateral as a *complete* quadrilateral $\square ABCD$, where we will assume that none of the sides of $ABCD$ are parallel and no three are concurrent. [2] The lines AC, BD and FG are known as the *diagonals* of $\square ABCD$. Let $AC \cap BD$ be denoted by E_{FG} and so on. Then, the triangle $\triangle E_{AC}E_{BD}E_{FG}$ formed by the diagonals of $\square ABCD$ is usually referred to as the *diagonal triangle* of $\square ABCD$ (see Figure 2). With these notations, we are now ready to prove Emelyanov's Theorem.

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[1]See [1, pp.110–111] for both the proof (which relies on two propositions proved earlier) and the authors' contention.

[2]Thus, for any quadrilateral $ABCD$ with F and G as above, $\square ABCD$, $\square AFCG$, and $\square BGDF$ and so on, all denote the same configuration.

2. Emelyanov's Theorem

We will prove Emelyanov's Theorem as a corollary to a slightly more general result. For this we first need the following lemma (see Figure 1).

Lemma 1. *For any complete quadrilateral* $\square ABCD$ (as defined above), *let* F_{BC} *be the unique point on* AD *such that* $F_{BC}E_{FG}$ *is parallel to* BC *and let* F_{DA}, G_{AB} *and* G_{CD} *be defined similarly. Finally, let* F_G *and* G_F *be the midpoints of* FE_{FG} *and* GE_{FG}, *respectively. Then* F_{BC}, F_{DA}, G_{AB}, G_{CD} *all four lie on the line* F_GG_F.

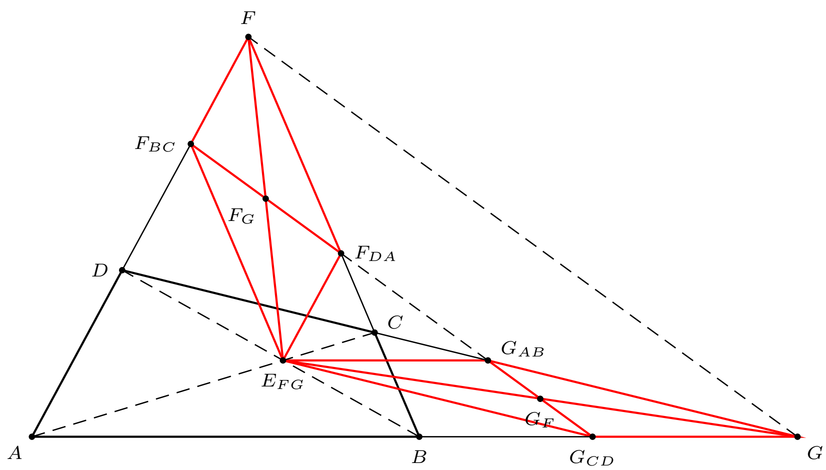

Figure 1. Collinearity of F_{BC}, F_{DA}, G_{AB}, G_{CD} and of F_G, G_F

Proof. Note that by the harmonic property of quadrilaterals, the sides DA and BC are harmonically separated by FE_{FG} and FG. Therefore, the points F_{DA} and F_{BC} are harmonically separated by the points of intersection $FE_{FG} \cap F_{DA}F_{BC}$ and $FG \cap F_{DA}F_{BC}$. By the construction of F_{DA} and F_{BC}, $E_{FG}F_{DA}FF_{BC}$ is a parallelogram and therefore $FE_{FG} \cap F_{DA}F_{BC}$ coincides with F_G. As F_G is also the midpoint of $F_{DA}F_{BC}$, it follows that $FG \cap F_{DA}F_{BC}$ has to be the point at infinity of $F_{DA}F_{BC}$. In other words, FG and $F_{DA}F_{BC}$ are parallel. As F_GG_F is parallel to FG as well and F_G also lies on $F_{DA}F_{BC}$, it follows that $F_{DA}F_{BC}$ and F_GG_F coincide. By the same argument, $G_{AB}G_{CD}$ coincides with F_GG_F as well. It follows that the six points are collinear. $\square$

Corollary 2. *With the notation introduced above, the directed ratios* $\dfrac{F_{BC}D}{F_{BC}A}$ *and* $\dfrac{F_{DA}C}{F_{DA}B}$ *are equal, as are the ratios* $\dfrac{G_{CD}A}{F_{CD}B}$ *and* $\dfrac{G_{AB}D}{F_{AB}C}$.

Proof. It suffices to prove the first part of the statement. Note that by construction the ratio $\dfrac{F_{BC}D}{F_{BC}A}$ is equal to the cross ratio $[E_{FG}D, E_{FG}A; E_{FG}F_{BC}, E_{FG}F_{DA}]$ of the lines $E_{FG}D$, $E_{FG}A$, $E_{FG}F_{BC}$, and $E_{FG}F_{DA}$. Similarly, the ratio $\dfrac{F_{DA}C}{F_{DA}B}$

equals the cross ratio $[E_{FG}C, E_{FG}B; E_{FG}F_{DA}, E_{FG}F_{BC}]$. As ED is parallel to EB, while EA is parallel to EC, the two cross ratios are equal. Therefore, the two ratios are equal as well. $\square$

We are now ready to derive our main result. We start with a lemma about Miquel points, which we prefer to associate to a complete quadrilateral $\square ABCD$, rather than to $ABCD$.

Lemma 3. *For any quadrilateral $ABCD$* (with its sides in general position), *the Miquel points of $\square ABF_{DA}F_{BC}$ and $\square CDF_{BC}F_{DA}$ both coincide with the Miquel point M of $\square ABCD$.*

Proof. Let M be constructed as the second point of intersection (other than F) of the circumcircles of $\triangle FAB$ and $\triangle FCD$. By Corollary 2, the ratio of the power of F_{BC} with respect to the circumcircle of $\triangle FCD$ and the power of F_{BC} with respect to the circumcircle of $\triangle FAB$ equals the ratio of the power of F_{DA} with respect to the same two circles. This means that F_{BC} and F_{DA} lie on the same circle of the coaxal system generated by the circumcircles of $\triangle FCD$ and $\triangle FAB$. In other words, F, F_{BC}, F_{DA} and M are co-cyclic. Since M lies on both the circumcircle of $\triangle F_{BC}F_{DA}F$ and the circumcircle of $\triangle FAB$, it follows that M is also the Miquel point of $\square ABF_{DA}F_{BC}$. By a similar argument, M is the Miquel point of $\square CDF_{BC}F_{DA}$ as well. $\square$

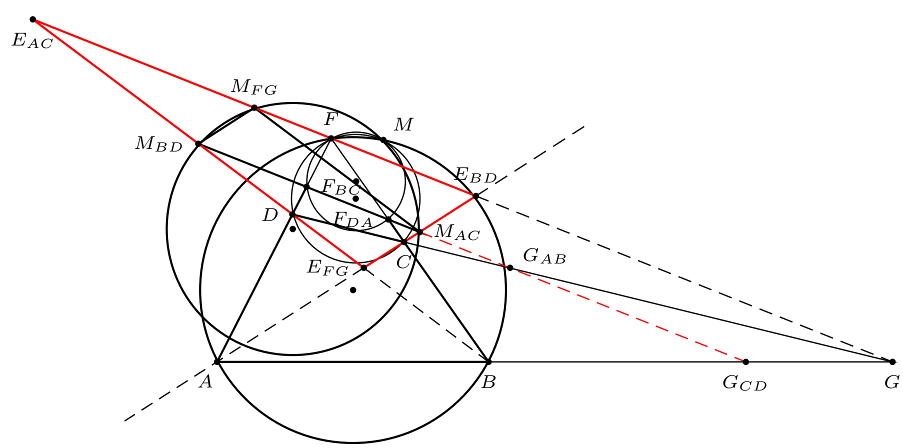

Figure 2. Coincidence of Miquel points

Corollary 4. *For any quadrilateral $ABCD$* (with sides in general position), *the (orthogonal) projection of M on $F_{BC}F_{DA}$ lies on the pedal line of $\square ABCD$.*

Proof. By Lemma 3 and the properties of Miquel points, the (orthogonal) projection of M on $F_{BC}F_{DA}$ is collinear with the (orthogonal) projections of M on AB, BF_{DA} and $F_{BC}A$, i.e. its projections on AB, BC, and DA. But for $ABCD$ in general position, the latter points do not all three coincide. As they also lie on the pedal line of $\square ABCD$, they therefore define the pedal line and the (orthogonal) projection of M on $F_{BC}F_{DA}$ has to lie on it. $\square$

Now, let M_{AC} be the midpoint of $E_{BD}E_{FG}$ and so on. Clearly, $M_{AC}M_{BD}$ coincides with $F_{BC}F_{DA}$. Furthermore, Corollary 4 applies to the quadrilaterals $AFCG$ and $BFDG$ as well. Since $\square AFCG$ and $\square BFDG$ coincide with $\square ABCD$, their Miquel points also coincide. These observations immediately lead to our main result.

Theorem 5. *For any quadrilateral* $ABCD$ (with sides in general position)*, the (orthogonal) projections of the Miquel point* M *of* $\square ABCD$ *on the sides of the triangle* $\triangle M_{AC}M_{BD}M_{FG}$ *all three lie on the pedal line of* $\square ABCD$*.*

Emelyanov's Theorem follows from Theorem 5 as a corollary.

Corollary 6 (Emelyanov)**.** *For any quadrilateral* $ABCD$ (with sides in general position)*, the Miquel point* M *of* $\square ABCD$ *lies on the nine-point circle of the diagonal triangle* $\triangle E_{AC}E_{BD}E_{FG}$ *of* $\square ABCD$*.*

Proof. Since the (orthogonal) projections of M on the sides of $\triangle M_{AC}M_{BD}M_{FG}$ are collinear, M has to lie on the circumcircle of $\triangle M_{AC}M_{BD}M_{FG}$. But this is the same as saying that M lies on the nine-point circle of $\triangle E_{AC}E_{BD}E_{FG}$. $\square$

3. Conclusion

In this note we derived an elementary proof of Emelyanov's Theorem as stated in [**?**] from a more general result. At this point, it is unclear to us whether this Theorem 5 may have any other implications than Emelyanov's Theorem, but it was not our goal to look for such implications. Similarly, we could have shortened our proof a little bit by noting that Corollary 2 implies that $F_{BC}F_{DA}$ is a tangent line to the unique inscribed parabola of $\square ABCD$. The same parabola therefore is also the inscribed parabola to $\square ABF_{DA}F_{BC}$ and $\square CDF_{BC}F_{DA}$. Since the focal point of the parabola inscribing a complete quadrilateral is the Miquel point of the same, Lemma 3 immediately follows. As stated in the introduction, however, our goal was to provide a proof of the theorem without using the theory of conic sections.

Reference

[1] A. V. Akopyan and A. A. Zaslavsky, *Geometry of Conics*, *Mathematical World*, Vol. 26, Amer. Math. Soc. 2007.

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