

A Generalization of Thébault's Theorem on the Concurrency of Three Euler Lines

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Abstract. We prove a generalization of Victor Thébault's theorem that if $H_aH_bH_c$ is the orthic triangle of ABC , then the Euler lines of triangles AH_cH_b , BH_aH_c , and CH_bH_a are concurrent at the center of the Jerabek hyperbola which is the isogonal transform of the Euler line.

In this note we generalize a theorem of Victor Thébault's as given in [1, Theorem 1]. Given a triangle ABC with orthic triangle $H_aH_bH_c$, the Euler lines of the triangles AH_bH_c , BH_cH_a , and CH_aH_b are concurrent at a point on the nine-point circle, which is the center of the Jerabek hyperbola, the isogonal transform of the Euler line of triangle ABC .

Since triangle AH_cH_b is similar to ABC , it is the reflection in the bisector of angle A of a triangle AB_aC_a , which is a homothetic image of ABC . Let P be a triangle center of triangle ABC . Its counterpart in AH_cH_b is the point P_a constructed as the reflection in the bisector of angle A of the point on AP which is the intersection of the parallels to BP , CP through C_a , B_a respectively (see Figure 1).

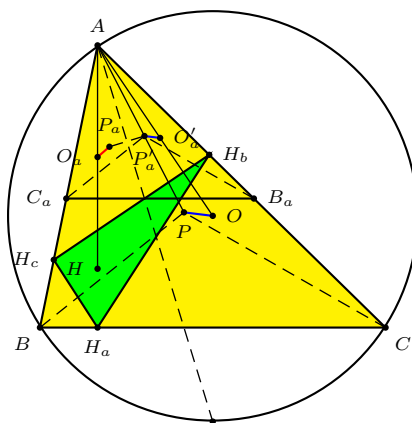


Figure 1.

Note that the circumcenter O_a of triangle AH_cH_b is the midpoint of AH . It is also the reflection (in the bisector of angle A) of the circumcenter O'_a of triangle AB_aC_a . The line O_aP_a is the reflection of $O'_aP'_a$ in the bisector of angle A .

Here is an alternative description of the line O_aP_a that leads to an interesting result. Consider the line ℓ'_a through A parallel to OP , and its reflection ℓ_a in the bisector of angle A . It is well known that ℓ_a intersects the circumcircle at a point Q' which is the isogonal conjugate of the infinite point of OP . Now, the line O_aP_a is clearly the image of ℓ_a under the homothety $h(H, \frac{1}{2})$. As such, it contains the midpoint Q of the segment HQ' .

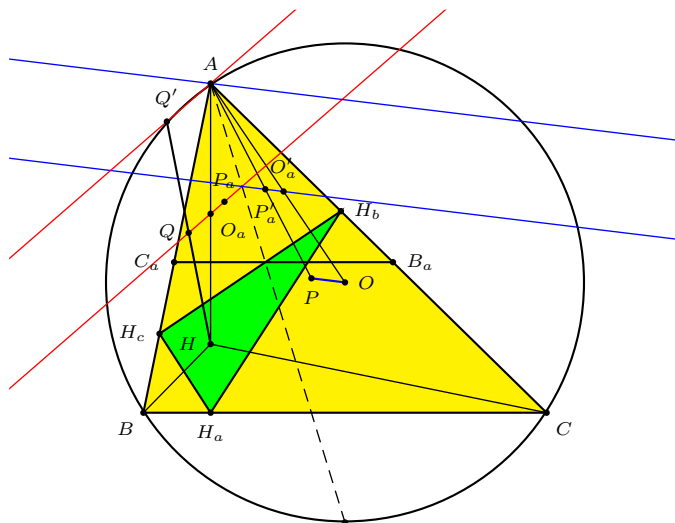


Figure 2.

The above reasoning applies to the lines O_bP_b and O_cP_c as well. The reflections of the parallels to OP through B and C in the respective angle bisectors intersect the circumcircle of ABC at the same point Q' , which is the isogonal conjugate of the infinite point of OP (see Figure 3). Therefore, the lines O_bP_b and O_cP_c also contain the same point Q , which is the image of the Q' under the homothety $h(H, \frac{1}{2})$. As such, it lies on the nine-point circle of triangle BAC . It is well known (see [3]) that Q is the center of the rectangular circum-hyperbola which is the isogonal transform of the line OP .

We summarize this in the following theorem.

Theorem. *Let P be a triangle center of triangle ABC . If P_a, P_b, P_c are the corresponding triangle centers in triangles $AH_cH_b, BH_aH_c, CH_bH_a$ respectively, the lines O_aP_a, O_bP_b, O_cP_c intersect at a point Q on the nine-point circle of ABC , which is the center of the rectangular circumhyperbola which is the isogonal transform of the line OP .*

Thébault's theorem is the case when P is the orthocenter.

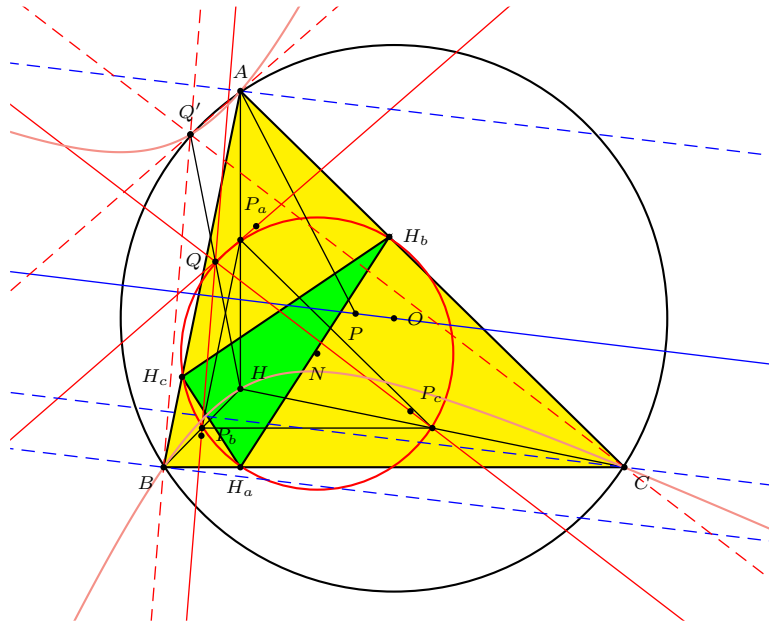


Figure 3.

We conclude with a record of coordinates. Suppose P has homogeneous barycentric coordinates $(u : v : w)$ in reference to triangle ABC . The line O_aP_a , O_bP_b , O_cP_c intersect at the point

$$\begin{aligned}
 Q &= ((b^2 - c^2)u + a^2(v - w))(c^2(a^2 + b^2 - c^2)v - b^2(c^2 + a^2 - b^2)w) \\
 &: (c^2 - a^2)v + b^2(w - u))(a^2(b^2 + c^2 - a^2)w - c^2(a^2 + b^2 - c^2)u) \\
 &: (a^2 - b^2)w + c^2(u - v))(b^2(c^2 + a^2 - b^2)u - a^2(b^2 + c^2 - a^2)v))
 \end{aligned}$$

on the nine-point circle, which is the center of the rectangular hyperbola through A, B, C, H and

$$\begin{aligned}
 Q' &= \left(\frac{a^2}{((b^2 - c^2)^2 - a^2(b^2 + c^2))u + a^2(b^2 + c^2 - a^2)(v + w)} \right. \\
 &: \frac{b^2}{((c^2 - a^2)^2 - b^2(c^2 + a^2))v + b^2(c^2 + a^2 - b^2)(w + u)} \\
 &: \left. \frac{c^2}{((a^2 - b^2)^2 - c^2(a^2 + b^2))w + c^2(a^2 + b^2 - c^2)(u + v)} \right).
 \end{aligned}$$

on the circumcircle. Here are some examples. The labeling of triangle centers follows [2].

P	Q on nine-point circle	Q' on circumcircle
Orthocenter X_4	Jerabek center X_{125}	X_{74}
Symmedian point X_6	Kiepert center X_{115}	X_{98}
Incenter X_1	Feuerbach point X_{11}	X_{104}
Nagel point X_8	X_{3259}	X_{953}
Spieker center X_{10}	X_{124}	X_{102}
X_{66}	X_{127}	X_{1297}
Steiner point X_{99}	X_{2679}	X_{2698}

References

- [1] N. Dergiades and P. Yiu, Antiparallels and Concurrent Euler Lines, *Forum Geom.*, 4(2004) 1–20.
- [2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at <http://faculty.evansville.edu/ck6/encyclopedia/ETC.html>.
- [3] P. Yiu, *Introduction to the Geometry of the Triangles*, Florida Atlantic University Lecture Notes, 2001.

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