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A Generalization of Thébault's Theorem on the Concurrency of Three Euler Lines

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Abstract. We prove a generalization of Victor Thébault's theorem that if $H_aH_bH_c$ is the orthic triangle of ABC, then the Euler lines of triangles AH_cH_b, BH_aH_c, and CH_bH_a are concurrent at the center of the Jerabek hyperbola which is the isogonal tranform of the Euler line.

In this note we generalize a theorem of Victor Thébault's as given in [1, Theorem 1]. Given a triangle ABC with orthic triangle $H_aH_bH_c$, the Euler lines of the triangles AH_bH_c, BH_cH_a, and CH_aH_b are concurrent at a point on the nine-point circle, which is the center of the Jerabek hyperbola, the isogonal transform of the Euler line of triangle ABC.

Since triangle AH_cH_b is similar to ABC, it is the reflection in the bisector of angle A of a triangle AB_aC_a, which is a homothetic image of ABC. Let P be a triangle center of triangle ABC. Its counterpart in AH_cH_b is the point P_a constructed as the reflection in the bisector of angle A of the point on AP which is the intersection of the parallels to BP, CP through C_a, B_a respectively (see Figure 1).

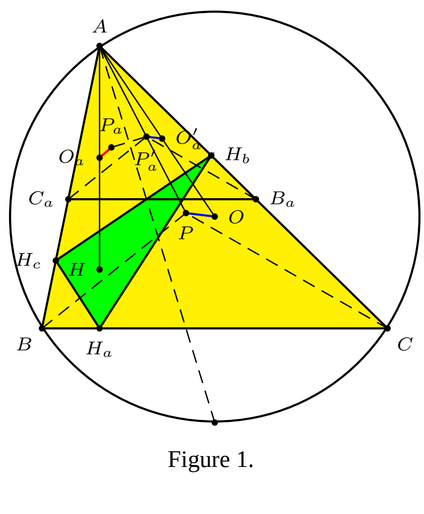

Figure 1.

Note that the circumcenter O_a of triangle AH_cH_b is the midpoint of AH. It is also the reflection (in the bisector of angle A) of the circumcenter O_a' of triangle AB_aC_a. The line O_aP_a is the reflection of $O_a'P_a'$ in the bisector of angle A.

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Here is an alternative description of the line O_aP_a that leads to an interesting result. Consider the line ℓ'_a through A parallel to OP, and its reflection ℓ_a in the bisector of angle A. It is well known that ℓ_a intersects the circumcircle at a point Q' which is the isogonal conjugate of the infinite point of OP. Now, the line O_aP_a is clearly the image of ℓ_a under the homothety $\mathsf{h}(H, \frac{1}{2})$. As such, it contains the midpoint Q of the segment HQ'.

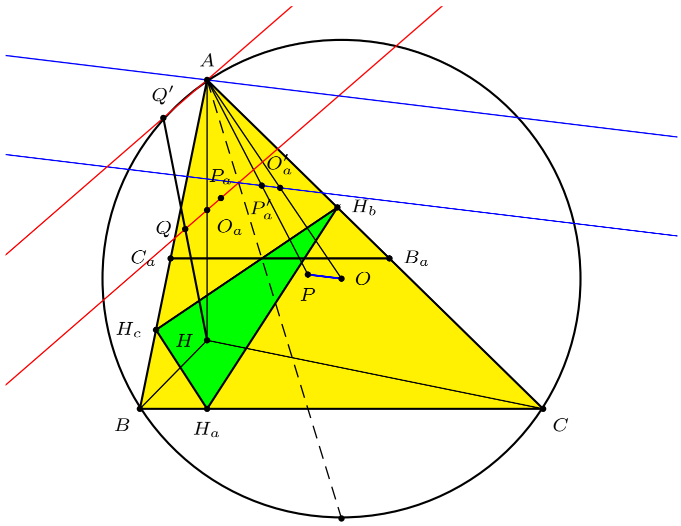

Figure 2.

The above reasoning applies to the lines O_bP_b and O_cP_c as well. The reflections of the parallels to OP through B and C in the respective angle bisectors intersect the circumcircle of ABC at the same point Q', which is the isogonal conjugate of the infinite point of OP (see Figure 3). Therefore, the lines O_bP_b and O_cP_c also contain the same point Q, which is the image of the Q' under the homothety $\mathsf{h}(H, \frac{1}{2})$. As such, it lies on the nine-point circle of triangle BAC. It is well known (see [3]) that Q is the center of the rectangular circum-hyperbola which is the isogonal transform of the line OP.

We summarize this in the following theorem.

Theorem. *Let P be a triangle center of triangle ABC. If P_a, P_b, P_c are the corresponding triangle centers in triangles AH_cH_b, BH_aH_c, CH_bH_a respectively, the lines O_aP_a, O_bP_b, O_cP_c intersect at a point Q on the nine-point circle of ABC, which is the center of the rectangular circumhyperbola which is the isogonal transform of the line OP.*

Thébault's theorem is the case when P is the orthocenter.

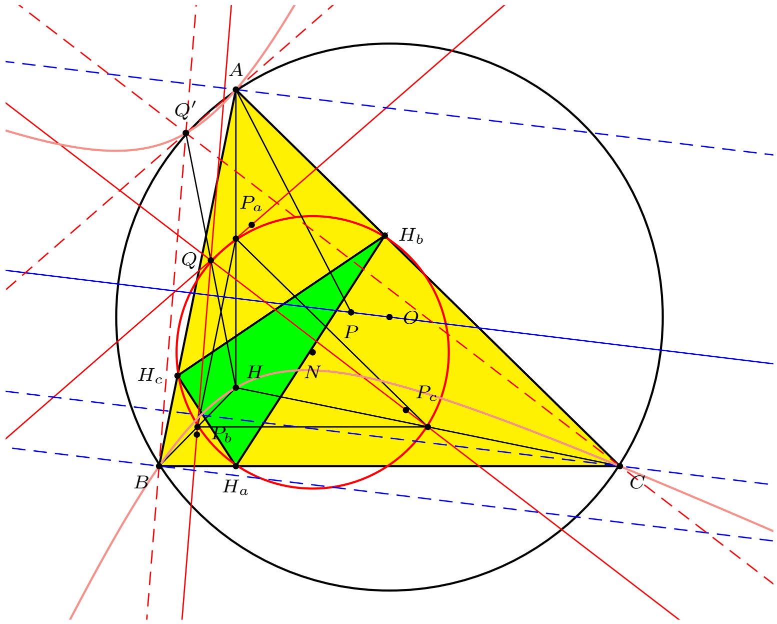

Figure 3.

We conclude with a record of coordinates. Suppose P has homogeneous barycentric coordinates $(u : v : w)$ in reference to triangle ABC. The line O_aP_a, O_bP_b, O_cP_c intersect at the point

$$\begin{aligned} Q = \big(&((b^2-c^2)u+a^2(v-w))(c^2(a^2+b^2-c^2)v-b^2(c^2+a^2-b^2)w) \\ &: (c^2-a^2)v+b^2(w-u))(a^2(b^2+c^2-a^2)w-c^2(a^2+b^2-c^2)u) \\ &: (a^2-b^2)w+c^2(u-v))(b^2(c^2+a^2-b^2)u-a^2(b^2+c^2-a^2)v)\big) \end{aligned}$$

on the nine-point circle, which is the center of the rectangular hyperbola through A, B, C, H and

$$\begin{aligned} Q' = \Bigg(&\frac{a^2}{((b^2-c^2)^2-a^2(b^2+c^2))u+a^2(b^2+c^2-a^2)(v+w)} \\ &: \frac{b^2}{((c^2-a^2)^2-b^2(c^2+a^2))v+b^2(c^2+a^2-b^2)(w+u)} \\ &: \frac{c^2}{((a^2-b^2)^2-c^2(a^2+b^2))w+c^2(a^2+b^2-c^2)(u+v)}\Bigg). \end{aligned}$$

on the circumcircle. Here are some examples. The labeling of triangle centers follows [2].

P	Q on nine-point circle	Q' on circumcircle
Orthocenter X_4	Jerabek center X_{125}	X_{74}
Symmedian point X_6	Kiepert center X_{115}	X_{98}
Incenter X_1	Feuerbach point X_{11}	X_{104}
Nagel point X_8	X_{3259}	X_{953}
Spieker center X_{10}	X_{124}	X_{102}
X_{66}	X_{127}	X_{1297}
Steiner point X_{99}	X_{2679}	X_{2698}

References

[1] N. Dergiades and P. Yiu, Antiparallels and Concurrent Euler Lines, *Forum Geom.*, 4(2004) 1–20.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[3] P. Yiu, *Introduction to the Geometry of the Triangles*, Florida Atlantic University Lecture Notes, 2001.

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