Forum Geometricorum
Volume 9 (2009) 1–17.

FORUM GEOM
ISSN 1534-1178

On n-Sections and Reciprocal Quadrilaterals

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Abstract. We introduce the notion of an n-section and reformulate a number of standard Euclidean results regarding angles in terms of 2-sections (with proof). Using 6-sections, we define the notion of reciprocal (complete) quadrangles and derive some properties of such quadrangles.

1. Introduction

While classical geometry is still admired as a model for mathematical reasoning, it is only fair to admit that following through an argument in Euclidean geometry in its full generality can be rather cumbersome. More often than not, a discussion of all manner of special cases is required. Specifically, Euclid's notion of an angle is highly unsatisfactory. With the rise of projective geometry in the 19th century, some of these issues (such as the role of points at infinity) were addressed. The need to resolve any of the difficulties connected with the notion of an angle was simply obviated by (largely) avoiding any direct appeal to the concept. By the end of the 19th century, as projective geometry and metric geometry aligned again and vectorial methods became commonplace, classical geometry saw the formal introduction of the notion of *orientation*. In the case of the concept of an angle, this led to the notion of a *directed (oriented, sensed)* angle. In France, the (elite) high school teacher and textbook author Louis Gérard was an early champion of this notion, as was Jacques Hadamard (1865-1963); in the USA, Roger Arthur Johnson (1890-1954) called for the use of such angles in classical geometry in two papers published in 1917.[1] Today, while the notion of a directed angle certainly has found its place in classical geometry research and teaching, it has by no means supplanted the traditional notion of an angle. Many college geometry textbooks still ignore the notion of orientation altogether.

In this paper we will use a notion very closely related to that of a directed angle. This notion was introduced by the Australian mathematician David Kennedy Picken (1879-1956) as the *complete angle* in 1922. Five years later and again in 1947, the New Zealand mathematician Henry George Forder (1889-1981) picked

Publication Date: January 26, 2009. Communicating Editor: Paul Yiu.
This paper is an extended version of a presentation with the same title at the Invited Paper Session: Classical Euclidean Geometry in MathFest, July 31–August 2, 2008 Madison, Wisconsin, USA.

[1]See [5], as well as [6] and [7]; Johnson also consistently uses directed angles in his textbook [8].

up on the idea, preferring the term *cross*.[2] Essentially, where we can look upon an angle as the configuration of two rays departing from the same point, the cross is the configuration of two intersecting lines. Here, we will refer to a *cross* as a 2-section and consider it a special case of the more general notion of an n-section.

We first define these n-sections and establish ground rules for their manipulation. Using these rules, we derive a number of classical results on angles in terms of 2-sections. In the process, to overcome some of the difficulties that Picken and Forder ran into, we will also bring the theory of circular inversion into the mix. After that, we will focus on 6-sections formed by the six sides of a complete quadrangle. This will lead us to the introduction of the reciprocal to a complete quadrangle as first introduced by James Clerk Maxwell (1831-1879). We conclude this paper by studying some of the properties of reciprocal quadrangles.

2. The notion of an n-section

In the Euclidean plane, let $\{l\}$ denote the equivalence class of all lines parallel to the line l. We will refer to $\{l\}$ as the *direction* of l. Now consider the ordered set of directions of a set of lines $l_1, ..., l_n$ ($n \geq 2$). We refer to such a set as an n-section (of lines), which we will write as $\{\ell_1, ..., \ell_n\}$.[3]. Clearly, any n-section is an equivalence class of all lines $m_1, ..., m_n$ each parallel to the corresponding of $l_1, ..., l_n$. Therefore, we can think of any n-section as represented by n lines all meeting in one point. Also note that any n-section corresponds to a configuration of points on the line at infinity.

We say that two n-sections $\{l_1, ..., l_n\}$ and $\{m_1, ..., m_n\}$ are directly congruent if for any representation of the two sections by means of concurrent lines there is a rotation combined with a translation that maps each line of the one representation onto the corresponding line of the other. We write $\{l_1, ..., l_n\} \cong_D \{m_1, ..., m_n\}$. If in addition a reflection is required, $\{l_1, ..., l_n\}$ is said to be *inversely* congruent to $\{m_1, ..., m_n\}$, which we write as $\{l_1, ..., l_n\} \cong_I \{m_1, ..., m_n\}$.

Generally, no two n-sections can be both directly and inversely congruent to each other. Particularly, as a rule, an n-section is not inversely congruent to itself. A notable exception is formed by the 2-sections. Clearly, a 2-section formed by two parallel lines is inversely as well as directly congruent to itself. We will refer to such a 2-section as trivial. Any non-trivial 2-section that is inversely congruent to itself is called *perpendicular* and its two directions are said to be perpendicular to each other. We will just assume here that for every direction there always is exactly one direction perpendicular to it.[4]

No other n-sections can be both directly and inversely congruent, except for such n-sections which only consist of pairs of lines that either all parallel or are perpendicular. We will generally ignore such sections.

[2]See [3] (pp.120-121+151-154), [4], [16], and [17]. The term *cross* seems to have been coined by Edward Hope Neville in [14]. Forder may actually have also used crosses in his two geometry textbooks from 1930 and 1931, but we have not been able to locate copies of these.

[3]We adapt this notation from [15].

[4]A proof using SAS is fairly straightforward.

The following basic principles for the manipulation of n-sections apply. We would like to insist here that these principles are just working rules and not axioms (in particular they are not independent) and serve the purpose of providing a shorthand for frequent arguments more than anything else.

Principle 1 (Congruency)**.** *Two n-sections are congruent if and only if all corresponding sub-sections are congruent, where the congruencies are either all direct or all inverse.*

Principle 2 (Transfer)**.** *For any three directions $\{a\}$, $\{b\}$, and $\{c\}$, there is exactly one direction $\{d\}$ such that $\{a,b\} \cong_D \{c,d\}$.*

Principle 3 (Chain Rule)**.** *If $\{a,b\} \cong \{a',b'\}$ and $\{b,c\} \cong \{b',c'\}$ then $\{a,c\} \cong \{a',c'\}$, where the congruencies are either all direct or all inverse.*

Principle 4 (Rotation)**.** *Two n-sections $\{a_1,...,a_n\}$ and $\{b_1,...,b_n\}$ are directly congruent if and only if all $\{a_i,b_i\}$ ($1 \leq i \leq n$) are directly congruent.*

Principle 5 (Reflection)**.** *Two n-sections $\{a_1,...,a_n\}$ and $\{b_1,...,b_n\}$ are inversely congruent if and only if there is a direction $\{c\}$ such that $\{a_i,c\} \cong_I \{b_i,c\}$ for all $1 \leq i \leq n$.*

Most of the usual triangle similarity tests are still valid (up to orientation) if we replace the notion of an angle by that of a cross or 2-section, except for Side-Cross-Side (SCS). Since we cannot make any assumptions about the orientation on an arbitrary line, SCS is ambiguous in terms of sections in that a 2-section with a length on each of its legs, (generally) determines two non-congruent triangles. The only situation in which SCS holds true (up to orientation) is for perpendicular sections. Since we are in the Euclidean plane, the *Dilation Principle* applies to any 2-section as well: For any triangle $\triangle ABC$ with P on CA and Q on CB, $\triangle PQC$ is directly similar to $\triangle ABC$ if and only if $\overline{CP}/\overline{CA} = \overline{CQ}/\overline{CB}$, where $\overline{CA}$ and so on denote *directed* lengths.

Once again, note that we do not propose to use the n-sections to completely replace the notion of an angle. The notion of n-sections just provides a uniform way to discuss the large number of problems in geometry that are really about configurations of lines rather than configurations of rays. Starting from our definition of a perpendicular section, for instance, the basic principles suffice to give a formal proof that all perpendicular sections are congruent. In other words, they suffice to prove that all perpendicular lines are made equal. Essentially this proof streamlines the standard proof (first given by Hilbert). Let $\{a,a'\}$ be a perpendicular section and let $\{b\}$ be arbitrary direction. Now, let $\{b'\}$ be such that (i) $\{a,b\} \cong_D \{a',b'\}$ (BP 2). Then, since $\{a,a'\} \cong_D \{a',a\}$, also $\{a',b\} \cong_D \{a,b'\}$ or (ii) $\{b',a\} \cong_D \{b,a'\}$ (BP 3). Combining (i) and (ii), it follows that $\{b,b'\} \cong_D \{b',b\}$ (BP 3). In other words, $\{b'\}$ is perpendicular to $\{b\}$. Finally, by BP 4, $\{b,b'\} \cong_D \{a,a'\}$.

The same rules also naturally allow for the introduction of both angle bisectors to an angle and do not distinguish the two. Indeed, note that the "symmetry" direction $\{c\}$ in BP 5 is not unique. If $\{a_i,c\} \cong_I \{b_i,c\}$, then the same is true for the direction $\{c'\}$ perpendicular to $\{c\}$ by BP 3. Conversely, for any direction $\{d\}$ such that $\{a_i, d\} \cong_I \{b_i, d\}$, it follows that $\{d, c\} \cong_D \{c, d\}$ by BP 3. In other words, $\{c, d\}$ is a perpendicular section. We will refer to the perpendicular section $\{c, c'\}$ as the *symmetry section* of the inversely congruent n-sections $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_n\}$. In the case of the inversely congruent systems $\{a_1, a_2\}$ and $\{a_2, a_1\}$, we speak of the symmetry section of the 2-section. Obviously, the directions of the latter section are those of the angle bisectors of the angle formed by any two rays on any two lines representing $\{a_1, a_2\}$.

Using the notion of a symmetry section, we can now easily prove Thales' Theorem (as it is known in the Anglo-Saxon world).

Theorem 6 (Thales)**.** *For any three distinct points A, B and C, the line AC is perpendicular to BC if and only if C lies on the unique circle with diameter AB.*

Proof. Let O be the center of the circle with diameter AB. Since both $\triangle AOC$ and $\triangle BOC$ are isosceles, the two line of the symmetry section of $\{AB, OC\}$ are each perpendicular to one of AC and BC. Consequently, by BP 4 (Rotation), $\{AC, BC\}$ is congruent to the symmetry section, *i.e.*, AC and BC are perpendicular. Conversely, let A', B', C' be the midpoints of BC, CA, AB, respectively. Then, by dilation, $C'A'$ and $C'B'$ are parallel to CA and CB, respectively. It follows that $C'A'CB'$ is a rectangle and therefore $|B'A'| = |C'C|$, but by dilation $|B'A'| = |AC'| = |BC'|$, *i.e.*, C lies on the unique circle with diameter AB. □

3. Circular inversion

To allow further comparison of n-sections, we need the equivalent of a number of the circle theorems from Book III of Euclid's *Elements*. It is easy to see how to state any of these theorems in terms of 2-sections. As Picken remarks, however, really satisfactory proofs (in terms of 2-sections) are not so obvious and probably impossible if we do not want to use rays and angles at all. Be that as it may, we can still largely avoid directly using angles.[5] In this paper we will have recourse to the notion of *circular inversion*, which allows for reasonably smooth derivations. This transformation of (most of) the affine plane is defined with respect to a given circle with radius r and center O. For any point P of the plane other than O, its image under inversion with respect to O and the circle of radius r is defined as the unique point P' such that $\overline{OP} \cdot \overline{OP'} = r^2$ (where $\overline{OP}$ and so on denote *directed* lengths). Note that by construction circular inversion is a closed (and bijective) operation on the affine plane (excluding O). Also, if A' and B' are the images of A and B under a circular inversion with respect to a point O, then by construction $\triangle A'B'O$ is inversely similar to $\triangle ABO$. The following fundamental lemma applies.

Lemma 7. *Let O be the center of a circular inversion. Then, under this inversion (i) any circle not passing through O is mapped onto a circle not passing through O, (ii) any line not passing through O is mapped onto a circle passing through O and vice versa (with the point at infinity of the line corresponding to O), (iii) any*

[5] See [16], p.190 and [4], p.231. Forder is right to claim that the difficulty lies with the lack of an ordering for crosses and that directed angles need to be used at some point.

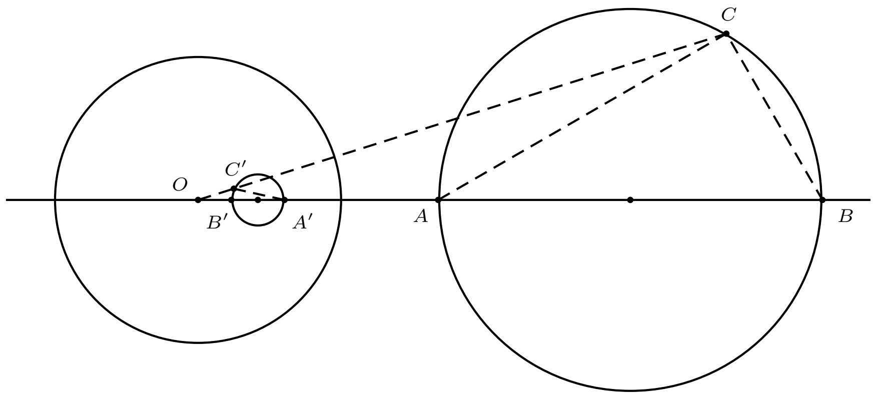

Figure 1. Circular Inversion of a Circle

line passing through O is mapped onto itself (with the point at infinity of the line again corresponding to O).

Proof. Starting with (i), draw the line connecting O with the center of the circle not passing through O and let the points of intersection of this line with the latter circle be A and B (see Figure 1). Then, by Theorem 6 (Thales), the lines AC and BC are perpendicular. Let A', B' and C' be the images of A, B, and C under the inversion. By the previous lemma $\{OA, AC, CO\}$ is indirectly congruent to $\{OC', C'A', A'O\}$. Likewise $\{OB, BC, CO\} \cong_I \{OC', C'B', B'O\}$. Since OA, OB, OA', and OB' coincide, it follows that $\{OB, BC, CA, CO\}$ is inversely congruent to $\{OC', C'B', C'A', B'O\}$. Therefore $\{BC, CA\}$ is indirectly congruent to $\{C'B', C'A'\}$. Consequently, $C'A'$ and $C'B'$ are perpendicular as well. This means that C' lies on the circle that has the segment $A'B'$ for a diameter. The second statement is proved in a similar way, while the third statement is immediate. □

We can now prove the following theorem, which is essentially a rewording in the language of sections of Propositions 21 and 22 from Book III of Euclid's *Elements* (with a trivial extension).

Theorem 8 (Equal Angle)**.** *For four points on either a circle or a straight line, let X, Y, Z, W be any permutation of A, B, C, D. Then, any 2-section $X\{Y, Z\}$ is directly congruent to the 2-section $W\{Y, Z\}$ and the sections are either trivial (in case the points are collinear) or non-trivial (in case the points are co-cyclic). Conversely, any four (distinct) points A, B, C, D for which there is a permutation X, Y, Z, W such that $X\{Y, Z\}$ and $W\{Y, Z\}$ are directly congruent either are co-cyclic (in case the sections are non-trivial) or collinear (in case the two sections are trivial).*

Proof. It suffices to prove both statements for one permutation of A, B, C, D. Assume that A, B, C, D are co-cyclic or collinear. Let B', C' and D' denote the images of B, C and D, respectively, under circular inversion with respect to A. Then, $\{DA, DC\} \cong_I \{C'A, D'C'\}$ and $\{BA, BC\} \cong_I \{C'A, B'C'\}$. Since by

Lemma 7, $B'C'$ coincides with $D'C'$, it follows that $\{DA, DC\} \cong_D \{BA, BC\}$. Conversely, assume that $\{DA, DC\} \cong_D \{BA, BC\}$. Then, $\{C'A, D'C'\} \cong_D \{C'A, B'C'\}$, *i.e.*, B', C' and D' are collinear. By Lemma 7 again, if the two 2-sections are non-trivial, A, B, C, D are co-cyclic. If not, the four points are collinear. □

Corollary 9. *Let A, B, C, D be any four co-cyclic points with $E = AC \cap BD$. Then the product of directed lengths $\overline{AE} \cdot \overline{CE}$ equals the product $\overline{BE} \cdot \overline{DE}$.*

Proof. Let A' and B' be the images under inversion of A and B with respect to E (and a circle of radius r). Then $\triangle A'B'E$ and $\triangle CDE$ are directly similar with two legs in common. Therefore $\overline{CE}/\overline{A'E} = \overline{DE}/\overline{B'E}$ or $\overline{CE} \cdot \overline{AE}/r^2 = \overline{DE} \cdot \overline{BE}/r^2$. □

For the sake of completeness, although we will not use it in this paper, we end with a sometimes quite useful reformulation of Propositions 20 and 32 from Book III of Euclid's *Elements*.

Lemma 10 (Bow, String and Arrow)**.** *For any triangle $\triangle ABC$, let C' be the midpoint of AB and let O be the circumcenter of the triangle and let $T_{AB,C}$ denote the tangent line to the circumcircle of $\triangle ABC$ at C. Then $C\{B, A\}$ is directly congruent to (i) both $O\{C', A\}$ and $O\{B, C'\}$ and (ii) $\{BA, T_{CB,A}\}$ and $\{T_{CA,B}, AB\}$.*

Proof. It suffices to prove the first statements of (i) and (ii). Let A' be the midpoint of BC. Since OC' is perpendicular to AB and OA' is perpendicular to BC, it follows that $\square C'OA'B$ is cyclic and therefore that $A'\{B, C'\} \cong_D O\{B, C'\}$. But $A'C'$ is parallel to CA and therefore $A'\{B, C'\} \cong_D C\{B, A\}$ as well, which proves the first statement of (i). As for (ii), since BA is perpendicular to OC' and $T_{CB,A}$ is perpendicular to OA, it follows that $\{BA, T_{CB,A}\}$ is directly congruent to $O\{C', A\}$ by BP 4 (Rotation). Since $O\{C', A\}$ is directly congruent to $C\{B, A\}$, the first statement of (ii) follows. □

The preceding results provide a workable framework for the application of n-sections to a great many problems in plane geometry involving configurations of circles and lines (as opposed to rays). The well-known group of circle theorems usually attributed to Steiner and Miquel as well as most theorems associated with the Wallace line are particularly amenable to the use of n-sections. Examples can be found in [16], [17], and [5].

4. 6-sections and complete quadrangles

So far we have essentially only used 2-sections and 3-sections. Note how any 3-section (with distinct directions) always corresponds to a unique class of directly similar triangles. Clearly, there is no such correspondence for 4-sections. To determine a quadrilateral, we need the direction of at least one of its diagonals as well. Therefore, it makes sense to consider the 6-sections and their connection to the so-called *complete quadrangles* $\boxtimes ABCD$, *i.e.*, all configurations of four points (with no three collinear) and the six lines passing through each two of them. Clearly any $\boxtimes ABCD$ defines a 6-section. Conversely, not every 6-section can be represented by the six sides of a complete quadrangle. In order to see under what condition a 6-section originates from a complete quadrangle, we need a little bit of projective geometry.

Any two n-sections are said to be *in perspective* or to form a *perspectivity* if for a representation of each of the sections by concurrent lines the points of intersection of the corresponding lines are collinear. Two sections are said to be *projective* if a representation by concurrent lines of the one section can be obtained from a similar representation of the other as a sequence of perspectivities. It can be shown that any two sections that are congruent are also projective. In the case of 2-sections and 3-sections all are actually projective. As for 4-sections, the projectivity of two sections is determined by their so-called *cross ratio*. Every 4-section $\{\ell_1, ..., \ell_4\}$ has an associated cross ratio $[\ell_1, ..., \ell_4]$. If $\underline{A}$ denotes the pencil of lines passing through A, represent the lines of any section by lines $\ell_i \in \underline{A}$. If ℓ_i has an equation $\mathcal{L}_i = 0$, we can write $\mathcal{L}_3$ as $\lambda_{31}\mathcal{L}_1 + \lambda_{32}\mathcal{L}_2$ and $\mathcal{L}_4$ as $\lambda_{41}\mathcal{L}_1 + \lambda_{42}\mathcal{L}_2$. We now (unambiguously) define the cross ratio $[\ell_1, \ldots, \ell_4]$ as the quotient $(\lambda_{31}/\lambda_{32}) : (\lambda_{41}/\lambda_{42})$. From this definition of a cross ratio it follows that its value does not change when the first pair of elements and the second pair are switched or when the elements within each pair are swapped. Note that for any two 3-sections $\{l_1, l_2, l_3\}$ and $\{m_1, m_2, m_3\}$ (with $\{l_1, l_2, l_3\}$ and $\{m_1, m_2, m_3\}$ each formed by three distinct directions), the cross ratio defines a bijective map φ between any two pencils $\underline{A}$ and $\underline{B}$, by choosing the l_i in $\underline{A}$ and the m_i in $\underline{B}$ and defining the image $\varphi(l)$ of any line $l \in \underline{A}$ as the line of $\underline{B}$ such that $[l_1, l_2; l_3, l]$ equals $[m_1, m_2; m_3, \varphi(l)]$. The map φ is called a projective map (of the pencil). It can be shown that any projective map can be obtained as a projectivity and vice versa. Therefore, two 4-sections are projective if and only if their corresponding cross ratios are equal. By the duality of projective geometry, all of the preceding applies to the points of a line instead of the lines of a pencil as well. Moreover, for any four points L_1, L_2, L_3, L_4 on a line ℓ and a point L_0 outside ℓ, the cross ratio $[L_1, L_2; L_3, L_4]$ is equal to $[L_0L_1, L_0L_2; L_0L_3, L_0L_4]$. By the latter property, we can associate any projective map defined by two sections of lines with a projective map from the line at infinity to itself.

The notion of a projective map can be extended to the projective plan where any such map φ maps any line to a straight line and the restriction of φ to a line and its image line is a projective map. Where a projective map between two lines is defined by two triples of (non-coinciding) points, a projective map between two planes requires two sets of four points, no three of which can be collinear. In other words, any two quadrilaterals define a projective map. Finally, we define an *involution* as a projective map which is its own inverse. In the case of an involution of a line or pencil, any two distinct pairs of elements (with the elements within each pair possibly coinciding) fully determine the map.

We can now formulate the following result.

Theorem 11. *An arbitrary* 6*-section* $\{l_1, l_2, m_1, m_2, n_1, n_2\}$ *can be formed from the sides of a complete quadrangle* $\boxtimes ABCD$ *(such that* l_1*,* l_2 *and so on are pairs* *of opposite sides) if and only if the three pairs of opposite sides can be rearranged such that* $\{l_1, l_2\}$ *is non-trivial and* $[l_1, l_2; m_1, n_2]$ *equals* $[l_1, l_2; n_1, m_2]$.

Proof. Since any two quadrilaterals determine a projective map, every complete quadrangle is projective to the configuration of a rectangle and its diagonals. Therefore the diagonal points of a complete quadrangle are never collinear and every quadrangle in the affine plane has at least one pair of opposite sides which are not parallel. Without loss of generality, we may assume that $\{l_1, l_2\}$ corresponds to this pair of opposite sides. Let $\underline{A}$, $\underline{B}$ denote the pencils of lines through A and B respectively. Now define a map φ from $\underline{A}$ to $\underline{B}$ by assigning the line AX to BX for all X on a line ℓ not passing through A or B. It is easily verified that φ is a projectivity, which assigns AB to itself and the line of $\underline{A}$ parallel to L to the corresponding parallel line of $\underline{B}$. Therefore, if C and D are distinct points on ℓ, the cross ratio $[AB, CD; AC, AD]$ equals the cross ratio $[AB, CD, BC, BD]$. Conversely, for any 6-section $\{l_1, l_2, m_1, m_2, n_1, n_2\}$ such that $[l_1, l_2; m_1, n_2]$ equals $[l_1, l_2; n_1, m_2]$, we can choose A and B such that AB is parallel to l_1 and let D be the point of intersection of the line of $\underline{A}$ parallel to m_1 and the line of $\underline{B}$ parallel to n_1. Likewise, let C be the point of intersection of the line of $\underline{A}$ parallel to m_2 and the line through D parallel to l_2. Then, since $\{l_1, l_2\}$ is non-trivial, the line BC has to be parallel to n_2. □

Note that the previous theorem is a projective version of Ceva's Theorem determining the concurrency of transversals in a triangle and the usual expression of that theorem can be readily derived from the condition above. We now have the following corollary.

Corollary 12. *For any complete quadrangle* $\boxtimes ABCD$*, there is an involution that pairs the points of intersection of its opposite sides with the line at infinity.*

Proof. Without loss of generality, we may assume that $\{AB, CD\}$ is non-trivial. Let $L_1 = AB \cap \ell_\infty$ and so on. Then $[L_1, L_2, M_1, N_2] = [L_1, L_2, N_1, M_2]$. Now let φ be the involution of ℓ_∞ determined by pairing L_1 with L_2 and M_1 with M_2. Then $[L_1, L_2, M_1, N_2]$ equals $[L_2, L_1, M_2, \varphi(N_2)]$. Since the former expression is also equal to $[L_2, L_1, M_2, N_1]$ (and L_1, L_2 and M_2 are distinct), it follows that $\varphi(N_2) = N_1$. In other words, the involution pairs N_1 and N_2 as well. □

In case $\boxtimes ABCD$ is a trapezoid, the point on the line at infinity corresponding to the parallel sides is a fixed point of the involution; in case the complete quadrangle is a parallelogram, the two points corresponding to the two pairs of parallel sides both are fixed points.

In the language of classical projective geometry, we say that a 6-section formed by the sides of any complete quadrangle defines an involution of six lines pairing the opposite sides of the quadrangle. Note that this statement implies what is known as Desargues' Theorem, which states that any complete quadrangle defines an involution (of points) on any line not passing through any of its vertices that pairs the points of intersection of that line with the opposite sides of the quadrangle. For this reason, we will say that any 6-section satisfying the condition of Theorem 11 is *Desarguesian*.

Corollary 13. *Any Desarguesian* 6*-section is associated with two similarity classes of quadrilaterals (which may coincide).*

Proof. Let the 6-section be denoted by $\{l_1, l_2, m_1, m_2, n_1, n_2\}$. If the cross ratio $[l_1, l_2; m_1, n_2]$ equals $[l_1, l_2; n_1, m_2]$, then the cross ratio $[l_2, l_1; m_2, n_1]$ also equals $[l_2, l_1; n_2, m_1]$. Whereas the quadrilateral constructed from the first equality contains a triangle formed by the lines l_2, m_2, n_2, while l_1, m_1, n_1 meet in one point, this is reversed for the quadrilateral formed from the second equality. Since $\{l_1, m_1, n_1\}$ and $\{l_2, m_2, n_2\}$ are not necessarily congruent, the two quadrilaterals will be different (but may coincide in some cases). □

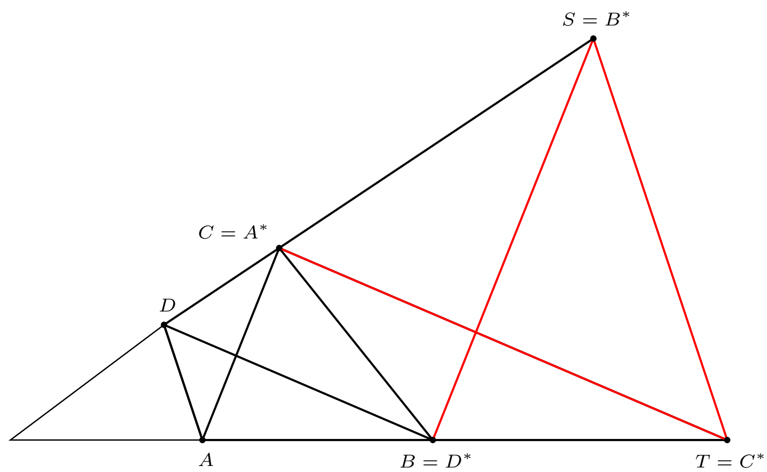

Figure 2. Constructing $\boxtimes A^*B^*C^*D^*$

If the complete quadrangle $\boxtimes ABCD$ is one of the two quadrangles forming a given 6-section, we can easily construct the other quadrangle $\boxtimes A^*B^*C^*D^*$. Indeed, let $\boxtimes ABCD$ be as in Figure 2. Then, draw the line through B parallel to AC, meeting CD in S. Likewise, draw the line through C parallel to BD meeting AB in T. Then, by construction ST is a parallel to AD and all the opposite sides of $\boxtimes ABCD$ are parallel to a pair of opposite sides of $\boxtimes BCST$. The two quadrangles, however, are generally not similar. Alternatively, we can consider the quadrangle formed by the circumcenters of the four circles circumscribing the four triangle formed by A, B, C, D. For this quadrangle, all three pairs of opposite sides are parallel to a pair of opposite sides of the original quadrangle. Again, it is easy to see that this quadrilateral is generally not similar to the original one. The latter construction was first systematically studied by Maxwell in [10] and [11], in which he referred to the quadrilateral of circle centers as a reciprocal figure. For this reason, we will refer to the two complete quadrangles associated with a Desarguesian 6-section as *reciprocal* quadrangles.

Relabeling the vertices of the preceding quadrangles as indicated in Figure 2, we will formally define two complete quadrilaterals $\boxtimes ABCD$ and $\boxtimes A^*B^*C^*D^*$ as directly/inversely reciprocal if and only if

$$\{AB, CD, AC, BD, DA, BC\} \cong \{C^*D^*, A^*B^*, B^*D^*, A^*C^*, B^*C^*, D^*A^*\},$$

where the congruence is either direct or inverse. From this definition, we immediately derive the following two corollaries.

Corollary 14. *A complete quadrangle is directly reciprocal to itself if and only if it is orthocentric.*

Proof. Since for any complete quadrangle directly reciprocal to itself all three 2-sections of opposite sides have to be both directly and inversely congruent, it follows that all opposite sides are perpendicular to each other. In other words, every vertex is the orthocenter of the triangle formed by the other three vertices, which is what orthocentric means. □

Corollary 15. *A complete quadrangle is inversely reciprocal to itself if and only if it is cyclic.*

Proof. Let $\boxtimes ABCD$ denote the complete quadrangle. Then, if $\boxtimes ABCD$ is inversely reciprocal to itself, $\{AB, AC\}$ has to be inversely congruent to $\{CD, BD\}$ or $A\{B, C\} \cong_D D\{B, C\}$. But this means that $\boxtimes ABCD$ is cyclic. The converse readily follows. □

Because of the preceding corollaries, when studying the relations between reciprocal quadrangles, we can often just assume that a complete quadrangle is neither orthocentric nor cyclic. Also, as a special case, note that if a complete quadrangle $\boxtimes ABCD$ has a pair of parallel opposite sides, then its reciprocal is directly congruent to $\boxtimes BADC$. For this reason, it is usually fine to assume that $\boxtimes ABCD$ does not have any parallel sides either.

Maxwell's application of his reciprocal figures to the study of statics contributed to the development of a heavily geometrical approach to that field (know as *graphostatics*) which ultimately made projective geometry a required course at many engineering schools until well into the 20th century. At the same time, the idea of "reciprocation" was largely ignored within the classical geometry community. This only changed in the 1890s, when (probably not entirely independently of Maxwell) Joseph Jean Baptiste Neuberg (1840-1826) reintroduced the concept of reciprocation under the name of *metapolarity*. This notion, however, seems to have been quickly eclipsed by the related notion of *orthology* that was introduced by Émile Michel Hyacinthe Lemoine (1840-1912) and others as a tool to study triangles. In this context, consider a triangle $\triangle ABC$ and a point P in the plane of the triangle. Now, construct a new triangle $\triangle A'B'C'$ such that each of its sides is perpendicular to the corresponding side of $\{CP, AP, BP\}$. In this new triangle, construct transversals each perpendicular to the corresponding line of $\triangle ABC$. Then, these three transversals will meet in a new point P'. The triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be *orthologic* with poles P and P'. Clearly, for any two orthologic triangles $\triangle ABC$ and $\triangle A'B'C'$ with poles P and P', $\boxtimes ABCP$ and $\boxtimes A'B'C'P'$ are reciprocal quadrangles. Conversely, for any two reciprocal quadrangles $\boxtimes ABCD$ and $\boxtimes A^*B^*C^*D^*$, $\triangle ABC$ and $\triangle A^*B^*C^*$ are orthologic with poles D and D^* (up to a rotation), and similarly for the three other pairs of triangles contained in the two quadrangles. It is in the form of some variation of orthology that the notion of reciprocation is best known today.[6]

A nice illustration of the use of reciprocal quadrangles (or orthology, in this case) is the following problem from a recent International Math Olympiad Training Camp.[7]

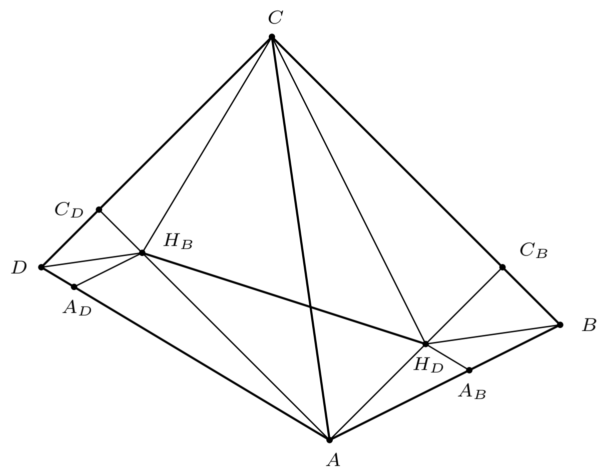

Figure 3. $\boxtimes ABCD$ with H_B and H_D

Problem (IMOTC 2005)**.** Let $ABCD$ be a quadrilateral, and H_D the orthocenter of triangle $\triangle ABC$. The parallels to the lines AD and CD through the point H_D meet the lines AB and BC at the points C_B and A_B, respectively. Prove that the perpendicular to the line C_BA_B through the point H_D passes through the orthocenter H_B of triangle $\triangle ACD$.

Solution. The proposition still has to be true if we switch the role of B and D. Now note that the complete quadrangles $\boxtimes H_BC_DDA_D$ and $\boxtimes BC_BH_DA_B$ have five parallel corresponding sides. Therefore, they are similar. Moreover, five of the sides of the complete quadrangle $\boxtimes H_BCH_DA$ are perpendicular to the opposite of the corresponding sides of $\boxtimes H_BC_DDA_D$ and $\boxtimes BC_BH_DB$. We conclude that $\boxtimes H_BCH_DA$ is directly reciprocal to $\boxtimes H_BC_DDA_D$ and $\boxtimes BC_BH_DA_B$. Consequently, its sixth side H_BH_D is perpendicular to A_DC_D and A_BC_B.

5. Some relations between reciprocal quadrangles

In order to study the relations between reciprocal quadrangles, we note yet another way to generate a reciprocal to a given complete quadrangle. In fact,

[6] On metapolar quadrangles, see e.g. [12] and [13] or (more accessibly) Neuberg's notes to [18] (p.458). On orthology, see [9]. In 1827, well before Lemoine (and Maxwell), Steiner had also outlined the idea of orthology (see [19], p.287, Problem 54), but nobody seems to have picked up on the idea at the time. Around 1900, the Spanish mathematician Juan Jacobo Durán Loriga (1854-1911) extended the notion of orthology to that of *isogonology*, which concept was completely equivalent to reciprocation. Durán-Loriga's work, however, met with the same fate as Neuberg's metapolarity.

[7] See [2] and the references there.

let $\boxtimes ABCD$ be a complete quadrangle with diagonal points $E = AC \cap BD$, $F = BC \cap DA$, $G = AB \cap CD$, with A, B, C, D, and, F in the affine plane. Now let A^* be the image of D under circular inversion with respect to F (see Figure 4). Likewise let D^* be the image of A under the same inversion. Similarly B^* is the image of C and C^* is the image of B. Then, using the properties of inversion it is easily verified that $\boxtimes A^*B^*C^*D^*$ is inversely reciprocal to $\boxtimes ABCD$. We can use this construction to derive the following two lemmata.

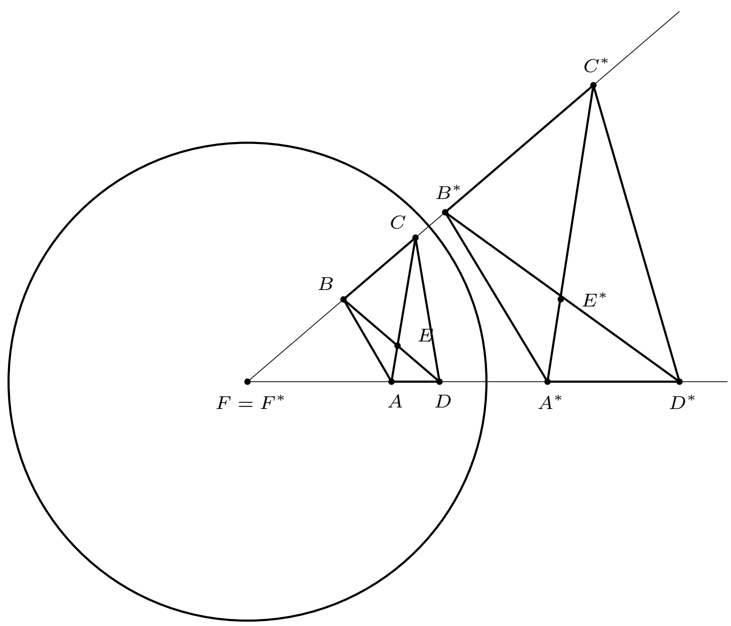

Figure 4. Constructing $\boxtimes A^*B^*C^*D^*$ by Inversion

Lemma 16 (Invariance of Ratios)**.** *Let $\boxtimes ABCD$ and $\boxtimes A^*B^*C^*D^*$ be a pair of (affine) reciprocal quadrangles and diagonal points E, F, G and E^*, F^*, G^*, respectively. Moreover, let X, Y, and Z be any collinear triple of two vertices and a diagonal point of $\boxtimes ABCD$ with X^*, Y^*, Z^* the corresponding triple of $\boxtimes A^*B^*C^*D^*$. Then*

$$\frac{\overline{XY}}{\overline{YZ}} = \frac{\overline{X^*Y^*}}{\overline{Y^*Z^*}},$$

where $\overline{XY}$ denotes the directed length of the line segment XY and so on.

Proof. The statement is trivial for any diagonal point on ℓ_∞. Without loss of generality, let us assume that the diagonal point F is in the affine plane. It now suffices to prove the statement for B, C and F. Under inversion with respect to F and a circle of radius r, we find that $\overline{B^*F^*} = r^2/\overline{CF}$ and $\overline{C^*F^*} = r^2/\overline{BF}$. The statement of the lemma now immediately follows. □

Lemma 17 (Maxwell)**.** *Let $\boxtimes ABCD$ and $\boxtimes A^*B^*C^*D^*$ be a pair of reciprocal quadrangles. Then*

$$\frac{|AB||CD|}{|A^*B^*||C^*D^*|} = \frac{|AC||BD|}{|A^*C^*||B^*D^*|} = \frac{|AD||CB|}{|A^*D^*||C^*B^*|},$$

where $|AB|$ denotes the absolute length of the segment AB and so on.

Proof. Assume again that the point F is in the affine plane. Under inversion with respect to F and a circle of radius r, we find $|A^*D^*| = |r^2/|FD| - r^2/|FA|| = r^2|AD|/(|FA||FD|)$ and $|B^*C^*| = |r^2/|FC| - r^2/|FB|| = r^2|BC|/(|FB||FD|)$. Similarly, $|A^*B^*| = r^2|AB|/(|FA||FB|)$ and $|C^*D^*| = r^2|CD|/(|FC||FD|)$, while $|A^*C^*| = r^2|AC|/(|FA||FC|)$ and $|B^*D^*| = r^2|BD|/(|FB||FD|)$. Combining these expressions shows the equality of the three expressions. □

Note that for any three collinear points, the ratio $|AC|/|BC|$ equals the cross ratio $[A, B, C, I_{AB}]$, where I_{AB} denotes the point at infinity of the line AB. Now, for any pair of reciprocal quadrangles $\boxtimes ABCD$ and $\boxtimes A^*B^*C^*D^*$, let φ be the unique projective map sending A to A^* and so on. Then, φ maps the line AB to the line A^*B^* and $[A, B, G, I_{AB}] = [A^*, B^*, G^*, \varphi(I_{AB})]$ (where $G = AB \cap CD$). By Lemma 16, $[A, B, G, I_{AB}]$ also equals $[A^*, B^*, G^*, I_{A^*B^*}]$. Therefore, since G is distinct from A and B, φ maps I_{AB} to $I_{A^*B^*}$. Likewise, the points at infinity of BC and CA are mapped to the points at infinity of B^*C^* and C^*A^*, respectively. But then, φ must map the whole line at infinity onto itself. Therefore, any map defined by "reciprocation" of a complete quadrangle is an *affine* map. Conversely, any affine map can be modeled by a reciprocation of a complete quadrangle (which we may assume not to have any parallel sides). To see this, we first need another lemma.

Lemma 18. *For a given triangle $\triangle ABC$ and any non-trivial 3-section $\{l, m, n\}$ not inversely congruent to $\{BC, CA, AB\}$ there is exactly one point D in the plane of $\triangle ABC$ (and not on the sides of $\triangle ABC$) such that $\{AD, BD, CD\}$ is directly congruent to $\{l, m, n\}$. In case $\{BC, CA, AB\} \cong_I \{l, m, n\}$, $\{AD, BD, CD\}$ will be directly congruent to $\{l, m, n\}$ for any point D on the circumcircle of $\triangle ABC$.*

Proof. Without loss of generality, we may assume that l, m, and n are concurrent at a point Q. Let a point L be a fixed point on l and let M be a variable point on m. Now construct a triangle $\triangle LMN$ directly similar to $\triangle ABC$. Then, the locus of N as M moves along m is a straight line as N is obtained from M by a fixed dilation followed by a rotation over a fixed angle. Therefore, this locus will intersect n in exactly one point as long as $\{AC, AB\}$ is not directly congruent to $\{n, m\}$. The point D we are looking for now has the same position with respect to $\triangle ABC$ as has Q with respect to $\triangle LMN$. If the two 2-sections are directly congruent, we can repeat the process starting with M or N. This means that we cannot find a point D as stated in the lemma using the procedure above only if $\triangle ABC$ is inversely congruent to $\{l, m, n\}$. But if the latter is the case, we can take any point D on the circumcircle of $\triangle ABC$ by Cor. 15. □

As an aside, note that for $\{l, m, n\}$ directly congruent to either $\{AB, BC, CA\}$ or $\{CA, AB, BC\}$, this construction also guarantees the existence of the two so-called Brocard points Ω^+ and Ω^- of $\triangle ABC$. Moreover, it is easily checked that $\boxtimes ABC\Omega^+$ and $\boxtimes BCA\Omega^-$ are reciprocal quadrangles. This explains the congruence of the two Brocard angles. We are now ready to prove the following theorem.

Theorem 19. *A projective map of the plane is affine if and only if it can be obtained by reciprocation of a complete quadrangle $\boxtimes ABCD$ with no parallel sides. Any such map reverses orientation if $\boxtimes ABCD$ is convex and retains orientation when not. The map is Euclidean if and only if $\boxtimes ABCD$ is orthocentric (in which case the map retains orientation) or cyclic (in which case the map reverses orientation).*

Proof. We already proved the if-part above. For an affine map, consider a triangle $\triangle ABC$ and its image $\triangle A^*B^*C^*$. By the previous lemma there is at least one point D (not on the sides of $\triangle ABC$) such that $\{AD, BD, CD\}$ is directly congruent to $\{B^*C^*, C^*A^*, A^*B^*\}$. The reciprocation of $\boxtimes ABCD$ that A maps to $\triangle ABC$ maps to $\triangle A^*B^*C^*$, then, must be the affine map. The connection between convexity of $\boxtimes ABCD$ follows from the various constructions (and relabeling) of a reciprocal quadrangle. The last statement follows immediately. In case $\boxtimes ABCD$ has parallel opposite sides, note that the affine map (after a rotation aligning one pair of parallel sides with their images) induces a map on the line at infinity with either one or two fixed points (if not just a translation combined with a dilation), corresponding to a glide or a dilation in two different directions. This means that if we choose the sides of $\triangle ABC$ such that they are not parallel to the directions represented by the fixed points on the line at infinity, no opposite sides of $\boxtimes ABCD$ will be parallel. $\square$

Finally, note that if a complete quadrangle $\boxtimes ABCD$ is cyclic, then its reciprocal $\boxtimes A^*B^*C^*D^*$ is as well. Likewise, by Lemma 17, if for a complete quadrangle the product of the lengths of a pair of opposite sides equals that of the lengths of another pair, the same is true for the corresponding pairs of its reciprocal. More surprisingly perhaps, reciprocation also retains inscribability, *i.e.*, if $\square ABCD$ has an incircle, then so has $\square A^*B^*C^*D^*$. To see this, we can use the following generalization of a standard result.

Lemma 20 (Generalized Ptolemy)**.** *For any six points A, B, C, D, P, and Q in the (affine) plane*

$$\begin{aligned} &|\triangle PAB||\triangle QCD| + |\triangle PCD||\triangle QAB| \\ &+ |\triangle PAD||\triangle QBC| + |\triangle PBC||\triangle QAD| \\ &= |\triangle PAC||\triangle QBD| + |\triangle PBD||\triangle QAC|. \end{aligned}$$

Proof. We represent the points A, B, C, D, P, and Q by vectors $\vec{a} = (a_1, a_2, 1)$ and so on. Now consider the vectors $(\vec{a} \oplus \vec{a})^T$, ..., $(\vec{d} \oplus \vec{d})^T$, as well as the vectors $(\vec{p} \oplus i\vec{p})^T$ and $(i\vec{q} \oplus \vec{q})^T$. Then clearly, the 6×6-determinant formed by these six vectors equals zero. If we now evaluate this determinant as the sum of the signed product of every 3×3-determinant contained in the three first rows and its complementary 3×3-determinant in the three bottom rows, we obtain exactly the identity of the lemma. □

Note that the imaginary numbers are necessary to ensure that no two of the products automatically cancel against each other. Also, note that this result really is about octahedrons in 3-space and can immediately be extended to their analogs in any dimension. Ptolemy's Theorem follows by letting P and Q coincide and assuming this point is on the circumcircle of $\boxtimes ABCD$.

Corollary 21. *For any complete quadrangle $\boxtimes ABCD$ and $E = AC \cap BD$ and a point P both in the (affine) plane of the quadrangle,*

$$|\triangle PDA| \cdot |\triangle EBC| + |\triangle PBC| \cdot |\triangle EDA|$$
$$= |\triangle PCD| \cdot |\triangle EAB| + |\triangle PAB| \cdot |\triangle ECD|,$$

where E is the point of intersection of AC with BD.

Proof. Let Q coincide with E. □

Now, let $\boxtimes ABCD$ be convex. Then $E = AC \cap BD$ is in the affine plane and we can obtain $\boxtimes A^*B^*C^*D^*$ by circular inversion with respect to E. Also, $\square A^*B^*C^*D^*$ is convex by Theorem 19. Therefore, the equality of $|A^*B^*| + |C^*D^*|$ and $|D^*A^*| + |B^*C^*|$ is both necessary and sufficient for the quadrangle to be inscribable. By the properties of inversion, this condition is equivalent to the condition

$$\frac{|AB|}{|EA||EB|} + \frac{|CD|}{|EC||ED|} = \frac{|DA|}{|ED||EA|} + \frac{|BC|}{|EB||EC|},$$

or

$$DA \cdot |\triangle EBC| + BC \cdot |\triangle EDA| = CD \cdot |\triangle EAB| + BA \cdot |\triangle ECD|.$$

If $\square ABCD$ is inscribable, this condition can also be written in the form

$$|\triangle IDA| \cdot |\triangle EBC| + |\triangle IBC| \cdot |\triangle EDA| = |\triangle ICD| \cdot |\triangle EAB| + |\triangle IAB| \cdot |\triangle ECD|.$$

But this equality is true by Cor. 21. We conclude that if $\square ABCD$ is inscribable, then so is $\square A^*B^*C^*D^*$.

Alternatively, we can use a curious result that received some on-line attention in recent years, but which is probably considerably older.

Theorem 22. *For any convex quadrilateral $\square ABCD$ with $E = AC \cap BD$, let I_{AB} be the incenter of $\triangle EAB$ and so on. Then $\boxtimes I_{AB}I_{BC}I_{CD}I_{DA}$ is cyclic if and only if $\square ABCD$ is inscribable.*

Proof. See [1] and the references there. The convexity requirement might not be necessary. □

Let us assume again that $\square ABCD$ is inscribable. This means that the quadrangle is convex and that $E = AC \cap BD$ is in the affine plane. Also, note that $E = I_A I_C \cap I_B I_D$. Therefore, $\overline{EI_{AB}} \cdot \overline{EI_{CD}}$ equals $\overline{EI_{BC}} \cdot \overline{EI_{DA}}$ by Theorem 22. Now, let $\boxtimes A^*B^*C^*D^*$ be a reciprocal of $\boxtimes ABCD$ obtained by circular inversion with respect to E and a circle with radius r. As we assumed that $\boxtimes ABCD$ is convex, so is $\boxtimes A^*B^*C^*D^*$ by Theorem 19. Since $\triangle EA^*B^*$ is inversely similar to $\triangle ECD$ while $|A^*B^*| = r^2|BD|/(|EB||ED|)$, it follows that $\overline{EI_{A^*B^*}} = r^2\overline{EI_{CD}}/(|EC||ED|)$ and so on. Consequently, $\overline{EI_{A^*B^*}} \cdot \overline{EI_{C^*D^*}} = \overline{EI_{B^*C^*}} \cdot \overline{EI_{D^*A^*}}$. Therefore $\boxtimes I_{A^*B^*}I_{B^*C^*}I_{C^*D^*}I_{D^*A^*}$ is cyclic and $\square A^*B^*C^*D^*$ is inscribable by Theorem 22 again.

As a third proof, it is relatively straightforward to actually construct a reciprocal $\boxtimes A^*B^*C^*D^*$ with its sides tangent to the incircle of $\square ABCD$. More generally, this approach proves that the existence of any tangent circle to a quadrangle implies the existence of one for its reciprocal. This construction can actually be looked upon as a special case of yet another way to construct reciprocal quadrangles. The proof of the validity of this more general construction, however, seems to require a property of reciprocal quadrangles that we have not touched upon in this paper. We plan to discuss this property (and the specific construction of reciprocal quadrangles that follows from it) in a future paper.

6. Conclusions

In this paper we outlined how in many cases the concept of an angle can be replaced by the more rigorous notion of an n-section. Other than the increased rigor, one advantage of n-sections over angles is that reasoning with the former is somewhat more similar to the kind of reasoning one might see in other parts of mathematics, particularly in algebra. Although perhaps a little bit of an overstatement, Picken did have a point when he claimed that his paper did not have diagrams because they were "quite unnecessary."[8] Also, the formalism of n-sections provides a natural framework in which to study geometrical problems involving multiple lines and their respective inclinations. As such, it both provides a clearer description of known procedures and is bound to lead to questions that the use of the notion of angles would not naturally give rise to. As a case in point, we showed how the notion of n-section suggests both a natural description of the procedure involving orthologic triangles in the form of the notion of reciprocal quadrangles and give rise to the question what properties of a complete quadrangle are retained under the "reciprocation" of quadrangles.

At the same time, the fact that the "reciprocation" of quadrangles does not favor any of the vertices of the figures involved comes at a cost. Indeed, its use does not naturally give rise to certain types of questions that the use of orthologic triangles does lead to. For instance, it is hard to see how an exclusive emphasis on the notion of reciprocal quadrangles could ever lead to the study of antipedal triangles and similar constructions. In short, the notion of reciprocal quadrangles should be seen as a general notion underlying the use of orthologic triangles and not as a replacement of the latter.

[8] See [16], p.188.

References

[1] A. Bogomolny, Cyclic incenters, *Interactive Mathematics Miscellany and Puzzles* = http://www.cut-the-knot.org/Curriculum/Geometry/CyclicIncenters.shtml.

[2] A. Bogomolny, Orthologic triangles in a quadrilateral, *Interactive Mathematics Miscellany and Puzzles* = http://www.cut-the-knot.org/Curriculum/Geometry/OrthologicQuadri.shtml.

[3] H. G. Forder, *Foundations of Euclidean Geometry*, Cambridge: CUP, 1927, also reprinted by Dover without any changes in 1958.

[4] H. G. Forder, The cross and the foundations of Euclidean Geometry, *Mathematical Gazette* 31:296 (Oct., 1947) 227–233.

[5] M. Guillerault, L'affermissement des démonstrations en géométrie élémentaire au tournant du XXème siècle, *Preuve. International Newsletter on the Teaching and Learning of Mathematical Proof* (Sept./Oct., 1998) = http://www.lettredelapreuve.it/Newsletter/980910.html.

[6] R. A. Johnson, Directed angles in elementary geometry, *Amer. Math. Monthly*, 24 (1917) 101–105.

[7] R. A. Johnson, Directed angles and inversion, with a proof of Schoute's Theorem, *Amer. Math. Monthly*, 24(1917) 313–317.

[8] R. A. Johnson, *Modern Geometry - An Elementary Treatise on the Geometry of the Triangle and the Circle*, New York: Houghton Mifflin, 1929; reprinted as *Advanced Euclidean Geometry*, Dover, 2007.

[9] E. Lemoine, Sur les triangles orthologiques et sur divers sujets de la géométrie du triangle, *Compte-Rendu de la 19ième Séance de l'Association Française pour l'Avancement des Sciences. Deuxième Partie: Notes et Mémoires*, (Paris, 1891) 111–140 (lu 9 août, 1890).

[10] J. C. Maxwell, On reciprocal figures and diagrams of force, *Philosophical Magazine* 27 (1864) 250–261.

[11] J. C. Maxwell, On reciprocal figures, frames and diagrams of force, *Transactions of the Royal Society of Edinburgh* 26 (1872) 1–40.

[12] J. J. B. Neuberg and A. Gob, Sur les foyers de Steiner d'un triangle, *Compte-Rendu de la 18ième Séance de l'Association Française pour l'Avancement des Sciences. Deuxième Partie: Notes et Mémoires*, (Paris, 1890) 179- -196 (lu 12 août 1889).

[13] J. J. B. Neuberg, Sur les quadrangles complets, *Mathesis*, 2 (1891) 33–35,67–70,81–82,189–195.

[14] E. H. Neville, Complete angle or cross?, *Mathematical Gazette* 11:163 (March, 1923) 275.

[15] D. Pedoe, *Geometry. A Comprehensive Course*, Cambridge University Press, 1970; also available as Dover pocket.

[16] D. K. Picken, The complete angle and geometrical generality, *Mathematical Gazette* 11:161 (Dec., 1922), 188–193.

[17] D. K. Picken, Euclidean geometry of angle, *Proceedings of the London Mathematical Society* (2) 23(1924) 45–55.

[18] E. Rouché et C. de Camberousse, *Traité de Géométrie*, 7th edition, Paris: Gauthier-Villars, 1900.

[19] J. Steiner, Aufgaben und Lehrsätze, erstere aufzulösen, letztere zu beweisen, *Journal für die reine und angewandte Mathematik* 2 (1827) 287–292 (problems 54–65).

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