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The Feuerbach Point and Reflections of the Euler Line

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Abstract. We investigate some results related to the Feuerbach point, and use a theorem of Hatzipolakis to give synthetic proofs of the facts that the reflections of OI in the sidelines of the intouch and medial triangle all concur at the Feuerbach point. Finally we give some results on certain reflections of the Feuerbach point.

1. Poncelet point

We begin with a review of the Poncelet point of a quadruple of points W, X, Y, Z. This is the point of concurrency of
(i) the nine-point circles of triangles WXY, WXZ, XYZ, WYZ,
(ii) the four pedal circles of W, X, Y, Z with respect to XYZ, WYZ, WXZ, WXY respectively.

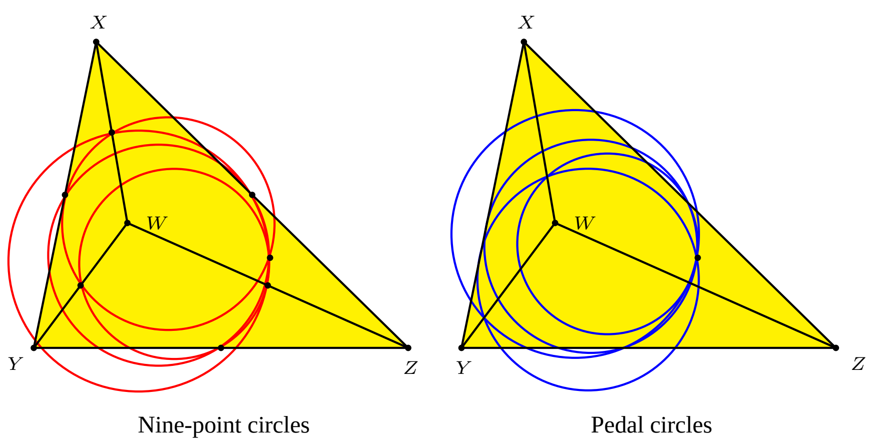

Figure 1.

Basic properties of the Poncelet point can be found in [4]. Let I be the incenter of triangle ABC. The Poncelet point of I, A, B, C is the famous Feuerbach point F_{e}, as we show in Theorem 1 below. In fact, we can find a lot more circles passing through F_{e}, using the properties mentioned in [4].

Theorem 1. *The nine-point circles of triangles AIB, AIC, BIC are concurrent at the Feuerbach point F_{e} of triangle ABC.*

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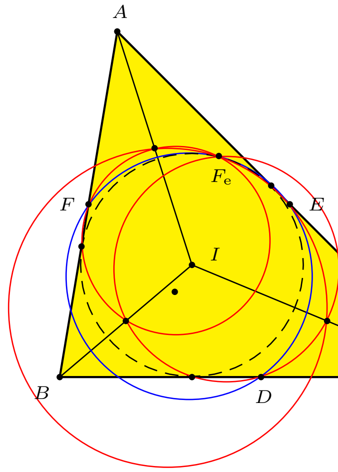

Figure 2.

Proof. The Poncelet point of A, B, C, I must lie on the pedal circle of I with respect to triangle ABC, and on the nine-point circle of triangle ABC (see Figure 1). Since these two circles have only the Feuerbach point $F_{\rm e}$ in common, it must be the Poncelet point of A, B, C, I. □

A second theorem, conjectured by Antreas Hatzipolakis, involves three curious triangles which turn out to have some very surprising and beautiful properties. We begin with an important lemma, appearing in [9] as Lemma 2 with a synthetic proof. The midpoints of BC, AC, AB are labeled D, E, F.

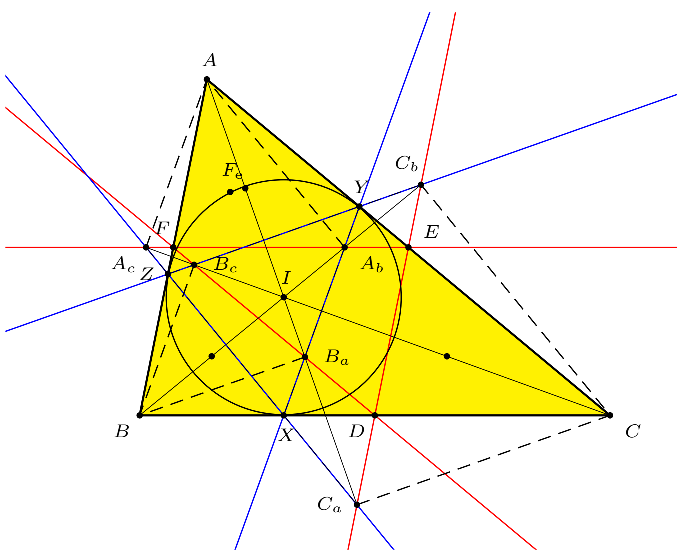

Figure 3.

We shall adopt the notations of [9]. Given a triangle ABC, let D, E, F be the midpoints of the sides BC, CA, AB, and X, Y, Z the points of tangency of the incircle with these sides. Let A_b and A_c be the orthogonal projections of A on the bisectors BI and CI respectively. Similarly define B_c, B_a, C_a, C_b (see Figure 3).

Lemma 2. (a) *A_b and A_c lie on EF.*
(b) *A_b lies on XY, A_c lies on XZ.*
Similar statements are true for B_a, B_c and C_a, C_b.

We are now ready for the second theorem, stated in [6]. An elementary proof was given by Khoa Lu Nguyen in [7]. We give a different proof, relying on the Kariya theorem (see [5]), which states that if X', Y', Z' are three points on IX, IY, IZ with $\frac{IX'}{IX} = \frac{IY'}{IY} = \frac{IZ'}{IZ} = k$, then the lines AX', BY', CZ' are concurrent. For $k = -2$, this point of concurrency is known to be X_{80}, the reflection of I in F_{e}.

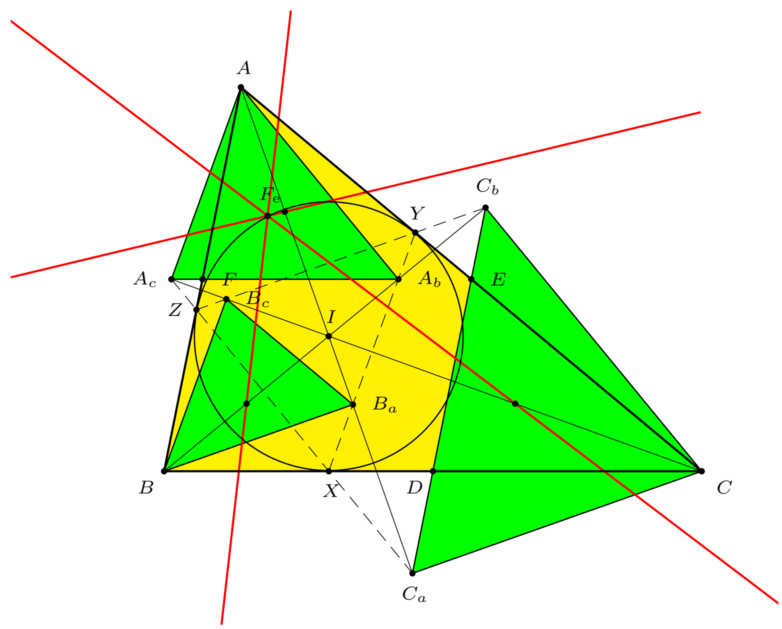

Figure 4.

Theorem 3 (Hatzipolakis). *The Euler lines of triangles AA_bA_c, BB_aB_c, CC_aC_b are concurrent at F_{e}* (see Figure 4).

Proof. If X' is the antipode of X in the incircle, O_a the midpoint of A and I, H_a the orthocenter of triangle AA_bA_c, then clearly H_aO_a is the Euler line of triangle AA_bA_c. Also, $\angle A_bAA_c = \pi - \angle A_cIA_b = \frac{B+C}{2}$. Because AI is a diameter of the circumcircle of triangle AA_bA_c, it follows that $AH_a = AI \cdot \cos\frac{B+C}{2} = AI \cdot \sin\frac{A}{2} = r$, where r is the inradius of triangle ABC. Clearly, $IX' = r$, and it follows from Lemma 1 that $AH_a \parallel IX'$. Triangles AH_aO_a and $IX'O_a$ are congruent, and X' is the reflection of H_a in O_a. Hence X' lies on the Euler line of triangle AA_bA_c.

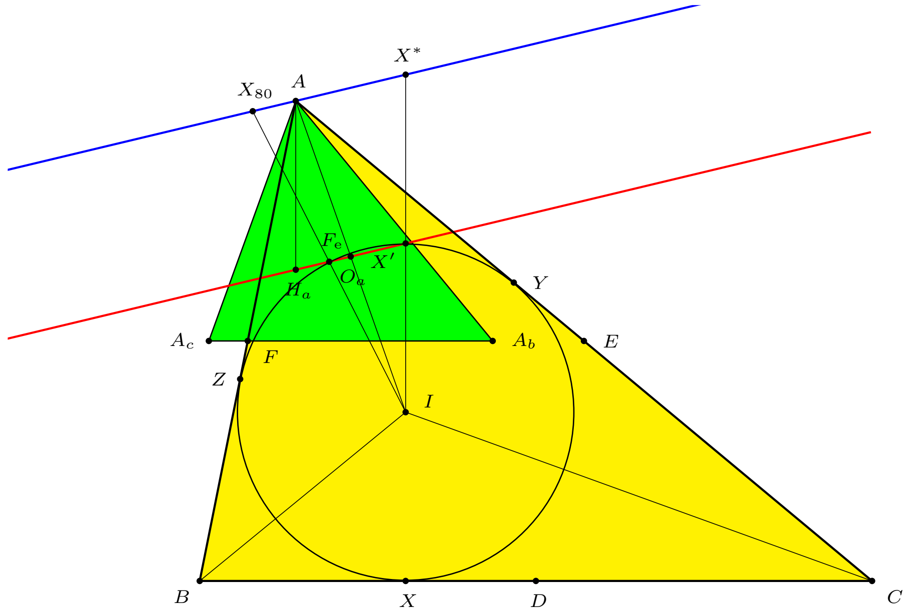

Figure 5.

If X^* is the reflection of I in X', we know by the Kariya theorem that A, X^*, and X_{80} are collinear. Now the homothety $\mathsf{h}(I, \frac{1}{2})$ takes A to O_a, X^* to X', and X_{80} to the Feuerbach point F_e. □

We establish one more theorem on the Feuerbach point. An equivalent formulation was posed as a problem in [10].

Theorem 4. *If X'', Y'', Z'' are the reflections of X, Y, Z in AI, BI, CI, then the lines DX'', EY'', FZ'' concur at the Feuerbach point F_e* (see Figure 6).

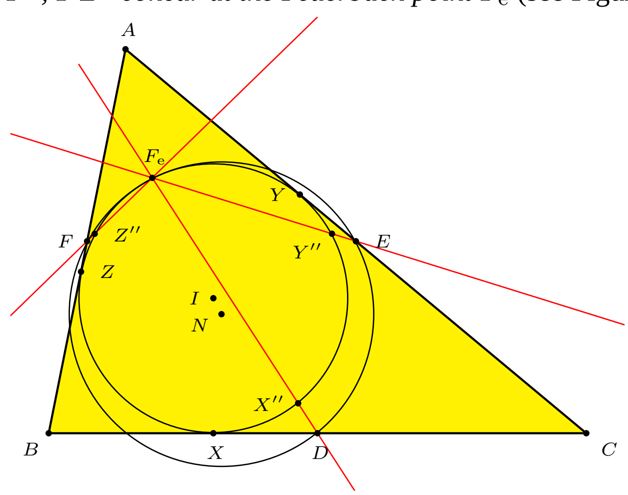

Figure 6.

Proof. We show that the line DX'' contains the Feuerbach point F_e. The same reasoning will apply to EY'' and FZ'' as well.

Clearly, X'' lies on the incircle. If we call N the nine-point center of triangle ABC, then the theorem will follow from $IX'' \parallel ND$ since F_e is the external center of similitude of the incircle and nine-point circle of triangle ABC. Now, because $IX \parallel AH$, and because O and H are isogonal conjugates, $IX'' \parallel AO$. Furthermore, the homothety $\mathsf{h}(G, -2)$ takes D to A and N to O. This proves that $ND \parallel AO$. It follows that $IX'' \parallel ND$. □

2. The Euler reflection point

The following theorem was stated by Paul Yiu in [11], and proved by barycentric calculation in [8]. We give a synthetic proof of this result.

Theorem 5. *The reflections of OI in the sidelines of the intouch triangle DEF are concurrent at the Feuerbach point of triangle ABC* (see Figure 7).

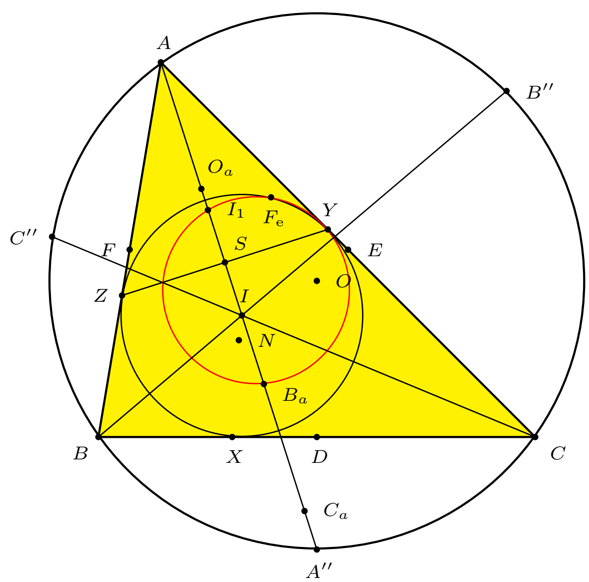

Figure 7.

Proof. Let us call I_1 the reflection of I in YZ. By Theorem 1, the nine-point circle of triangle AIC, which clearly passes through Y, O_a, C_a, also passes through F_e. If S is the intersection of YZ and AI, then clearly A is the inverse of S with respect to the incircle. Because $2 \cdot IO_a = IA$ and $2 \cdot IS = II_1$, it follows that O_a is the inverse of I_1 with respect to the incircle. Because C_a lies on XZ, its polar line must pass through B and be perpendicular to AI. This shows that B_a is the inverse of C_a with respect to the incircle.

Now invert the nine-point circle of triangle AIC with respect to the incircle of triangle ABC. This circle can never pass through I since $\angle AIC > \frac{\pi}{2}$, so the image is a circle. This shows that $YI_1F_eB_a$ is a cyclic quadrilateral, so it follows that $\angle F_eI_1B_a = \angle F_eYX = \angle F_eX'X$.

If we call $A''B''C''$ the circumcevian triangle of I, then we notice that $\angle AA_bA_c = \angle AIA_c = \angle A''IC$. Now, it is well known that $A''C = A''I$, so it follows that $\angle AA_bA_c = \angle ICA'' = \angle C''B''A''$. Similar arguments show that triangle AA_bA_c and triangle $A''B''C''$ are inversely similar.

As we have pointed out before as a consequence of Lemma 2, AH_a and IX' are parallel. By Theorem 3, F_eX' is the Euler line of triangle AA_bA_c. Therefor, $\angle F_eX'X = \angle O_aX'X = \angle O_aH_aA$. We know that triangle AA_bA_c is inversely similar to triangle $A''B''C''$. Since O and I are the circumcenter and orthocenter of triangle $A''B''C''$, it follows that $\angle O_aH_aA = \angle A''IO = \angle OIA$.

We conclude that $\angle F_eI_1S = \angle F_eI_1B_a = \angle F_eYX = \angle F_eX'X = \angle AIO = \angle SIO$. This shows that the reflection of OI in EF passes through F_e. Similar arguments for the reflections of OI in XY and XZ complete the proof. □

A very similar result is stated in the following theorem. We give a synthetic proof, similar to the proof of the last theorem in many ways. First, we will need another lemma.

Lemma 6. *The vertices of the polar triangle of DEF with respect to the incircle are the orthocenters of triangles BIC, AIC, AIB. Furthermore, they are the reflections of the excenters in the respective midpoints of the sides.*

This triangle is the main subject of [9], in which a synthetic proof can be found.

Theorem 7. *The reflections of OI in the sidelines of the medial triangle DEF are concurrent at the Feuerbach point of triangle ABC* (see Figure 8).

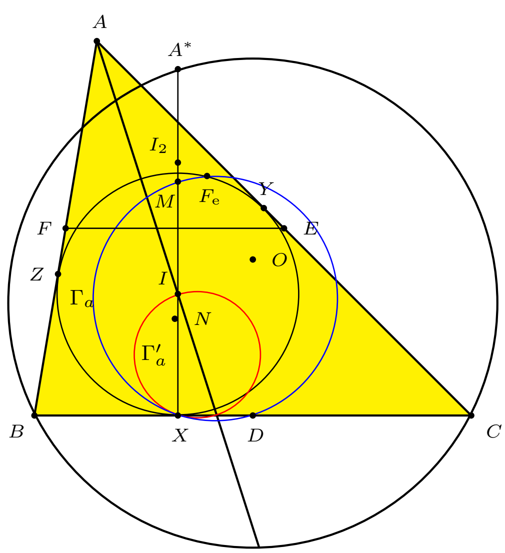

Figure 8.

Proof. Call I_2 the reflection of I in EF, and A^* the orthocenter of triangle BIC. The midpoint of I and A^* is called M. Using Lemma 6, we know that EF is the polar line of A^* with respect to the incircle. A similar argument as the one we gave in the proof of Theorem 5 shows that I_2 is the inverse of M with respect to the incircle.

Clearly, $F_{\rm e}$, M, X, D all lie on the nine-point circle of triangle BIC. Call this circle Γ_a and call Γ'_a the circumcircle of triangle IXX''. Clearly, the center of Γ'_a is on AI. Because I is the orthocenter of triangle BA^*C, we have that the reflection of I in D is the antipode of A^* in the circumcircle of A^*BC. Call this point L'. Consider the homothety $\mathsf{h}(I,2)$, MD is mapped, and hence is parallel, to A^*L'. We know that A^* is the reflection in D of the A-excenter of triangle ABC (see [9]), so A^*L' is also parallel to AI. It follows that AI and MD are parallel.

If we call T the intersection of AI and BC, then it is clear that T lies on Γ'_a. Because IT and MD are parallel diameters of two circles, there exists a homothety centered at X which maps Γ'_a to Γ_a. Because X lies on both circles, we now conclude that X is the point of tangency of Γ_a and Γ'_a. Inverting these two circles in the incircle, we see that XX'' is tangent to the circumcircle of $XF_{\rm e}I_2$.

Finally, $\angle MIO = \angle AIO + \angle MIA = \angle F_{\rm e}X'X + \angle IMD = \angle F_{\rm e}XD + \angle XF_{\rm e}D = \angle F_{\rm e}XD + \angle DXX'' = \angle F_{\rm e}XX'' = \angle F_{\rm e}I_2X$, where the last equation follows from the alternate segment theorem. This proves that $I_2F_{\rm e}$ is the reflection of OI in EF. Similar arguments for DF and DE prove the theorem. □

The following theorem gives new evidence for the strong correlation between the nature of the Feuerbach point and the Euler reflection point.

Theorem 8. *The three reflections of H_aO_a in the sidelines of triangle AA_bA_c and the line OI are concurrent at the reflection E_a of $F_{\rm e}$ in A_bA_c. Similar theorems hold for triangles BB_aB_c, CC_aC_b* (see Figure 9).

Proof. The 3 reflections of H_aO_a in the sidelines of triangle AA_bA_c are concurrent at the Euler reflection point of triangle AA_bA_c. We will first show that this point is the reflection of $F_{\rm e}$ in A_bA_c.

The circle with diameter XH_a clearly passes through A_b, A_c by definition of A_b, A_c. It also passes through $F_{\rm e}$, since $H_aF_{\rm e} = X'F_{\rm e} \perp XF_{\rm e}$, so we conclude that $F_{\rm e}$, A_c, X, A_b are concyclic. Because AA_cXA_b is a parallellogram, we see that the reflection in the midpoint of A_b and A_c of the circle through A_b, A_c, X, $F_{\rm e}$ is in fact the circumcircle of triangle AA_bA_c. We deduce that the reflection of $F_{\rm e}$ in A_bA_c lies on the circumcircle of triangle AA_bA_c. Since $F_{\rm e} \neq H_a$ lies on the Euler line of triangle AA_bA_c and E_a lies on the circumcircle of triangle AA_bA_c, we have proven that the reflection of $F_{\rm e}$ in A_bA_c is the Euler reflection point of triangle AA_bA_c.

By theorem 7, it immediately follows that E_a lies on OI. This completes the proof. □

We know that we can see E_a as an intersection point of the perpendicular to A_bA_c through $F_{\rm e}$ with the circumcircle of triangle AA_bA_c. This line intersects the

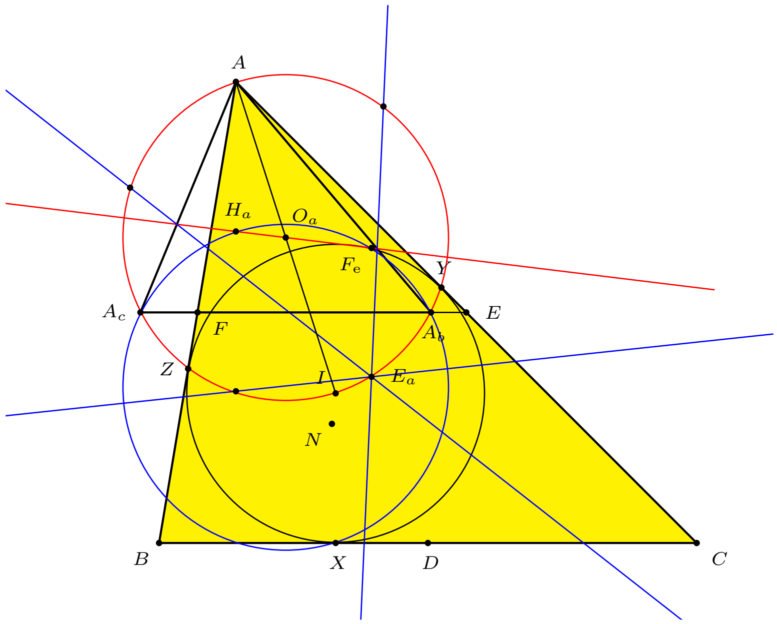

Figure 9.

circle in another point, which we will call U. Similarly define V and W on the circumcircles of triangles BB_aB_c and CC_aC_b.

Theorem 9. *The lines AU, BV, CW are concurrent at X_{80}, the reflection of I in* $F_{\rm e}$ (see Figure 10).

Proof. The previous theorem tells us that E_a lies on OI. It follows that $\angle E_aIO_a = \angle OIA$. In the proof of Theorem 5, we prove that $\angle AIO = \angle AH_aO_a$. Since F_eE_a and AH_a are parallel, we deduce that E_a, I, O_a and F_e are concyclic. If we call U' the intersection of $E_aF_{\rm e}$ and the line through A parallel to $O_aF_{\rm e}$, then we have that $\angle E_aIA = \angle E_aF_{\rm e}O_a = \angle E_aU'A$. It follows that A, U', E_a, I are concyclic, so $U \equiv U'$.

Now consider a homothety centered at I with factor 2. Clearly, $O_aF_{\rm e}$ is mapped to a parallel line through A, which is shown to pass through U. The image of $F_{\rm e}$ however is X_{80}, so AU passes through X_{80}. Similar arguments for BV, CW complete the proof. □

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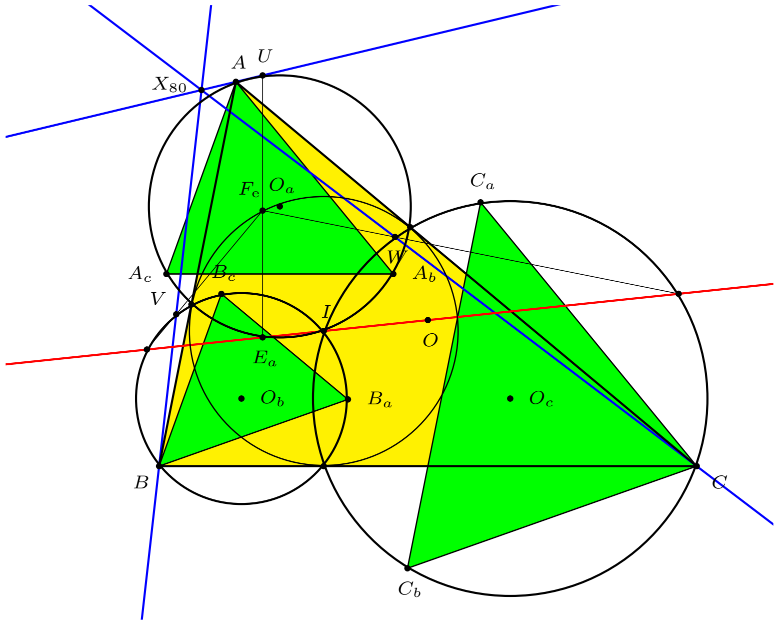

Figure 10.

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