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On the Newton Line of a Quadrilateral

Paris Pamfilos

Abstract. We introduce the idea of the conjugate polygon of a point relative to another polygon and examine the closing property of polygons inscribed in others and having sides parallel to a conjugate polygon. Specializing for quadrangles we prove a characterization of their Newton line related to the possibility to inscribe a quadrangle having its sides parallel to the sides of a conjugate one.

1. Introduction

Given two quadrangles $a = A_1A_2A_3A_4$ and $b = B_1B_2B_3B_4$ one can ask whether it is possible to inscribe in the first a quadrangle $c = C_1C_2C_3C_4$ having its sides parallel to corresponding sides of the second. It is also of importance to know how many solutions to the problem exist and which is their structure. The corresponding problem for triangles is easy to solve, well known and has relations to pivoting around a pivot-point of which there are twelve in the generic case ([9, p. 297], [8, p. 109]). Here I discuss the case of quadrangles and in some extend the case of arbitrary polygons. While in the triangle case the inscribed one is *similar* to a given triangle, for quadrangles and more general polygons this is no more possible. I start the discussion by examining properties of polygons inscribed in others to reveal some general facts. In this frame it is natural to introduce the class of *conjugate polygons* with respect to a point, which generalize the idea of the *precevian* triangle, having for vertices the *harmonic associates* of a point [12, p.100]. Then I discuss some properties of them, which in the case of quadrangles relate the inscription-problem to the Newton line of their associated *complete* quadrilateral (in this sense I speak of the *Newton line of the quadrangle* [13, p.169], [6, p.76], [3, p.69], [4], [7]). After this preparatory discussion I turn to the examination of the case of quadrangles and prove a characteristic property of their Newton line (§5, Theorems 11, 14).

2. Periodic polygon with respect to another

Consider two closed polygons $a = A_1 \cdots A_n$ and $b = B_1 \cdots B_n$ and pick a point C_1 on side A_1A_2 of the first. From this draw a parallel to side B_1B_2 of the second polygon until it hits side A_2A_3 to a point C_2 (see Figure 1). Continue in this way picking points C_i on the sides of the first polygon so that C_iC_{i+1} is parallel to side B_iB_{i+1} of the second polygon (indices $i > n$ are reduced modulo n if corresponding points X_i are not defined). In the last step draw a parallel to B_nB_1 from C_n until it hits the initial side A_1A_2 at a point C_{n+1}. I call polygon $c = C_1 \cdots C_{n+1}$ *parallel to* b *inscribed in* a *and starting at* C_1. In general polygon c is not closed. It can even have self-intersections and/or some side(s) degenerate to

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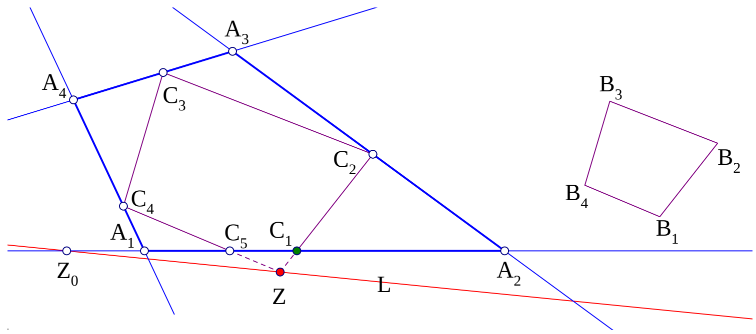

Figure 1. Inscribing a polygon

points (identical with vertices of a). One can though create a corresponding closed polygon by extending segment C_nC_{n+1} until to hit C_1C_2 at a point Z. Polygon $ZC_2\cdots C_n$ has sides parallel to corresponding sides of $B_1\cdots B_n$. Obviously triangle $C_{n+1}ZC_1$ has fixed angles and remains similar to itself if the place of the starting point C_1 changes on A_1A_2. Besides one can easily see that the function expressing the coordinate y of C_{n+1} in terms of the coordinate x of C_1 is a linear one $y = ax + b$. This implies that point Z moves on a fixed line L ([10, Tome 2, p. 10]) as point C_1 changes its position on line A_1A_2 (see Figure 1). This in turn shows that there is, in general, a unique place for C_1 on side A_1A_2 such that points C_{n+1}, C_1 coincide and thus define a *closed* polygon $C_1\cdots C_n$ inscribed in the first polygon and having its sides parallel to corresponding sides of the second. This place for C_1 is of course the intersection point Z_0 of line L with side A_1A_2. In the exceptional case in which L is parallel to A_1A_2 there is no such polygon. By the way notice that, for obvious reasons, in the case of triangles line L passes through the vertex opposite to side A_1A_2.

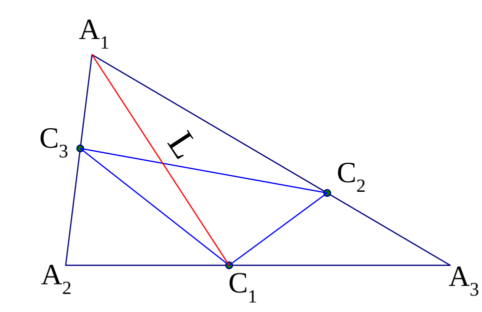

Figure 2. The triangle case

This example shows that the answer to next question is not in general in the affirmative. The question is: Under which conditions for the two polygons is line L identical with side A_1A_2, so that the above procedure produces always closed polygons $C_1\cdots C_n$? If this is the case then I say that polygon $B_1\cdots B_n$ is *periodic* with respect to $A_1 \cdots A_n$. Below it will be shown that this condition is independent of the side A_1A_2 selected. If it is satisfied by starting points C_1 on this side and drawing a parallel to B_1B_2 then it is satisfied also by picking the starting point C_i on side A_iA_{i+1}, drawing a parallel to B_iB_{i+1} and continuing in this way.

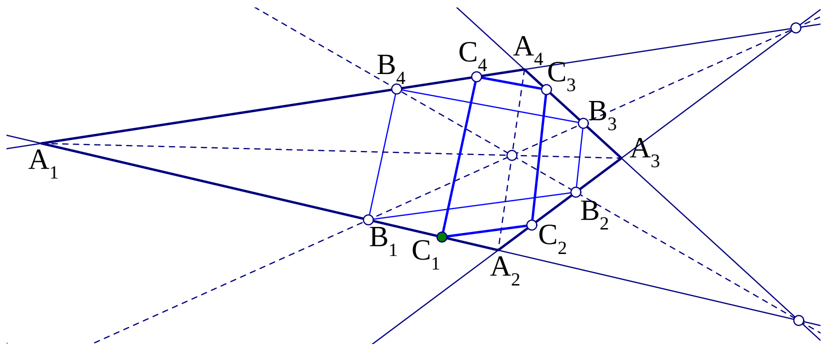

Figure 3. $B_1B_2B_3B_4$ periodic with respect to $A_1A_2A_3A_4$

There are actually plenty of examples of pairs of polygons satisfying the periodicity condition. For instance take an arbitrary quadrangle $A_1A_2A_3A_4$ and consider its $dual$ quadrangle $B_1B_2B_3B_4$, created through the intersections of its sides with the lines joining the intersection of its diagonals with the two intersection points of its pairs of opposite sides (see Figure 3). For every point C_1 on A_1A_2 the procedure described above closes and defines a quadrangle $C_1C_2C_3C_4$ inscribed in $A_1A_2A_3A_4$ and having its sides parallel to $B_1B_2B_3B_4$. This will be shown to be a consequence of Theorem 11 in combination with Proposition 16. It should be noticed though that periodicity, as defined here, is a relation depending on the *ordered* sets of vertices of two polygons. $B_1 \cdots B_n$ can be periodic with respect to $A_1 \cdots A_n$ but $B_2 \cdots B_nB_1$ not. Figure 4 displays such an example.

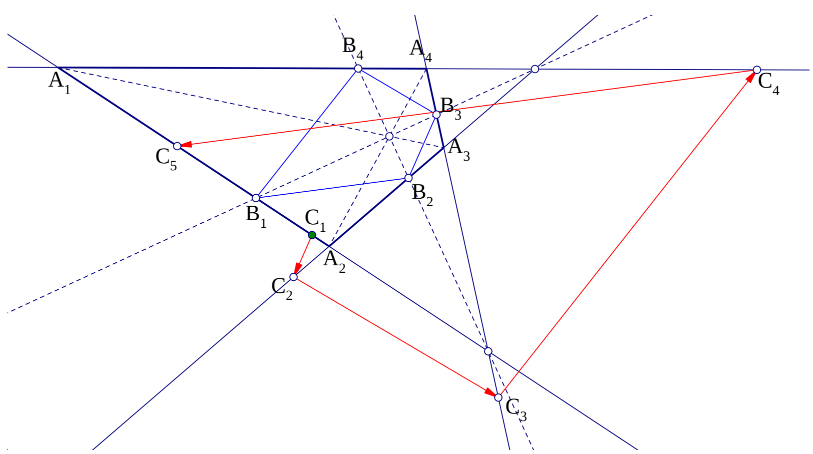

Figure 4. $B_2B_3B_4B_1$ not periodic with respect to $A_1A_2A_3A_4$

To handle the question in a systematic way I introduce some structure into the problem, which obviously is affinely invariant ([1], [2, vol.I, pp.32–66], [5]). I will consider the correspondence $C_1 \longmapsto C_{n+1}$ as the restriction on line A_1A_2 of a globally defined affine transformation G_1 and investigate the properties of this map. Figure 5 shows how transformation G_1 is constructed. It is the composition of *affine reflections* F_i ([5, p. 203]). The affine reflection F_i has its *axis* along A_iY_i which is the harmonic conjugate line of A_iX_i with respect to the two adjacent sides $A_{i-1}A_i$, A_iA_{i+1} at A_i. Its *conjugate direction* is A_iX_i which is parallel to side $B_{i-1}B_i$. By its definition map F_i corresponds to each point X the point Y such that the line-segment XY is parallel to the conjugate direction A_iX_i and has its middle on the axis A_iY_i of the map.

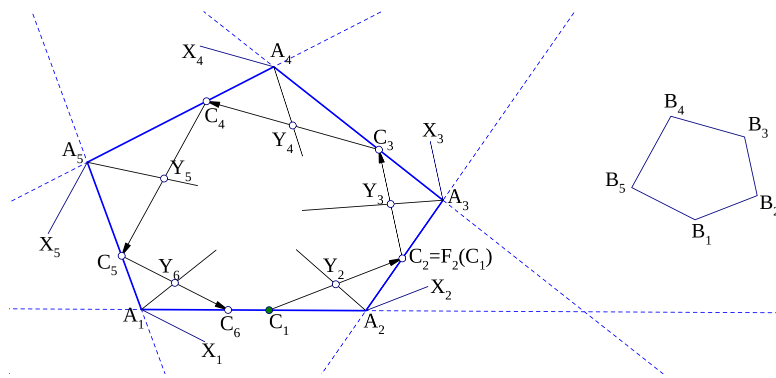

Figure 5. An affine transformation

The map $G_1 = F_1 \circ F_n \circ F_{n-1} \circ \cdots \circ F_2$ is a globally defined affine transformation, which on line A_1A_2 coincides with correspondence $C_1 \longmapsto C_{n+1}$. I call it *the first recycler of* b *in* a. Line A_1A_2 remains invariant by G_1 as a whole and each solution to our problem having $C_1 = C_{n+1}$ represents a fixed point of G_1. Thus, if there are more than one solutions, then line A_1A_2 will remain pointwise fixed under G_1. Assume now that G_1 leaves line A_1A_2 pointwise fixed. Then it is either an affine reflection or a *shear* ([5, p.203]) or it is the identity map, since these are the only affine transformations fixing a whole line and having determinant ± 1. Since G_1 is a product of affine reflections, its kind depends only on the number n of sides of the polygon. Thus for n even it is a shear or the identity map and for n odd it is an affine reflection. For n even it is shown by examples that both cases can happen: map G_1 can be a shear as well as the identity. In the second case I call $B_1 \cdots B_n$ *strongly periodic* with respect to $A_1 \cdots A_n$. The strongly periodic case delivers closed polygons $D_1 \cdots D_n$ with sides parallel to those of $B_1 \cdots B_n$ and the position of D_1 can be arbitrary. To construct such polygons start with an arbitrary point D_1 of the plane and define $D_2 = F_2(D_1), D_3 = F_3(D_2), \cdots, D_n = F_n(D_{n-1})$. The previous example of the dual of a quadrangle is a strongly periodic one (see Figure 6).

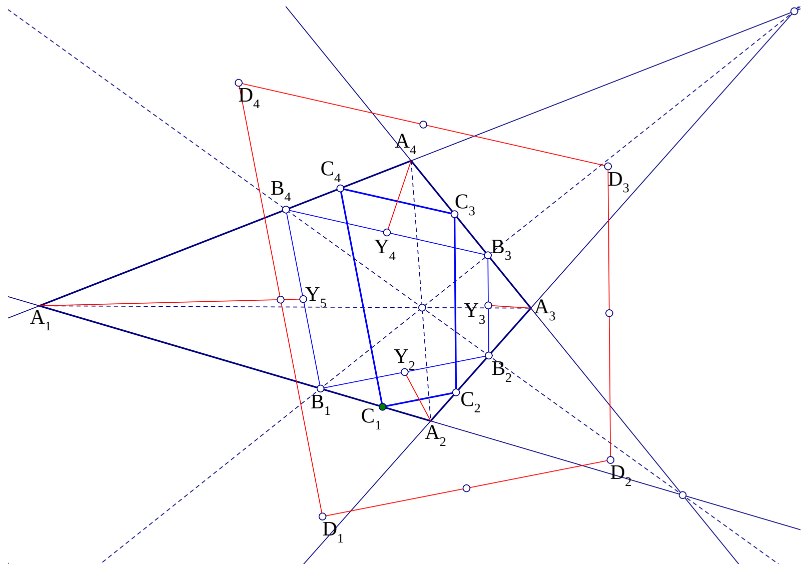

Figure 6. Strongly periodic case

Another case delivering many strongly periodic examples is that of a square $A_1A_2A_3A_4$ and the inscribed in it quadrangle $B_1B_2B_3B_4$, resulting by projecting an arbitrary point X on the sides of the square (see Figure 7).

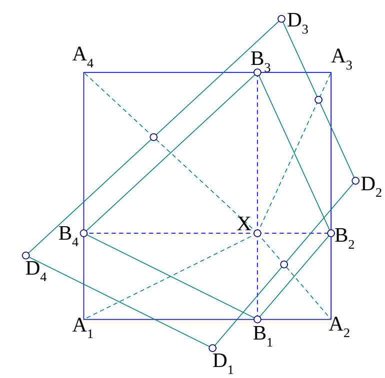

Figure 7. Strongly periodic case II

Analogously to G_1 one can define the affine map $G_2 = F_2 \circ F_1 \circ F_n \circ F_{n-1} \circ \cdots \circ F_3$, which I call *second recycler of* b *in* a. This does the same work in constructing a polygon $D_2 \cdots D_n D_1$ inscribed in $A_1 \cdots A_n$ and with sides parallel to those of $B_2 \cdots B_n B_1$ but now the starting point D_2 is to be taken on side A_2A_3, whereas the sides will be parallel successively to $B_2B_3, B_3B_4, \cdots$. Analogously are defined the affine maps $G_i, i = 3, \cdots, n$ (*i-th recycler of* b *in* a). It follows immediately from their definition that G_i are conjugate to each other. Obviously, since the F_i are involutive, we have $G_2 = F_2 \circ G_1 \circ F_2$ and more general $G_k = F_k \circ G_{k-1} \circ F_k$.

Thus, if there is a fixed point X_1 of G_1 on side A_1A_2, then $X_2 = F_2(X_1)$ will be a fixed point of G_2 on A_2A_3 and more general $X_k = F_k \circ \cdots \circ F_2(X_1)$ will be a fixed point of G_k on side A_kA_{k+1}. Corresponding property will be also valid in the case A_1A_2 remains pointwise fixed under G_1. Then every side A_kA_{k+1} will remain fixed under the corresponding G_k. The discussion so far is summarized in the following proposition.

Proposition 1. (1) *Given two closed polygons $a = A_1 \cdots A_n$ and $b = B_1 \cdots B_n$ there is in the generic case only one closed polygon $c = C_1 \cdots C_n$ having its vertex C_i on side A_iA_{i+1} and its sides C_iC_{i+1} parallel to B_iB_{i+1} for $i = 1, \cdots, n$. If there are two such polygons then there are infinite many and their corresponding point C_1 can be an arbitrary point of A_1A_2. In this case b is called periodic with respect to a.*

(2) *Using the sides of polygons a and b one can construct an affine transformation G_1 leaving invariant the side A_1A_2 and having the property: b is periodic with respect to a precisely when G_1 leaves side A_1A_2 pointwise fixed.*

(3) *In the periodic case, if n is odd then G_1 is an affine reflection with axis (mirror) line A_1A_2 and if n is even then it is a shear with axis A_1A_2 or the identity map. In the last case b is called strongly periodic with respect to a.*

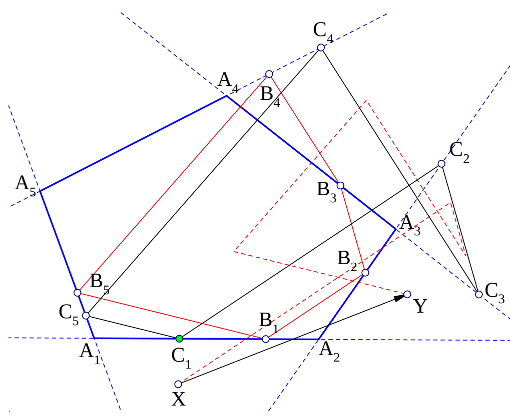

Figure 8. Periodic pentagons

Figure 8 shows a periodic case for $n = 5$. The figure shows also a typical pair $Y = G_1(X)$ of points related by the affine reflection G_1 resulting in this case.

3. Conjugate polygon

Given a closed polygon $a = A_1 \cdots A_n$ and a point P not lying on the side-lines of a, consider for each $i = 1, \ldots, n$ the harmonic conjugate line A_iX_i of line A_iP with respect to the two adjacent sides of a at A_i. The polygon $b = B_1 \cdots B_n$ having sides these lines is called *conjugate of a with respect to P*. The definition generalizes the idea of the *precevian triangle* of a triangle $a = A_1A_2A_3$ with respect to a point P, which is the triangle $B_1B_2B_3$ having vertices the *harmonic associates* B_i of P with respect to a ([12, p.100]).

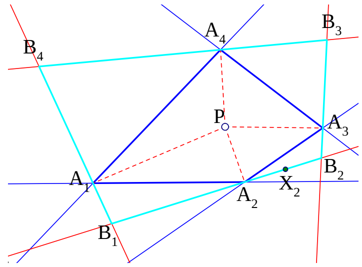

Figure 9. Conjugate quadrangle with respect to P

Proposition 2. *Given a closed polygon $a = A_1 \cdots A_n$ with n odd and a point P not lying on its side-lines, let $b = B_1 \cdots B_n$ be the conjugate polygon of a with respect to P. Then the transformation G_1 is an affine reflection the axis of which passes through P and its conjugate direction is that of line A_1A_2.*

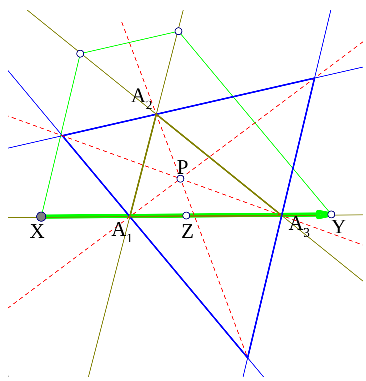

Figure 10. G_1 is an affine reflection

That point P remains fixed under G_1 is obvious, since G_1 is a composition of affine reflections all of whose axes pass through P. From this, using the preservation of proportions by affinities and the invariance of A_1A_2 follows also the that the parallels to A_1A_2 remain also invariant under G_1. Let us introduce coordinates (x, y) with origin at P and x-axis parallel to A_1A_2. Then G_1 has a representation of the form $\{x' = ax + by, y' = y\}$. Since its determinant is -1 it follows that $a = -1$. Thus, on every line $y = y_0$ parallel to A_1A_2 the transformation acts through $x' = -x + by_0 \Leftrightarrow x' + x = by_0$, showing that the action on line $y = y_0$ is a point symmetry at point Z with coordinates $(by_0/2, y_0)$, which remains also fixed by G_1 (see Figure 10). Then the whole line PZ remains fixed by G_1, thus showing it to be an affine reflection as claimed. The previous proposition completely solves the initial problem of inscription for conjugate polygons with n sides and n odd. In fact, as noticed at the beginning, such an inscription possibility corresponds to a fixed point of the map G_1 and this has a unique such point on A_1A_2. Thus we have next corollary.

Corollary 3. *If $b = B_1 \cdots B_n$ is the conjugate of the closed polygon $a = A_1 \cdots A_n$ with respect to a point P not lying on its side-lines and n is odd, then there is exactly one closed polygon $C_1 \cdots C_n$ with $C_i \in A_i A_{i+1}$ for every $i = 1, \cdots, n$ and sides parallel to corresponding sides of b. In particular, for n odd there are no periodic conjugate polygons.*

The analogous property for conjugate polygons and n even is expressed by the following proposition.

Proposition 4. *Given a closed polygon $a = A_1 \cdots A_n$ with n even and a point P not lying on its side-lines, let $b = B_1 \cdots B_n$ be the conjugate polygon of a with respect to P. Then the transformation G_1 either is a shear the axis of which is the parallel to side $A_1 A_2$ through P, or it is the identity map.*

The proof, up to minor changes, is the same with the previous one, so I omit it. The analogous corollary distinguishes now two cases, the second corresponding to G_1 being the identity. Periodicity and strong periodicity coincide when n is even and when b is the conjugate of a with respect to some point.

Corollary 5. *If $b = B_1 \cdots B_n$ is the conjugate of the closed polygon $a = A_1 \cdots A_n$ with respect to a point P not lying on its side-lines and n is even, then there is either no closed polygon $C_1 \cdots C_n$ with $C_i \in A_i A_{i+1}$ for every $i = 1, \cdots, n$ and sides parallel to corresponding sides of b, or b is strongly periodic with respect to a.*

Remark. Notice that the existence of even one fixed point not lying on the parallel to $A_1 A_2$ through P (the axis of the shear) imply that G_1 is the identity or equivalently, the corresponding conjugate polygon is strongly periodic.

The next propositions deal with some properties of conjugate polygons needed, in the case of quadrangles, in relating the periodicity to the Newton's line.

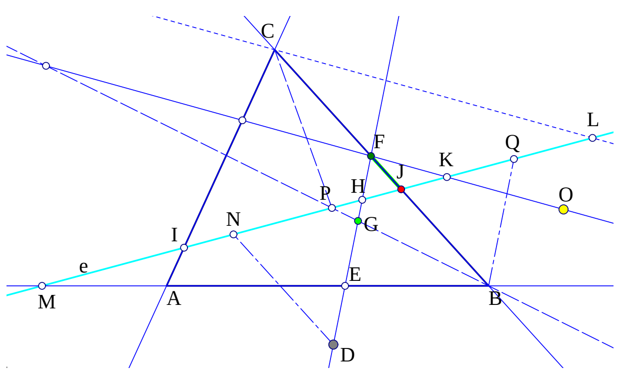

Figure 11. Fixed point O

Lemma 6. *Let $\{ABC, D, e\}$ be correspondingly a triangle, a point and a line. Consider a variable line through D intersecting sides AB, BC correspondingly at points E, F. Let G be the middle of EF and P the intersection point of lines*

e and BG. Let further CL be the harmonic conjugate of line CP with respect to CA, CB. Then the parallel to CL from F passes through a fixed point O.

To prove the lemma introduce affine coordinates with axes along lines $\{BC, e\}$ and origin at J, where $I = e \cap CA, J = e \cap CB$ (see Figure 11). The points on line e are: $M = e \cap AB, N = e \cap (||BC, D), H = e \cap DE, Q = e \cap (||DE, B)$, where the symbol $(||XY, Z)$ means: *the parallel to XY from Z*. Denote abscissas/ordinates by the small letters corresponding to labels of points, with the exceptions of $a = DN$, the abscissa x of F and the ordinate y of K. The following relations are easily deduced.

$$h = \frac{hx}{x+a}, \quad q = b\frac{h}{x}, \quad p = \frac{mq}{2q-m}, \quad l = \frac{pi}{2p-i}, \quad y = \frac{lx}{c}.$$

Successive substitutions produce a homographic relation between variables x, y:

$$p_1 x + p_2 y + p_3 xy = 0,$$

with constants (p_1, p_2, p_3), which is equivalent to the fact that line FK passes through point O with coordinates $(-\frac{p_2}{p_3}, -\frac{p_1}{p_3})$.

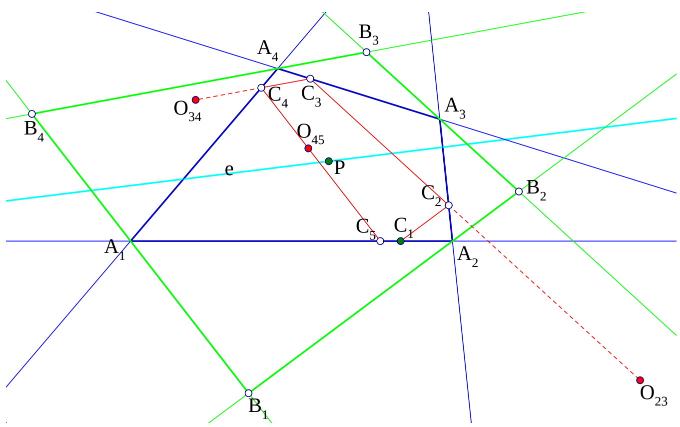

Figure 12. Sides through fixed points

Lemma 7. *Let $\{A_1 \cdots A_n, C_1, e\}$ be correspondingly a closed polygon, a point on side A_1A_2 and a line. Consider a point P varying on line e and the corresponding conjugate polygon $b = B_1 \cdots B_n$. Construct the parallel to b polygon $c = C_1 \cdots C_{n+1}$ starting at C_1. As P varies on e, every side of polygon c passes through a corresponding fixed point.*

The proof results by inductively applying the previous lemma to each side of c, starting with side C_1C_2, which by assumption passes through C_1 (see Figure 12). Next prove that side C_2C_3 passes through a point O_{23} by applying previous lemma to the triangle with sides A_1A_2, A_2A_3, A_3A_4 and by taking C_1 to play the role of D in the lemma. Then apply the lemma to the triangle with sides A_2A_3, A_3A_4, A_4A_5 taking for D the fixed point O_{23} of the previous step. There results a fixed point

O_{34} through which passes side C_3C_4. The induction continues in the obvious way, using in each step the fixed point obtained in the previous step, thereby completing the proof.

Lemma 8. *Let $\{A_1 \cdots A_n, C_1, e\}$ be correspondingly a closed polygon, a point on side A_1A_2 and a line. Consider a point P varying on line e, the corresponding conjugate polygon $b = B_1 \cdots B_n$ and the corresponding parallel to b polygon $c = C_1 \cdots C_{n+1}$ starting at C_1. Then the correspondence $P \longmapsto C_{n+1}$ is either constant or a projective one from line e onto line A_1A_2.*

Assume that the correspondence is not a constant one. Proceed then by applying the previous lemma and using the fixed points $O_{23}, O_{34}, \cdots$ through which pass the sides of the inscribed polygons c as P varies on line e. It is easily shown inductively that correspondences $f_1 : P \mapsto C_2,\ f_2 : P \mapsto C_3,\ \ldots, f_n : P \mapsto C_{n+1}$ are projective maps between lines. That f_1 is a projectivity is a trivial calculation. Map f_2 is the composition of f_1 and the perspectivity between lines A_3A_2, A_3A_4 from O_{23}, hence also projective. Map f_3 is the composition of f_2 and the perspectivity between lines A_4A_3, A_4A_5 from O_{34}, hence also projective. The proof is completed by the obvious induction.

4. The case of parallelograms

The only quadrangles not possessing a Newton line are the parallelograms. For these though the periodicity question is easy to answer. Next two propositions show that parallelograms are characterized by the strong periodicity of their conjugates with respect to *every* point not lying on their side-lines.

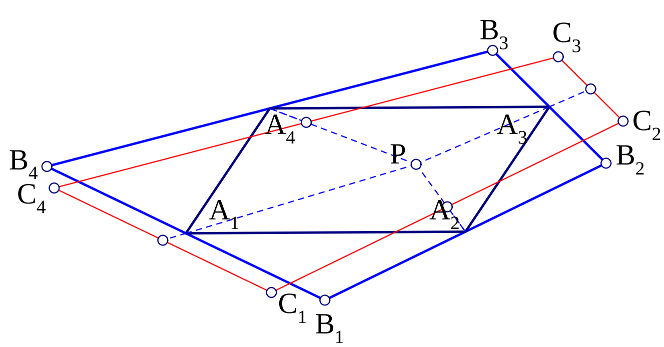

Figure 13. Parallelograms and periodicity

Proposition 9. *For every parallelogram $a = A_1A_2A_3A_4$ and every point P not lying on its side-lines the corresponding conjugate quadrangle $b = B_1B_2B_3B_4$ is strongly periodic.*

The proposition (see Figure 13) is equivalent to the property of the corresponding first recycler G_1 to be the identity. To prove this it suffices to show that G_1 fixes a point not lying on the parallel to A_1A_2 through P (see the remark after corollary 5 of previous paragraph). In the case of parallelograms however it is easily seen that a is the parallelogram of the middles of the sides of the conjugates b.

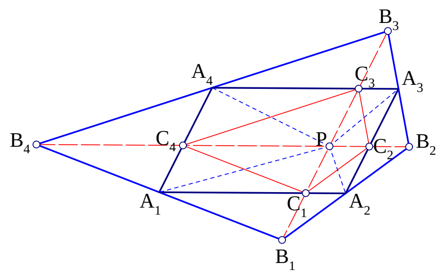

Figure 14. C_1 fixed by G_1

In fact, let $b = B_1B_2B_3B_4$ be the conjugate of a with respect to P and consider the intersection points $C_1, C_2, \cdots$ of the sides $A_1A_2, A_2A_3, \cdots$ of the parallelogram correspondingly with lines $PB_1, PB_2, \ldots$ (see Figure 14). The bundles of lines $A_1(B_1, P, C_1, A_4)$ at A_1 and $A_2(B_1, P, C_1, A_3)$ at A_2 are harmonic by the definition of b. Besides their three first rays intercept on line PB_1 correspondingly the same three points B_1, P, C_1 hence the fourth harmonic of these three points is the intersection point of their fourth rays A_1A_4, A_2A_3, which is the point at infinity. Consequently C_1 is the middle of PB_1. The analogous property for C_2, C_3, C_4 implies that quadrangle $c = C_1C_2C_3C_4$ has its sides parallel to those of b and consequently lines PA_i are the medians of triangles $PB_{i-1}B_i$. Thus point B_1 is a fixed point of G_1 not lying on its axis, consequently G_1 is the identity.

Proposition 10. *If for every point P not lying on the side-lines of the quadrangle $a = A_1A_2A_3A_4$ the corresponding conjugate quadrangle $b = B_1B_2B_3B_4$ is strongly periodic, then a is a parallelogram.*

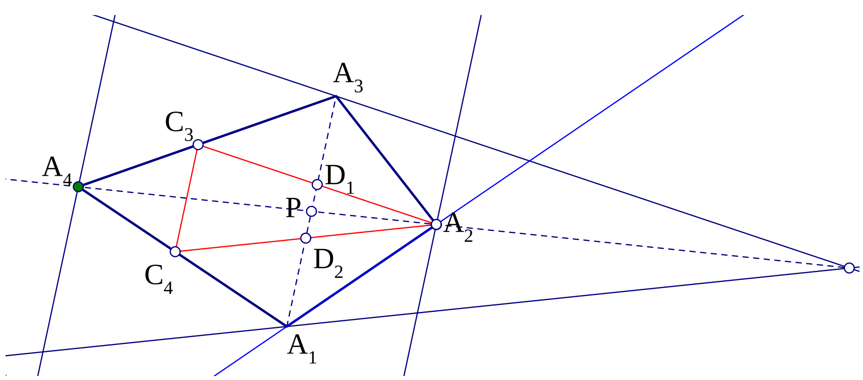

Figure 15. Parallelogram characterization

This is seen by taking for P the intersection point of the diagonals of the quadrangle. Consider then the parallel to b polygon starting at A_2. By assumption this must be closed, thus defining a triangle $A_2C_3C_4$ (see Figure 15). The middles D_1, D_2 of the sides of the triangle are by definition on the diagonal A_1A_3, which is parallel to C_3C_4. Thus the diagonal A_1A_3 is parallel to the conjugate direction of the other diagonal A_2A_4, consequently P is the middle of A_1A_3. Working in the same way with side A_2A_3 and the recycler G_2 it is seen that P is also the middle of A_2A_4, hence the quadrangle is a parallelogram.

5. A property of the Newton line

By the convention made above the *Newton line* of a quadrangle, which is not a parallelogram, is the line passing through the middles of the diagonals of the associated *complete* quadrilateral. In this paragraph I assume that the quadrangle of reference is not a parallelogram, thus has a Newton line. The points of this line are then characterized by having their corresponding conjugate quadrangle strongly periodic.

Theorem 11. *Given a non-parallelogramic quadrangle $a = A_1A_2A_3A_4$ and a point P on its Newton-line, the corresponding conjugate quadrangle $b = B_1B_2B_3B_4$ with respect to P is strongly periodic.*

Before starting the proof I supply two lemmata which reduce the periodicity condition to a simpler geometric condition that can be easily expressed in projective coordinates.

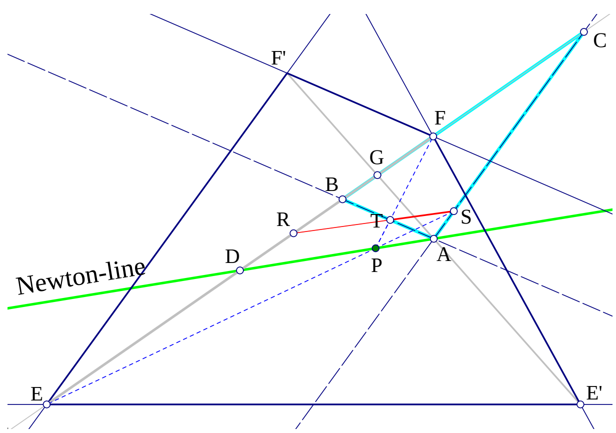

Figure 16. A fixed point

Lemma 12. *Let $a = EE'FF'$ be a quadrangle with diagonals $EF, E'F'$ and corresponding middles on them D, A. Draw from A parallels AB, AC correspondingly to sides $FF', F'E$ intersecting the diagonal EF correspondingly at points B, C. For every point P on the Newton-line AD of the quadrangle lines PE, PF*

intersect correspondingly lines AC, AB at points S, T. Line ST intersects the diagonal EF always at the same point R, which is the harmonic conjugate of the intersection point G of the diagonals with respect to B, C.

The proof is carried out using barycentric coordinates with respect to triangle ABC. Then points D, E, F, ... on line BC are represented using the corresponding small letters for parameters $D = B + dC$, $E = B + eC$, $F = B + fC$, ... (see Figure 16). In addition P is represented through a parameter p in $P = D + pA$. First we calculate E', F' in terms of these parameters:

$$E' = (f + g + 2fg)A - fB - (fg)C,$$
$$F' = (g - f)A + fB + fgC.$$

Then the coordinates of S, T are easily shown to be:

$$S = pA + (d - e)C,$$
$$T = (pf)A + (f - d)B.$$

From these the intersection point R of line ST with BC is seen to be:

$$R = (d - f)B + (f(d - e))C.$$

This shows that R is independent of the value of parameter p hence the same for all points P on the Newton-line. Some more work is needed to verify the claim about its precise location on line AB. For this the parallelism EF' to AC and the fact that D is the middle of EF are proved to be correspondingly equivalent to the two conditions:

$$g = \frac{f(1 + e)}{1 + f}, \quad d = \frac{f(e + 1) + e(f + 1)}{(e + 1) + (f + 1)}.$$

These imply in turn the equation

$$g = \frac{f(e - d)}{d - f},$$

which is easily shown to translate to the fact that R is the harmonic conjugate of G with respect to B, C.

Lemma 13. *Let $a = ABCD$ be a quadrangle with diagonals AC, BD and corresponding middles on them M, N. Draw from M parallels ME, MF correspondingly to sides AB, AD intersecting the diagonal BD at points E, F. Let P be a point of the Newton-line MN and S, T correspondingly the intersections of line-pairs $(PB, ME), (PD, MF)$. The conjugate quadrangle of P is periodic precisely when the harmonic conjugate of AP with respect to AB, AD is parallel to ST.*

In fact, consider the transformation $G_1 = F_4 \circ F_3 \circ F_2 \circ F_1$ composed by the affine reflections with corresponding axes PC, PD, PA, PB. By the discussion in the previous paragraph, the periodicity of the conjugate quadrilateral to P is equivalent to G_1 being the identity. Since G_1 is a shear and acts on BC in general as a translation by a vector $\mathbf{v}$ to show that $\mathbf{v} = \mathbf{0}$ it suffices to show that it fixes

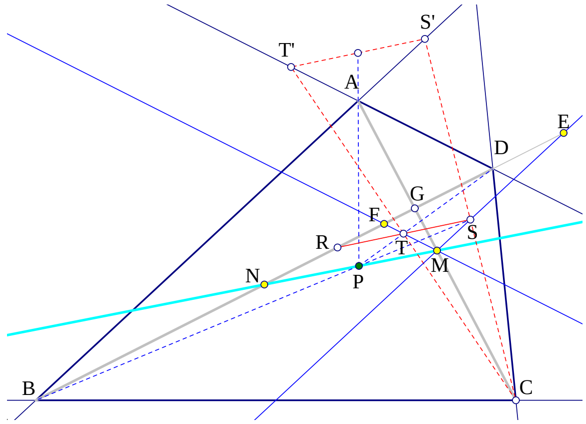

Figure 17. Equivalent problem

an arbitrary point on BC. This criterion applied to point C means that for $T' = F_2(C), S' = F_3(T')$ point $C' = F_4(S')$ is identical with C (see Figure 17). Since T, S are the middles of CT', CS', this implies the lemma.

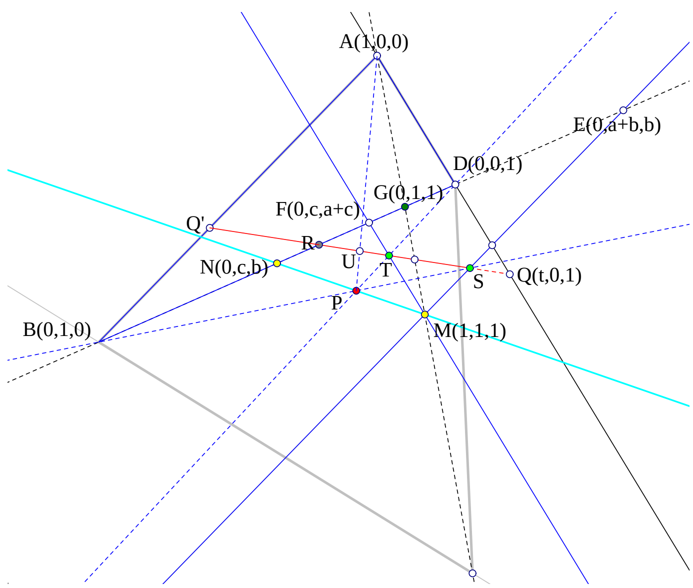

Figure 18. Representation in coordinates

Proof of the theorem: Because of the lemmata 12 and 13 one can consider the variable point P not as an independent point varying on the Newton-line MN but as a construct resulting by varying a line through R which is the harmonic conjugate of the intersection point G of the diagonals with respect to E, F. Such a line intersects the parallels ME, MF to sides AB, AD at S, T and determines P as intersection of lines BS, DT. Consider the coordinates defined by the projective basis (see Figure 18) $\{A(1,0,0), B(0,1,0), D(0,0,1), M(1,1,1)\}$. Assume further that the line at infinity is represented by an equation in the form

$$ax + by + cz = 0.$$

Then all relevant points and lines of the figure can be expressed in terms of the constants (a, b, c). In particular

$$ax - (a + c)y + cz = 0, \quad ax + by - (a + b)z = 0, \quad (c - b)x + by - cz = 0,$$

are the equations of lines MF, ME and the Newton-line MN. Point R has coordinates $(0, a', b')$, where $a' = (c - a - b)$, $b' = (a + c - b)$. Assume further that the parametrization of a line through R is done by a point $Q(t, 0, 1)$ on line AD. This gives for line RQ the equation $RQ : a'x + (tb')y - (ta')z = 0$. Point S has coordinates (a'', b'', c'') where $a'' = t(a'b-b'(a+b)),\ b'' = a'(a+b)-taa',\ c'' = a'b - tab'$. This gives for P the coordinates $(ba'', cc'' - a''(c - b), bc'')$ and the coordinates of the intersection point U of PA with RQ can be shown to be $U = c''Q - (c + at)Q'$, where $Q'(tb', -a', 0)$ is the intersection point of AB and RQ. From these follows easily that U is the middle of QQ' showing the claim according to Lemma 13.

Theorem 14. *For a non-parallelogramic quadrangle $a = A_1A_2A_3A_4$ only the points P on its Newton-line have the corresponding conjugate quadrangle $b = B_1B_2B_3B_4$ strongly periodic.*

The previous theorem guarantees that all points of the Newton line have a strongly periodic corresponding conjugate polygon b. Assume now that there is an additional point P_0, not on the Newton line, which has also a strongly periodic corresponding conjugate polygon. In addition fix a point C_1 on A_1A_2. Take then a point P_1 on the Newton line and consider line $e = P_0P_1$. By Lemma 8 the correspondence $f : e \to A_1A_2$ sending to each point $P \in e$ the end-point C_{n+1} of the polygon parallel to the conjugate b of a with respect to P starting at a fixed point C_1 is either a constant or a projective map. Since f takes for two points P_0, P_1 the same value (namely $f(P_0) = f(P_1) = C_1$) this map is constant. Hence the whole line e consists of points having corresponding conjugate polygon strongly periodic. This implies that any point of the plane has the same property. In fact, for an arbitrary point Q consider a line e_Q passing through Q and intersecting e and the Newton line at two points Q_0 and Q_1. By the same reasoning as before we conclude that all points of line e_Q have corresponding conjugate polygons strongly periodic, hence Q has the same property. By Proposition 10 of the preceding paragraph it follows that the quadrangle must be a parallelogram, hence a contradiction to the hypothesis for the quadrangle.

6. The dual quadrangle

In this paragraph I consider a non-parallelogramic quadrangle $a = A_1A_2A_3A_4$ and its *dual* quadrangle $b = B_1B_2B_3B_4$, whose vertices are the intersections of the sides of the quadrangle with the lines joining the intersection point of its diagonals with the intersection points of its two pairs of opposite sides. After a preparatory lemma, Proposition 16 shows that b is the conjugate polygon with respect to an appropriate point on the Newton line, hence b is strongly periodic.

Lemma 15. *Let $a = A_1A_2A_3A_4$ be a quadrangle and with diagonals intersecting at E. Let also $\{F, G\}$ be the two other diagonal points of its associated complete quadrilateral. Let also $b = B_1B_2B_3B_4$ be the dual quadrangle of a.*
(1) *Line EG intersects the parallel A_4N to the side B_1B_4 of b at its middle M.*
(2) *Side B_1B_2 of b intersects the segment MN at its middle O.*

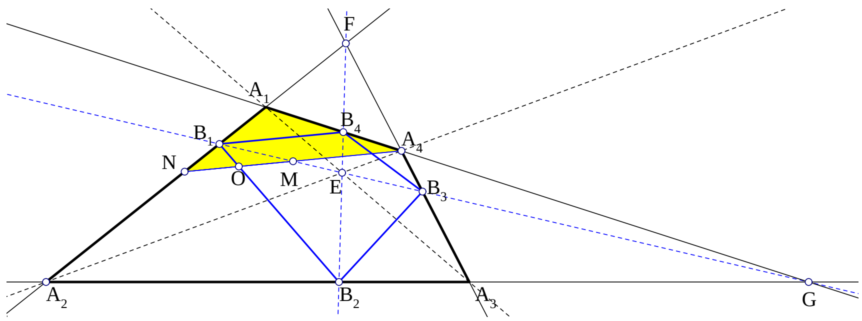

Figure 19. Dual property

$MN/MA_4 = 1$, since Menelaus theorem applied to triangle A_1NA_4 with secant line B_1B_3G gives $(B_1N/B_1A_1)(MA_4/MN)(GA_1/GA_4) = 1$. But $B_1N/B_1A_1 = B_4A_4/B_4A_1 = GA_4/GA_1$. Later equality because (B_4, G) are harmonic conjugate to (A_1, A_4). Also $ON/OM = 1$, since the bundle $B_1(B_2, B_4, E, F)$ is harmonic. Thus the parallel NM to line B_1B_4 of the bundle is divided in two equal parts by the other three rays of the bundle.

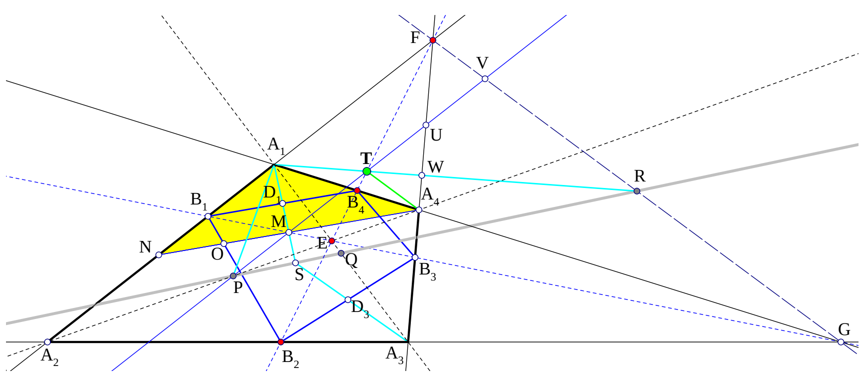

Figure 20. Dual is strongly periodic

Proposition 16. *Let $a = A_1A_2A_3A_4$ be a quadrangle and with diagonals intersecting at E. Let also $\{F, G\}$ be the two other diagonal points of its corresponding complete quadrilateral and $\{P, Q, R\}$ the middles of the diagonals $\{A_2A_4, A_1A_3, FG\}$ contained in the Newton line of the quadrilateral. Let $b =$*

$B_1B_2B_3B_4$ *be the dual quadrangle of* a
(1) *The four medians* $\{A_1D_1, A_2D_2, A_3D_3, A_4D_4\}$ *of triangles* $\{A_1B_1B_4$, $A_2B_2B_1$, $A_3B_3B_2$, $A_4B_4B_3\}$ *respectively meet at a point* S *on the Newton line.*
(2) S *is the harmonic conjugate of the diagonal middle* R *with respect to the two others* (P, Q).

Start with the intersection point T of diagonal B_2B_4 with line A_1R (see Figure 20). Draw from T line TV parallel to side A_1A_2 intersecting side A_3A_4 at U. Since the bundle $F(V, T, U, A_1)$ is harmonic and TV is parallel to ray FA_1 of it point U is the middle of TV. Since $A_4(A_1, W, T, R)$ is a harmonic bundle and R is the middle of FG, its ray A_4T is parallel to FG. It follows that A_4TFV is a parallelogram. Thus U is the middle of A_4F, hence the initial parallel TV to line A_1A_2 passes through the middles of segments having one end-point at A_4 and the other on line A_1A_2. Among them it passes through the middles of $\{A_1A_4, A_4N, A_4A_2\}$ the last being P the middle of the diagonal A_2A_4. Extend the median A_1D_1 of triangle $A_1B_1B_4$ to intersect the Newton line at S. Bundle $A_1(P, Q, S, R)$ is harmonic. In fact, using Lemma 15 it is seen that it has the same traces on line TV with those of the harmonic bundle $E(P, A_1, M, T)$. Thus S is the harmonic conjugate of R with respect to (P, Q).

Remarks. (1) Poncelet in a preliminary chapter [10, Tome I, p. 308] to his celebrated *porism* (see [2, Vol. II, pp. 203–209] for a modern exposition) examined the idea of *variable* polygons $b = B_1 \cdots B_n$ having *all but one* of their vertices on fixed lines (sides of another polygon) and restricted by having their sides to pass through corresponding fixed points $E_1, \cdots, E_n$. Maclaurin had previously shown that in the case of triangles ($n = 3$) the free vertex describes a conic([11, p. 248]). This generalizes to polygons with arbitrary many sides. If the fixed points through which pass the variable sides are *collinear* then the free vertex describes a line ([10, Tome 2, p. 10]). This is the case here, since the fixed points are the points on the line at infinity determining the directions of the sides of the inscribed polygons.

(2) In fact one could formulate the problem handled here in a somewhat more general frame. Namely consider polygons inscribed in a fixed polygon $a = A_1 \cdots A_n$ and having their sides passing through corresponding fixed *collinear* points. This case though can be reduced to the one studied here by a projectivity f sending the line carrying the fixed points to the line at infinity. The more general problem lives of course in the projective plane. In this frame the affine reflections F_i, considered above, are replaced by *harmonic homologies* ([5, p. 248]). The center of each F_i is the corresponding fixed point E_i through which passes a side B_iB_{i+1} of the variable polygon. The axis of the homology is the polar of this fixed point with respect to the side-pair $(A_{i_1}A_i, A_iA_{i+1})$ of the fixed polygon. The definitions of periodicity and the related results proved here transfer to this more general frame without difficulty.

(3) Though I am speaking all the time about a quadrangle, the property proved in §5 essentially characterizes the associated *complete quadrilateral*. If a point P has a periodic conjugate with respect to one, out of the three, quadrangles embedded

in the complete quadrilateral then it has the same property also with respect to the other two quadrangles embedded in the quadrilateral.

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Paris Pamfilos: Department of Mathematics, University of Crete, 71409 Herakleion, Greece
E-mail address: `pamfilos@math.uoc.gr`