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Folding a Square to Identify Two Adjacent Sides

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Abstract. The purpose of this paper is to establish some properties that appear in a square cut by two rays at 45 degrees passing through a vertex of the square. Elementary proofs and other interesting comments are provided.

1. A simple problem and a reformulation

The starting point of this work is the following problem from [3], partially discussed in [4]. [1]

Proposition 1. *Two points M and N on the hypotenuse BD of the isosceles, right-angled triangle ABD, with M between B and N, define an angle $\measuredangle MAN = 45^\circ$ if and only if $BM^2 + ND^2 = MN^2$* (see Figure 1).

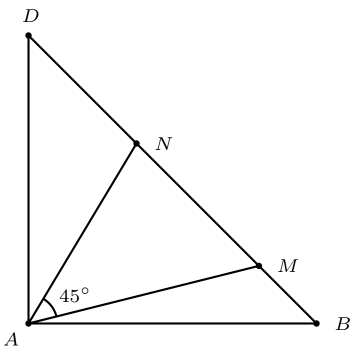

Figure 1

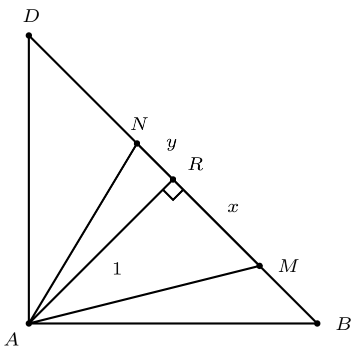

Figure 3

Proof. Let R be the midpoint of BD so that $AR = BR = BR$, and AR is an altitude of triangle ABD. We assume $AR = 1$ and denote $RM = x$, $RN = y$ (see Figure 3). Note that

$$\tan(\measuredangle MAN) = \tan(\measuredangle MAR + \measuredangle NAR) = \frac{x+y}{1-xy} = 1.$$

It follows that $\measuredangle MAN = 45^\circ$ if and only if $x + y = 1 - xy$. On the other hand, $BM^2 + ND^2 = MN^2$ if and only if $(1-x)^2 + (1-y)^2 = (x+y)^2$. Equivalently, $x + y = 1 - xy$, the same condition for $\measuredangle MAN = 45^\circ$. $\square$

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[1] This problem (erroneously attributed to another author in [1]) was considered by Boskoff and Suceavă as an example of an elliptic projectivity characterized by the Pythagorean relation.

This necessary and sufficient condition assumes new, interesting forms if we consider the isosceles right triangle as a half-square, and fold the adjacent sides AB and AC along the lines AM and AN. Without loss of generality we assume $AB = AC = 1$.

Theorem 2. *Let $ABCD$ be a unit square. Two half-lines through A meet the diagonal BD at M and N, and the sides BC, CD at M and P and Q respectively* (see Figure 2). *Assume $AP \not\equiv AQ$.*

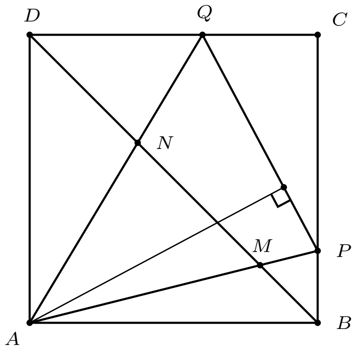

Figure 2

The following statements are equivalent:
(i) $\measuredangle PAQ = 45°$.
(ii) $MN^2 = BM^2 + ND^2$.
(iii) *The perimeter of triangle CPQ is equal to* 2.
(iv) $PQ = BP + QD$.
(v) *The distance from A to line PQ is equal to* 1.
(vi) *The area of triangle AMN is half of the area of triangle APQ.*
(vii) $PQ = \sqrt{2} \cdot MN$.
(viii) $PQ^2 = 2(BM^2 + ND^2)$.
(ix) *The line passing through A and $MQ \cap NP$ is perpendicular on PQ.*
(x) $AN = NP$.
(xi) $AM = MQ$.

Remark. In the excluded case $AP = AQ$, statement (ix) does not imply the other statements.

Proof of Theorem 2. With Cartesian coordinates $A(0,\ 0)$, $B(1,\ 0)$, $C(1,\ 1)$, $D(0,\ 1)$ and $P(1,\ a)$, $Q(b,\ 1)$ for some distinct $a, b \in (0,\ 1)$, we have $M(\frac{1}{1+a},\ \frac{a}{1+a})$ and $N(\frac{b}{1+b},\ \frac{1}{1+b})$. Then (i)-(xi) are each equivalent to

$$a + b + ab = 1. \tag{1}$$

This is clear from the following, which are obtained from routine calculations.
(i): $\tan \measuredangle PAQ = 1 - \frac{a+b+ab-1}{a+b}$.
(ii): $MN^2 - BM^2 - ND^2 = -\frac{2(a+b+ab-1)}{(b+1)(a+1)}$.

(iii, iv): $(PQ - BP - QD) + 2 = CP + PQ + QC = 2 - \frac{2(a+b+ab-1)}{a+b+\sqrt{(1-a)^2+(1-b)^2}}$.
(v): $\operatorname{dist}(A, PQ) = 1 + \frac{(1-a)(1-b)(a+b+ab-1)}{(1-ab+\sqrt{(1-a)^2+(1-a)^2})\sqrt{(1-a)^2+(1-b)^2}}$.
(vi): $\frac{\text{area}[AMN]}{\text{area}[APQ]} = \frac{1}{2} + \frac{a+b+ab-1}{2(1+a)(1+b)}$.
(vii): $PQ^2 - 2MN^2 = (ab + a + b - 1) \cdot \frac{(a+b)(a-b)^2+(ab^3+a^3b+a^2+b^2+2-6ab)}{(1+a)^2(1+b)^2}$.
(viii): $PQ^2 - 2(BM^2 + ND^2) = (a + b + ab - 1) \cdot f(a, b)$, where

$$f(a, b) := \frac{-4a - 4b - ab^2 - a^2b + ab^3 + a^3b - 10ab + a^2 + a^3 + b^2 + b^3 - 2}{(a + 1)^2(b + 1)^2}.$$

(ix): If O is the intersection of PN and QN, then

$$m_{AO}m_{PQ} = -1 + \frac{(a - b)(a + b + ab - 1)}{b(1 - b)(a + 1)}.$$

(x): $AN^2 - NP^2 = (a + b + ab - 1) \cdot \frac{1-a}{1+b}$.
(xi) $AM^2 - MQ^2 = (a + b + ab - 1) \cdot \frac{1-b}{1+a}$.

The expression (vii) is indeed equivalent with (1), if we take into account that

$$\frac{a^2 + b^2 + a^3b + ab^3 + 1 + 1}{6} > \sqrt[6]{a^2 \cdot b^2 \cdot a^3b \cdot ab^3 \cdot 1 \cdot 1} = ab.$$

For (viii), we prove that the $f(a, b) < 0$ for $a, b \in [0, 1]$. This is because, regarded as a function of $a \in [0, 1]$, $f''(a) = 6a + 6ab + 2(1 - b) > 0$. Since $f(0) < 0$ and $f(1) < 0$, we conclude that $f(a) < 0$ for $a \in [0, 1]$. $\square$

2. A simple geometric proof of (i) ⇔ (ii)

Statement (ii) clearly suggests a right triangle with sides congruent to BM, ND and MN. One way to do this is indicated in Figure 5, where M' is chosen such that the segment DM' is perpendicular to BD and is congruent to BM. Under the hypothesis (ii), we have $M'N = MN$. Moreover, $\Delta AMB \equiv \Delta AM'D$, and $\measuredangle MAM' = 90°$. It also follows that the triangles AMN and $AM'N$ have three pairs of equal corresponding sides, and are congruent. From this, $\measuredangle MAN = \measuredangle NAM' = 45°$. This shows that (ii) $\implies$ (i).

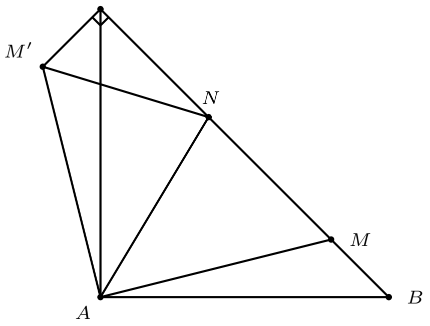

Figure 5

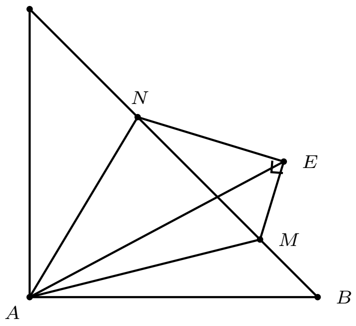

Figure 6

Another idea is to build an auxiliary right triangle with the hypotenuse MN, whose legs have lengths equal to BM and ND. This is based on the simple idea of folding the half-square ABD along AM and AN to identify the adjacent sides AB and AC. Let E be the reflection of B in the line AM (see Figure 6). Note that $BM = ME$. Assuming $\measuredangle MAN = 45^\circ$, we see that E is also the reflection of D in the line AN. Now the triangles AMB and AME are congruent, so are the triangles ANE and AND. Thus, $\measuredangle MEN = \measuredangle MEA + \measuredangle NEA = \measuredangle MBA + \measuredangle NDA = 45^\circ + 45^\circ = 90^\circ$. By the Pythagorean theorem, $MN^2 = ME^2 + EN^2 = BM^2 + ND^2$. This shows that (i) $\implies$ (ii).

3. A generalization

V. Proizolov has given in [6] the following nice result illustrating the beauty of the configuration of Theorem 2.

Proposition 3. *If M and N are points inside a square $ABCD$ such that $\measuredangle MAN = \measuredangle MCN = 45^\circ$, then $MN^2 = BM^2 + ND^2$* (see Figure 8).

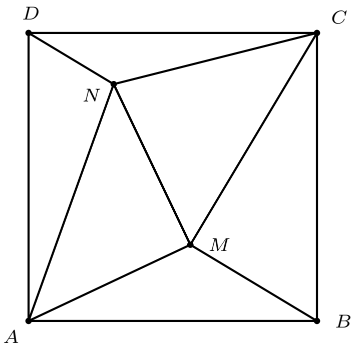

Figure 8

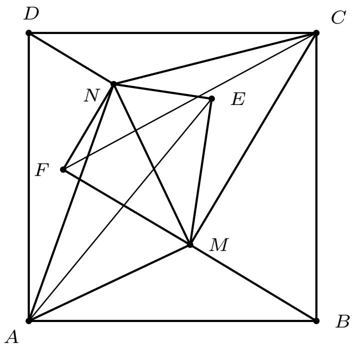

Figure 8A

This situation can be viewed as a surprising extension from the case of triangle ABD in Figure 6 is distorted into the polygon $ABMND$. In fact, by considering the symmetric of triangle ABD with respect to hypotenuse BD in Figure 1, a particular case of Proposition 3 is obtained. This analogy carries over to the general case. More precisely, we try to use the auxiliary construction from Figure 6, namely to consider the point E such that the triangles ANE and AND are symmetric and also the triangles AME and AMB are symmetric.

Let F be analogue defined, starting from the vertex C (see Figure 8A).

It follows that $\measuredangle MEN + \measuredangle MFN = 180^\circ$, as the sum of the angles $\measuredangle B$ and $\measuredangle D$ of the square. But the triangles MEN and MFN are congruent, so $\measuredangle MEN = \measuredangle MFN = 90^\circ$. The conclusion follows now from Pythagorean theorem applied in triangle MEN.

4. Rotation of the square

We show how to use the above auxiliary constructions to establish further interesting results. Complete the right triangle ABD from Figure 5 to an entire square $ABCD$. Triangle ADM' is obtained by rotating triangle ABM about A, through $90°$. This fact suggests us to make a clockwise rotation with center A of the entire figure to obtain the square $ADST$ (see Figure 9).

Denote the points corresponding to M, N, P, Q by M', N', P', Q' respectively. Assume that $\measuredangle PAQ = 45°$, or equivalently, $MN^2 = MB^2 + ND^2$.

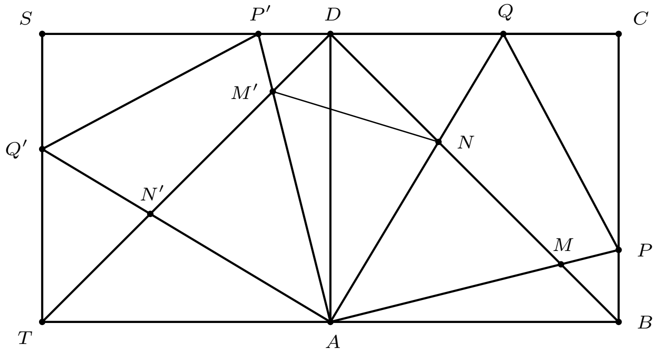

Figure 9

From $\Delta APQ \equiv \Delta AP'Q$ it follows that $PQ = P'Q$. If $AB = 1$, then

$$2 = SC = SP' + P'Q + QC = CP + PQ + QC$$

and we obtained the implication (i) $\Longrightarrow$ (iii).

The converse (iii) $\Longrightarrow$ (i) was first stated by A. B. Hodulev in [2].

5. Secants, tangents and lines external to a circle

We begin this section with an interesting question. Assuming $ABCD$ a unit square, how can we construct points P, Q such that the perimeter of triangle PQC is equal to 2? As we have already seen, one method is to make $\measuredangle PAQ = 45°$. Alternatively, note that the perimeter of triangle PQC is equal to 2 if and only if $PQ = BP + DQ$. This characterization allows us to construct points P, Q on the sides with the required property.

If we draw the arc with center A, passing through B and D, then every tangent line meeting the circle at T and the sides at P and Q determines the triangle ΔPQC of perimeter 2, because $PT = PB$ and $QT = QD$ (see Figure 10).

Moreover, if PQ does not meet the arc, then the length of PQ is less than the parallel tangent $P'Q'$ to the circle (see Figure 11). Consequently, if a segment PQ does not meet the circle, then $\measuredangle PAQ < 45°$. On the other hand, if PQ meets the circle twice, then $\measuredangle PAQ > 45°$.

We summarize these in the following theorem.

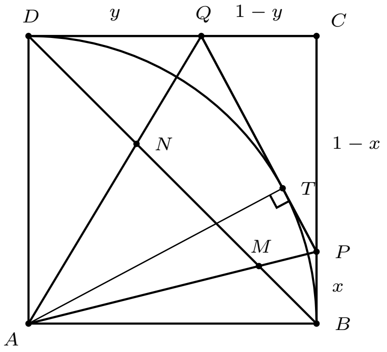

Figure 10

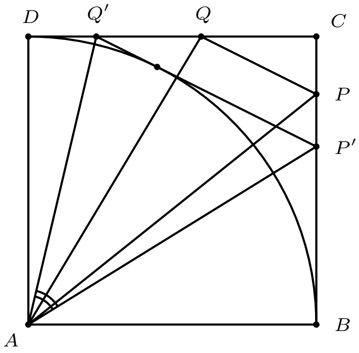

Figure 11

Theorem 4. *Let $ABCD$ be a unit square, and P, Q be points on the sides BC and CD respectively. Consider the quadrant ω of the circle with center A, passing through B and D.*
(a) *$\measuredangle PAQ = 45^\circ$ if and only if PQ is tangent to ω. Equivalently, the perimeter of triangle PQC is equal to 2.*
(b) *$\measuredangle PAQ > 45^\circ$ if and only if PQ intersects ω at two points. Equivalently, the perimeter of triangle PQC is greater than 2.*
(c) *$\measuredangle PAQ < 45^\circ$ if and only if PQ is exterior to ω. Equivalently, the perimeter of triangle PQC is less than 2.*

6. Comparison of areas

The implication (i) $\Longrightarrow$ (vi) was first discovered by Z. G. Gotman in [1].

In Figure 12 below, observe that the quadrilaterals $ABPN$ and $ADQM$ are cyclic, respectively because $\measuredangle NAP = \measuredangle NBP$ and $\measuredangle MAQ = \measuredangle MDQ$.

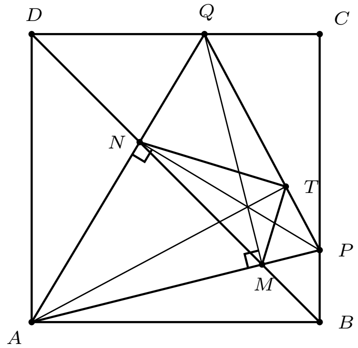

Figure 12

Consequently, AMQ and ANP are isosceles right-angled triangles. Hence,

$$\frac{S_{AMN}}{S_{APQ}} = \frac{AM \cdot AN}{AP \cdot AQ} = \frac{AM}{AQ} \cdot \frac{AN}{AP} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}.$$

Now we establish the implication (i) $\implies$ (ix).

In triangle ΔAPQ, QM and PN are altitudes, so the radius AT from Figure 10 is in fact the third altitude of the triangle ΔAPQ.

We can continue with the identifications, making use of the congruences $\Delta APB \equiv \Delta APT$ and $\Delta AQD \equiv \Delta AQT$. We deduce that $TM = MB$ and $TN = ND$. It follows that

$$MN^2 = MT^2 + TN^2 = BM^2 + ND^2.$$

Remark. The point E from Figure 6, coinciding with the point T from Figure 12, is more interesting than we have initially thought. It lies on the circumcircle of the given triangle ABD.

7. Two pairs of congruent segments

The implications (i) $\implies$ (x) and (xi) follow from the fact that ANP and AMQ are isosceles right-angled triangles.

For the converses, let us assume by way of contradiction that $\measuredangle MAN_1 = 45°$, with N_1 in BD, distinct from N. Then $AN_1 = N_1P$. As we have also $AN = NP$, it follows that NN_1 and consequently BD is the perpendicular bisector of AP, which is absurd.

8. Concluding remarks

Now let us return for a short time to the opposite angles drawn in Figure 8. It is the moment to celebrate the contribution of V. Proizvolov which proves in [5] the following nice result.

Proposition 5. *If M and N are points inside a square $ABCD$ such that $\measuredangle MAN = \measuredangle MCN = 45°$, then*

$$S_{MCN} + S_{MAB} + S_{NAD} = S_{MAN} + S_{MBC} + S_{NCD}.$$

Having at hand the previous construction from Figure 8A (where F is defined by the conditions $\Delta CND \equiv \Delta CNF$ and $\Delta CMB \equiv \Delta CMF$), we have

$$S_{MCN} + S_{MAB} + S_{NAD} = S_{MCN} + S_{AMEN} = S_{AMCN} + S_{MEN}.$$

Similarly, $S_{MAN} + S_{MBC} + S_{NCD} = S_{AMCN} + S_{MFN}$ and the conclusion follows from the congruence of the triangles MEN and MFN.

We mention for example that the idea of folding a square as in Figure 6 leads to new results under weaker hypotheses. Indeed, if we consider that piece of paper as an isosceles triangle, not necessarily right-angled, then similar results hold. Thus, if triangle ABD is isosceles, then in triangle MEN, the angle $\measuredangle MEN$ is the sum of angles $\measuredangle ABD$ and $\measuredangle ADB$. Consequently, by applying the law of cosines to triangle MEN, we obtain the following extension of Proposition 1.

Proposition 6. *Let M and N be two points on side BD of the isosceles triangle ABD such that the angle $\measuredangle MAN = \frac{1}{2}\measuredangle BAD$. Then*

$$MB^2 - MN^2 + DN^2 = -2MB \cdot DN \cos A.$$

Another interesting extension is the following problem proposed by the author at the 5th Selection Test of the Romanian Team participating at 44th IMO Japan 2003.

Problem. Find the angles of a rhombus $ABCD$ with $AB = 1$ given that on sides CD (CB) *there exist points* P, *respective* Q *such that the angle* $\measuredangle PAQ = \frac{1}{2}\measuredangle BAD$ and the perimeter of triangle CPQ is equal to 2.

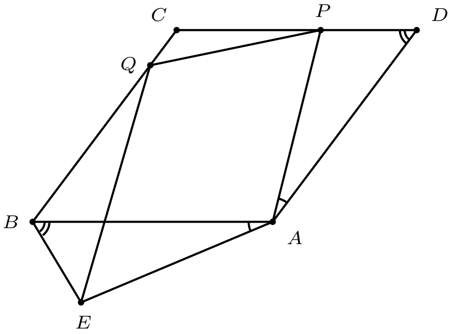

Figure 13

Let E be as in Figure 13 such that $\Delta APD \equiv \Delta AEB$. In fact we rotate triangle APD about A and what it is interesting for us is that $PQ = QE$ and $PD = BE$. Now, the equality $PQ = PD + QB$ can be written as $QE = BE + QB$, so the points Q, B, E are collinear.

This is possible only when $ABCD$ is square.

Finally, we consider replacing the square in Theorem 2 by a rhombus. Proposition 7 below was proposed by the author as a problem for the 12th Edition of the Clock-Tower School Competition, Râmnicu Vâlcea, Romania, 2009, then given at the first selection test for the Romanian team participating at the Junior Balkan Mathematical Olympiad, Neptun-Constanta, April, 15-th, 2009.

Proposition 7. *Let* $ABCD$ *be a rhombus. Two rays through* A *meet the diagonal* BD *at* M, N, *and the sides* BC *and* CD *at* P, Q *respectively* (see Figure 14). *Then* $AN = NP$ *if and only if* $AM = MQ$.

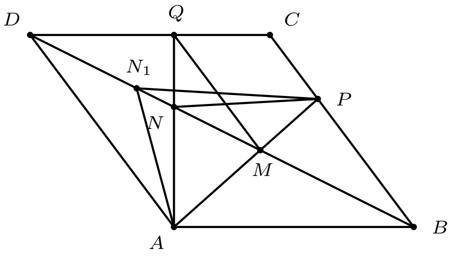

Figure 14

Proof. The key idea is that the statements $AN = NP$ and $AM = MQ$ are equivalent to $\measuredangle PAQ = \frac{1}{2}\measuredangle ABC$.

First, if $\measuredangle PAQ = \frac{1}{2}\measuredangle ABC$, then $\measuredangle NAP = \measuredangle NBP$, and the quadrilateral $ABPN$ is cyclic. As $\measuredangle ABN = \measuredangle PBN$, we have $AN = NP$.

For the converse, we consider N_1 on BD such that $\measuredangle PAN_1 = \frac{1}{2}\measuredangle BAD$. As above, we get $AN_1 = N_1P$. But $AN = NP$ so that BD must be the perpendicular bisector of the segment AP. This is absurd. □

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