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An Extension of Triangle Constructions from Located Points

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Abstract. W. Wernick has tabulated 139 triangle construction problems using a list of sixteen points associated with the triangle. We add four points to his list and find an additional 140 construction problems.

William Wernick [3] and Leroy Meyers [2] discussed the problem of constructing a triangle with ruler and compass given the location of three points associated with the triangle. Wernick tabulated all the significantly distinct problems that could be formed from the following list of sixteen points:

$A,\ B,\ C$	Three vertices
$M_a,\ M_b,\ M_c$	Three midpoints of the sides
$H_a,\ H_b,\ H_c$	Three feet of the altitudes
$T_a,\ T_b,\ T_c$	Three feet of the internal angle bisectors
$G,\ H,\ I,\ O$	The centroid, orthocenter, incenter and circumcenter

Wernick found 139 triples that could be made from these points. They can be divided into the following four distinct types:

R – Redundant. Given the location of two of the points of the triple, the location of the third point is determined. An example would be: A, B, M_c.

L – Locus Restricted. Given the location of two points, the third must lie on a certain locus. Example: A, B, O.

S – Solvable. Known ruler and compass solutions exist for these triples.

U – Unsolvable. By using algebraic means, it is possible to prove that no ruler and compass solution exists for these triples. Example: O, H, I; see [1, 4].

To extend the work of Wernick and Meyers, we add the following four points to their list:

$E_a,\ E_b,\ E_c$	Three Euler points, which are the midpoints between the vertices and the orthocenter
N	The center of the nine-point circle.

Tabulated below, along with their types, are all of the 140 significantly distinct triples that can be formed by adding our new points to the original sixteen. Problems that remain unresolved as to type are left blank. In keeping with the spirit

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of Wernick's article, we have listed all of the possible combinations of points that are significantly distinct, even though many of them are easily converted, using redundancies, to problems in Wernick's list. We point out that although many of the problems are quite simple, a few provide a fine challenge. Our favorites include $A,\ E_b,\ G$ (Problem 17) and $E_a,\ E_b,\ O$ (Problem 50).

1. A, B, E_a	**S**	36. A, M_a, N	**S**	71. E_a, H, T_b	**U**	106. E_a, M_b, T_c	
2. A, B, E_c	**S**	37. A, M_b, N	**S**	72. E_a, H_a, H_b	**S**	107. E_a, N, O	**S**
3. A, B, N	**S**	38. A, N, O	**S**	73. E_a, H_a, I	**S**	108. E_a, N, T_a	
4. A, E_a, E_b	**S**	39. A, N, T_a		74. E_a, H_a, M_a	**L**	109. E_a, N, T_b	
5. A, E_a, G	**S**	40. A, N, T_b		75. E_a, H_a, M_b	**S**	110. E_a, O, T_a	
6. A, E_a, H	**R**	41. E_a, E_b, E_c	**S**	76. E_a, H_a, N	**L**	111. E_a, O, T_b	
7. A, E_a, H_a	**L**	42. E_a, E_b, G	**S**	77. E_a, H_a, O	**S**	112. E_a, T_a, T_b	
8. A, E_a, H_b	**L**	43. E_a, E_b, H	**S**	78. E_a, H_a, T_a	**L**	113. E_a, T_b, T_c	
9. A, E_a, I	**S**	44. E_a, E_b, H_a	**S**	79. E_a, H_a, T_b		114. G, H, N	**R**
10. A, E_a, M_a	**S**	45. E_a, E_b, H_c	**S**	80. E_a, H_b, H_c	**L**	115. G, H_a, N	**S**
11. A, E_a, M_b	**S**	46. E_a, E_b, I	**U**	81. E_a, H_b, I		116. G, I, N	**U**
12. A, E_a, N	**S**	47. E_a, E_b, M_a	**L**	82. E_a, H_b, M_a	**L**	117. G, M_a, N	**S**
13. A, E_a, O	**S**	48. E_a, E_b, M_c	**S**	83. E_a, H_b, M_b	**S**	118. G, N, O	**R**
14. A, E_a, T_a	**S**	49. E_a, E_b, N	**L**	84. E_a, H_b, M_c	**S**	119. G, N, T_a	**U**
15. A, E_a, T_b	**U**	50. E_a, E_b, O	**S**	85. E_a, H_b, N	**L**	120. H, H_a, N	**S**
16. A, E_b, E_c	**S**	51. E_a, E_b, T_a		86. E_a, H_b, O	**S**	121. H, I, N	**U**
17. A, E_b, G	**S**	52. E_a, E_b, T_c	**U**	87. E_a, H_b, T_a		122. H, M_a, N	**S**
18. A, E_b, H	**S**	53. E_a, G, H	**S**	88. E_a, H_b, T_b	**U**	123. H, N, O	**R**
19. A, E_b, H_a	**S**	54. E_a, G, H_a	**S**	89. E_a, H_b, T_c		124. H, N, T_a	**U**
20. A, E_b, H_b	**L**	55. E_a, G, H_b	**S**	90. E_a, I, M_a	**S**	125. H_a, H_b, N	**L**
21. A, E_b, H_c	**S**	56. E_a, G, I		91. E_a, I, M_b		126. H_a, I, N	**S**
22. A, E_b, I		57. E_a, G, M_a	**S**	92. E_a, I, N	**S**	127. H_a, M_a, N	**L**
23. A, E_b, M_a	**S**	58. E_a, G, M_b	**S**	93. E_a, I, O		128. H_a, M_b, N	**L**
24. A, E_b, M_b	**S**	59. E_a, G, N	**S**	94. E_a, I, T_a		129. H_a, N, O	**S**
25. A, E_b, M_c	**S**	60. E_a, G, O	**S**	95. E_a, I, T_b		130. H_a, N, T_a	
26. A, E_b, N	**S**	61. E_a, G, T_a		96. E_a, M_a, M_b	**L**	131. H_a, N, T_b	
27. A, E_b, O	**S**	62. E_a, G, T_b		97. E_a, M_a, N	**R**	132. I, M_a, N	**S**
28. A, E_b, T_a		63. E_a, H, H_a	**L**	98. E_a, M_a, O	**S**	133. I, N, O	**U**
29. A, E_b, T_b		64. E_a, H, H_b	**L**	99. E_a, M_a, T_a		134. I, N, T_a	
30. A, E_b, T_c		65. E_a, H, I	**S**	100. E_a, M_a, T_b		135. M_a, M_b, N	**L**
31. A, G, N	**S**	66. E_a, H, M_a	**S**	101. E_a, M_b, M_c	**S**	136. M_a, N, O	**S**
32. A, H, N	**S**	67. E_a, H, M_b	**S**	102. E_a, M_b, N	**L**	137. M_a, N, T_a	
33. A, H_a, N	**S**	68. E_a, H, N	**S**	103. E_a, M_b, O	**S**	138. M_a, N, T_b	
34. A, H_b, N	**S**	69. E_a, H, O	**S**	104. E_a, M_b, T_a		139. N, O, T_a	**U**
35. A, I, N		70. E_a, H, T_a	**S**	105. E_a, M_b, T_b		140. N, T_a, T_b	

Many of the problems in our list can readily be converted to one in Wernick's list. Here are those by the application of a redundancy.

Problem	5	7	8	9	10	11	13	14	15	31	32	38
Wernick	40	45	50	57	24	33	16	55	56	16	16	16

Problem	53	63	64	65	66	67	69	70	71			
Wernick	40	45	50	57	24	33	16	55	56			

Problem	115	116	117	119	120	121	122	124	129	133	136	139
Wernick	75	80	66	79	75	80	66	79	75	80	66	79

A few solutions follow.

Problem 41. Given points E_a, E_b, E_c.
Solution. The orthocenter of triangle $E_aE_bE_c$ is also the orthocenter, H, of triangle ABC. Since E_a is the midpoint of AH, A can be found. Similarly, B and C.

Problem 50. Given points E_a, E_b, O.
Solution. Let P and Q be the midpoints of E_aO and E_bO, respectively. Let R be the reflection of P through Q. The line through E_b, perpendicular to E_aE_b, intersects the circle with diameter OR at M_a. The circumcircle, with center O and radius E_aM_a, intersects M_aR at B and C. The line through E_a perpendicular to BC intersects the circumcircle at A. There are in general two solutions.

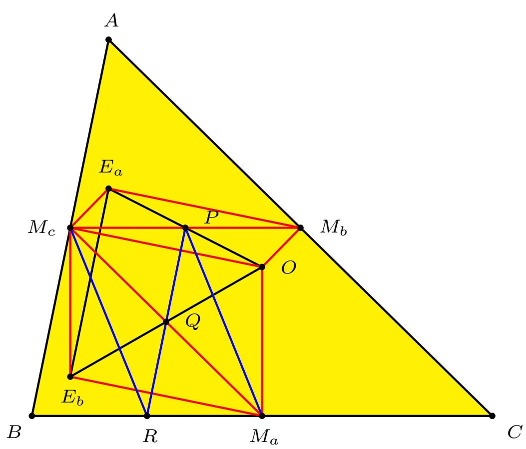

Figure 1.

Proof. In parallelogram $OM_aE_bM_c$, since diagonals bisect each other, Q is the midpoint of M_aM_c (see Figure 1). Similarly, P is the midpoint of M_bM_c. Since Q is also the midpoint of PR, PM_aRM_c is also a parallelogram and R must lie on BC. Therefore, the circle with diameter OR is a locus for M_a. Since M_aE_b is perpendicular to E_aE_b, the line through E_b perpendicular to PQ is a second locus for M_a. □

Problem 72. Given points E_a, H_a, H_b.
Solution. The line through Ha perpendicular to the line E_aH_a is the side BC. All three given points lie on the nine-point circle, so it can be found. The second intersection of the nine-point circle with BC gives M_a. The circle with M_a as

center and passing through H_b intersects the side BC at B and C. Finally, CH_b intersects E_aH_a at A.

Problem 103. Given points E_a, M_b, O.
Solution. The line through M_b, perpendicular to M_bO is AC. Reflecting AC through E_a, then dilating this line with O as center and ratio $\frac{1}{2}$ and finally intersecting this new line with the perpendicular bisector of E_aM_b gives N. Reflecting O through N gives H. E_aH intersects AC at A. The circumcircle, with center O passing through A, intersects AC again at C. The perpendicular from H to AC intersects the circumcircle at B.

References

[1] J. Stern, Euler' triangle determination problem, *Forum Geom.*, 7 (2007) 1–9.
[2] L. F. Meyers, Update on William Wernick's "Triangle constructions with three located points", *Math. Mag.*, 69 (1996) 46–49.
[3] W. Wernick, Triangle constructions with three located points, *Math. Mag.*, 55 (1982) 227–230.
[4] P. Yiu, Conic solution of Euler's triangle determination problem, *Journal for Geometry and Graphics*, 12 (2008) 75–80.

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