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A Note on the Anticomplements of the Fermat Points

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Abstract. We show that each of the anticomplements of the Fermat points is common to a triad of circles involving the triangle of reflection. We also generate two new triangle centers as common points to two other triads of circles. Finally, we present several circles passing through these new centers and the anticomplements of the Fermat points.

1. Introduction

The Fermat points $F_\pm$ are the common points of the lines joining the vertices of a triangle $\mathbf{T}$ to the apices of the equilateral triangles erected on the corresponding sides. They are also known as the isogonic centers (see [2, pp.107, 151]) and are among the basic triangle centers. In [4], they appear as the triangle centers X_{13} and X_{14}. Not much, however, is known about their anticomplements, which are the points $P_\pm$ which divide $F_\pm G$ in the ratio $F_\pm G : GP_\pm = 1 : 2$.

Given triangle $\mathbf{T}$ with vertices A, B, C,
(i) let A', B', C' be the reflections of the vertices A, B, C in the respective opposite sides, and
(ii) for $\varepsilon = \pm 1$, let A_ε, B_ε, C_ε be the apices of the equilateral triangles erected on the sides BC, CA, AB of triangle ABC respectively, on opposite or the same sides of the vertices according as $\varepsilon = 1$ or -1 (see Figures 1A and 1B).

Theorem 1. *For $\varepsilon = \pm 1$, the circumcircles of triangles $A'B_\varepsilon C_\varepsilon$, $B'C_\varepsilon A_\varepsilon$, $C'A_\varepsilon B_\varepsilon$ are concurrent at the anticomplement $P_{-\varepsilon}$ of the Fermat point $F_{-\varepsilon}$.*

2. Proof of Theorem 1

For $\varepsilon = \pm 1$, let $O_{a,\varepsilon}$ be the center of the equilateral triangle $A_\varepsilon BC$; similarly for $O_{b,\varepsilon}$ and $O_{c,\varepsilon}$.

(1) We first note that $O_{a,-\varepsilon}$ is the center of the circle through A', B_ε, and C_ε. Rotating triangle $O_{a,\varepsilon}AB$ through B by an angle $\varepsilon \cdot \frac{\pi}{3}$, we obtain triangle $O_{a,-\varepsilon}C_\varepsilon B$. Therefore, the triangles are congruent and $O_{a,-\varepsilon}C_\varepsilon = O_{a,\varepsilon}A$. Similarly, $O_{a,-\varepsilon}B_\varepsilon = O_{a,\varepsilon}A$. Clearly, $O_{a,\varepsilon}A = O_{a,-\varepsilon}A'$. It follows that $O_{a,-\varepsilon}$ is the center of the circle through A', B_ε and C_ε. Figures 1A and 1B illustrate the cases $\varepsilon = +1$ and $\varepsilon = -1$ respectively.

(2) Let $A_1B_1C_1$ be the anticomplementary triangle of ABC. Since AA_1 and A_+A_- have a common midpoint, $AA_+A_1A_-$ is a parallelogram. The lines $AA_{-\varepsilon}$ and A_1A_ε are parallel. Since A_1 is the anticomplement of A, it follows that the line A_1A_ε is the anticomplement of the line $AA_{-\varepsilon}$. Similarly, B_1B_ε and C_1C_ε are the anticomplements of the lines $BB_{-\varepsilon}$ and $CC_{-\varepsilon}$. Since $AA_{-\varepsilon}$, $BB_{-\varepsilon}$, $CC_{-\varepsilon}$

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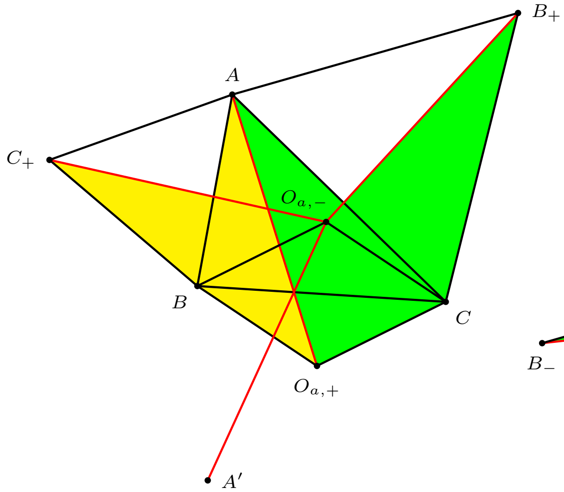

Figure 1A

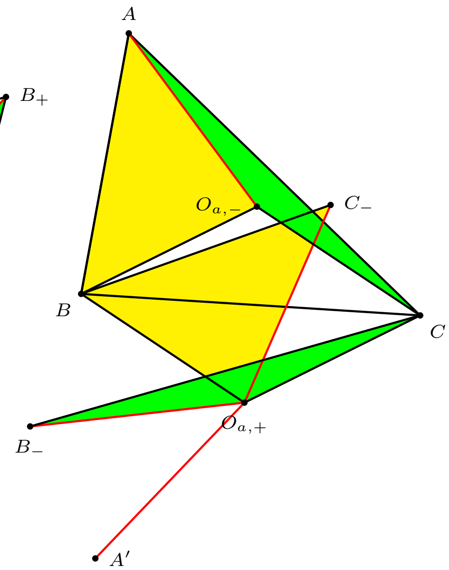

Figure 1B

concur at $F_{-\varepsilon}$, it follows that A_1A_ε, B_1B_ε, C_1C_ε concur at the anticomplement of $F_{-\varepsilon}$. This is the point $P_{-\varepsilon}$.

(3) $A_1B_1C_1$ is also the ε-Fermat triangle of $A_{-\varepsilon}B_{-\varepsilon}C_{-\varepsilon}$.

(i) Triangles $AB_\varepsilon C_\varepsilon$ and $CB_\varepsilon A_1$ are congruent, since $AB_\varepsilon = CB_\varepsilon$, $AC_\varepsilon = AB = CA_1$, and each of the angles $B_\varepsilon AC_\varepsilon$ and $B_\varepsilon CA_1$ is $\min\left(A + \frac{2\pi}{3},\ B + C + \frac{\pi}{3}\right)$. It follows that $B_\varepsilon C_\varepsilon = B_\varepsilon A_1$.

(ii) Triangles $AB_\varepsilon C_\varepsilon$ and BA_1C_ε are also congruent for the same reason, and we have $B_\varepsilon C_\varepsilon = A_1C_\varepsilon$.

It follows that triangle $A_1B_\varepsilon C_\varepsilon$ is equilateral, and $\angle C_\varepsilon A_1 B_\varepsilon = \frac{\pi}{3}$.

(4) Because $P_{-\varepsilon}$ is the second Fermat point of $A_1B_1C_1$, we may assume $\angle C_\varepsilon P_{-\varepsilon} B_\varepsilon = \frac{\pi}{3}$. Therefore, $P_{-\varepsilon}$ lies on the circumcircle of $A_1B_\varepsilon C_\varepsilon$, which is the same as that of $A'B_\varepsilon C_\varepsilon$. On the other hand, since the quadrilateral $AA_+A_1A_-$ is a parallelogram (the diagonals AA_1 and A_-A_+ have a common midpoint D, the midpoint of segment BC), the anticomplement of the line AA_- coincides with A_1A_+. It now follows that the lines A_1A_+, B_1B_+, C_1C_+ are concurrent at the anticomplement P_- of the second Fermat points, and furthermore, $\angle C_+P_-B_+ = \frac{\pi}{3}$. Since

$$
\begin{aligned}
\angle C_+AB_+ &= 2\pi - (\angle BAC_+ + \angle CAB + \angle B_+AC) \\
&= \frac{4\pi}{3} - \angle CAB \\
&= \frac{\pi}{3} + \angle ABC + \angle BCA \\
&= \angle ABC_+ + \angle ABC + \angle CBM \\
&= \angle C_+BM,
\end{aligned}
$$

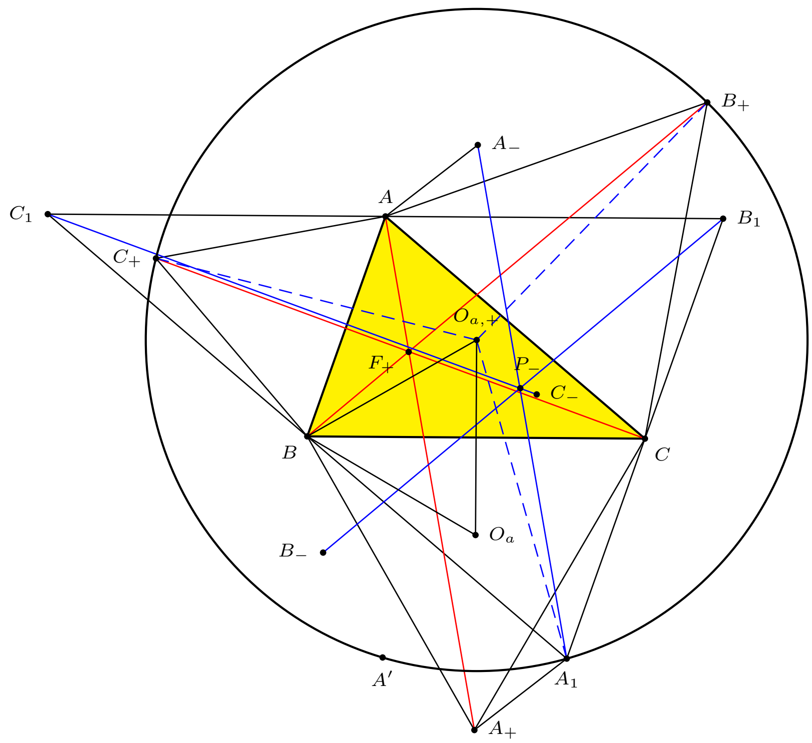

Figure 2

it follows that the triangles C_+AB_+ and C_+BM are congruent. Likewise, $\angle C_+AB_+ = \angle MCB_+$, and so the triangles C_+AB_+ and MCB_+ are congruent. Therefore, the triangle C_+MB_+ is equilateral, and thus $\angle C_+MB_+ = \frac{\pi}{3}$. Combining this with $\angle C_+P_-B_+ = 60°$, yields that the quadrilateral $MP_-B_+C_+$ is cyclic, and since $A'MB_+C_+$ is also cyclic, we conclude that the anticomplement P_- of the second Fermat point F_- lies on the circumcircle of triangle $A'B_+C_+$. Similarly, P_- lies on the circumcircles of triangles $B'C_+A_+$, and $C'A_+B_+$, respectively. This completes the proof of Theorem 1.

3. Two new triangle centers

By using the same method as in [6], we generate two other concurrent triads of circles.

Theorem 2. *For $\varepsilon = \pm 1$, the circumcircles of the triangles $A_\varepsilon B'C'$, $B_\varepsilon C'A'$, $C_\varepsilon A'B'$ are concurrent.*

Proof. Consider the inversion Ψ with respect to the anticomplement of the second Fermat point. According to Theorem 1, the images of the circumcircles of triangles $A'B_+C_+$, $B'C_+A_+$, $C'A_+B_+$ are three lines which bound a triangle $A'_+B'_+C'_+$, where A'_+, B'_+, C'_+ are the images of A_+, B_+, and C_+, respectively. Since the images A'', B'', C'' of A', B', C' under Ψ lie on the sidelines $B'_+C'_+$, $C'_+A'_+$, $A'_+B'_+$, respectively, by Miquel's theorem, we conclude that the circumcircles of triangles $A'_+B''C''$, $B'_+C''A''$, $C'_+A''B''$ are concurrent. Thus, the circumcircles of triangles $A_+B'C'$, $B_+C'A'$, $C_+A'B'$ are also concurrent (see Figure 3).

Similarly, inverting with respect to the anticomplement of the first Fermat point, by Miquel's theorem, one can deduce that the circumcircles of triangles $A_-B'C'$, $B_-C'A'$, $C_-A'B'$ are concurrent. □

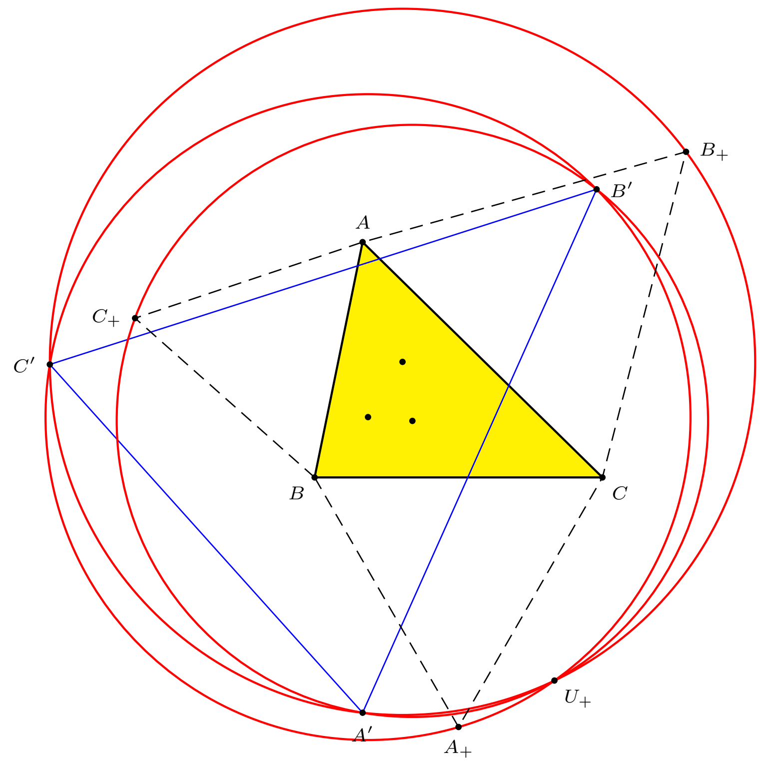

Figure 3.

Javier Francisco Garcia Capitan has kindly communicated that their points of concurrency do not appear in [4]. We will further denote these points by U_+, and U_-, respectively. We name these centers U_+, U_- the *inversive associates* of the anticomplements P_+, P_- of the Fermat points.

4. Circles around $P_\pm$ and their inversive associates

We denote by O, H the circumcenter, and orthocenter of triangle ABC. Let J_+, J_- be respectively the inner and outer isodynamic points of the triangle. Though the last two are known in literature as the common two points of the Apollonius circles, L. Evans [1] gives a direct relation between them and the Napoleonic configuration, defining them as the perspectors of the triangle of reflections $A'B'C'$

with each of the Fermat triangles $A_+B_+C_+$, and $A_-B_-C_-$. They appear as X_{15}, X_{16} in [4].

Furthermore, let W_+, W_- be the Wernau points of triangle ABC. These points are known as the common points of the folloing triads of circles: AB_+C_+, BC_+A_+, CA_+B_+, and respectively AB_-C_-, BC_-A_-, CA_-B_- [3]. According to the above terminology, W_+, W_- are the inversive associates of the Fermat points F_+, and F_-. They appear as X_{1337} and X_{1338} in [4].

We conclude with a list of concyclic quadruples involving these triangle centers. The first one is an immediate consequence of the famous Lester circle theorem [5]. The other results have been verified with the aid of Mathematica.

Theorem 3. *The following quadruples of points are concyclic:*
(i) P_+, P_-, O, H;
(ii) P_+, P_-, F_+, J_+;
(ii′) P_+, P_-, F_-, J_-;
(iii) P_+, U_+, F_+, O;
(iii′) P_-, U_-, F_-, O;
(iv) P_+, U_-, F_+, W_+;
(iv′) P_-, U_+, F_-, W_-;
(v) U_+, J_+, W_+, W_-;
(v′) U_-, J_-, W_+, W_-.

References

[1] L. Evans, Some configurations of triangle centers, *Forum Geom.*, 3 (2003) 49–56.
[2] W. Gallatly, *The Modern Geometry of the Triangle*, 2nd ed., London: Hodgson, 1913.
[3] D. Grinberg, Hyacinthos messages 6874, 6881, 6882, April, 2003.
[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.
[5] J. Lester, Triangles III: Complex Triangle Functions, *Aequationes Math.*, 53 (1997) 4–35.
[6] C. Pohoata, On the Parry reflection point, *Forum Geom.*, 8 (2008) 43–48.

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