Forum Geometricorum
Volume 9 (2009) 125–148.

FORUM GEOM
ISSN 1534-1178

Heptagonal Triangles and Their Companions

Paul Yiu

Abstract. A heptagonal triangle is a non-isosceles triangle formed by three vertices of a regular heptagon. Its angles are $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$. As such, there is a unique choice of a companion heptagonal triangle formed by three of the remaining four vertices. Given a heptagonal triangle, we display a number of interesting companion pairs of heptagonal triangles on its nine-point circle and Brocard circle. Among other results on the geometry of the heptagonal triangle, we prove that the circumcenter and the Fermat points of a heptagonal triangle form an equilateral triangle. The proof is an interesting application of Lester's theorem that the Fermat points, the circumcenter and the nine-point center of a triangle are concyclic.

1. The heptagonal triangle $\mathbf{T}$ and its companion

A heptagonal triangle $\mathbf{T}$ is one with angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$. Its vertices are three vertices of a regular heptagon inscribed in its circumcircle. Among the remaining four vertices of the heptagon, there is a unique choice of three which form another (congruent) heptagonal triangle $\mathbf{T}'$. We call this the companion of $\mathbf{T}$, and the seventh vertex of the regular heptagon the residual vertex of $\mathbf{T}$ and $\mathbf{T}'$ (see Figure 1). In this paper we work with complex number coordinates, and take the unit circle

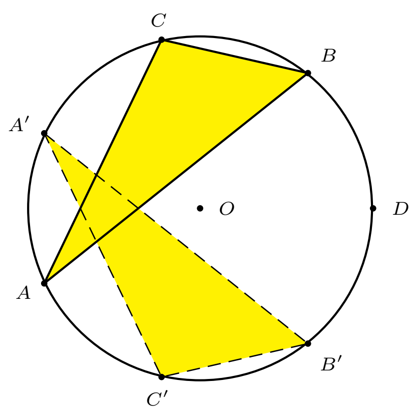

Figure 1. A heptagonal triangle and its companion

in the complex plane for the circumcircle of $\mathbf{T}$. By putting the residual vertex D at 1, we label the vertices of $\mathbf{T}$ by

$$A = \zeta^4, \qquad B = \zeta, \qquad C = \zeta^2,$$

Publication Date: June 22, 2009. Communicating Editor: Antreas P. Hatzipolakis.
The author thanks the Editor for suggesting the configuration studied in §8, leading to, among other results, Theorem 20 on six circles concurring at the Feuerbach point of the heptagonal triangle.

and those of $\mathbf{T}'$ by

$$A' = \zeta^3, \qquad B' = \zeta^6, \qquad C' = \zeta^5,$$

where $\zeta := \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ is a primitive 7-th root of unity.

We study the triangle geometry of $\mathbf{T}$, some common triangle centers, lines, circles and conics associated with it. We show that the Simson lines of A', B', C' with respect to $\mathbf{T}$ are concurrent (Theorem 4). We find a number of interesting companion pairs of heptagonal triangles associated with $\mathbf{T}$. For example, the medial triangle and the orthic triangle of $\mathbf{T}$ form such a pair on the nine-point circle (Theorem 5), and the residual vertex is a point on the circumcircle of $\mathbf{T}$. It is indeed the Euler reflection point of $\mathbf{T}$. In the final section we prove that the circumcenter and the Fermat points form an equilateral triangle (Theorem 22). The present paper can be regarded as a continuation of Bankoff-Garfunkel [1].

2. Preliminaries

2.1. *Some simple coordinates.* Clearly, the circumcenter O of $\mathbf{T}$ has coordinate 0, and the centroid is the point $G = \frac{1}{3}(\zeta + \zeta^2 + \zeta^4)$. Since the orthocenter H and the nine-point center N are points (on the Euler line) satisfying

$$OG : GN : NH = 2 : 1 : 3,$$

we have

$$\begin{aligned} H &= \zeta + \zeta^2 + \zeta^4, \\ N &= \frac{1}{2}(\zeta + \zeta^2 + \zeta^4). \end{aligned} \tag{1}$$

This reasoning applies to any triangle with vertices on the unit circle. The bisectors of angles A, B, C of $\mathbf{T}$ intersect the circumcircle at $-C'$, A', B' respectively. These form a triangle whose orthocenter is the incenter I of $\mathbf{T}$ (see Figure 2). This latter is therefore the point

$$I = \zeta^3 - \zeta^5 + \zeta^6. \tag{2}$$

Similarly, the external bisectors of angles A, B, C intersect the circumcircle at C', $-A'$, $-B'$ respectively. Identifying the excenters of $\mathbf{T}$ as orthocenters of triangles with vertices on the unit circle, we have

$$\begin{aligned} I_a &= -(\zeta^3 + \zeta^5 + \zeta^6), \\ I_b &= \zeta^3 + \zeta^5 - \zeta^6, \\ I_c &= -\zeta^3 + \zeta^5 + \zeta^6. \end{aligned} \tag{3}$$

Figure 2 shows the tritangent circles of the heptagonal triangle $\mathbf{T}$.

2.2. *Representation of a companion pair*. Making use of the simple fact that the complex number coordinates of vertices of a regular heptagon can be obtained from any one of them by multiplications by $\zeta, \dots, \zeta^6$, we shall display a companion pair of heptagonal triangle by listing coordinates of the center, the residual vertex and the vertices of the two heptagonal triangles, as follows.

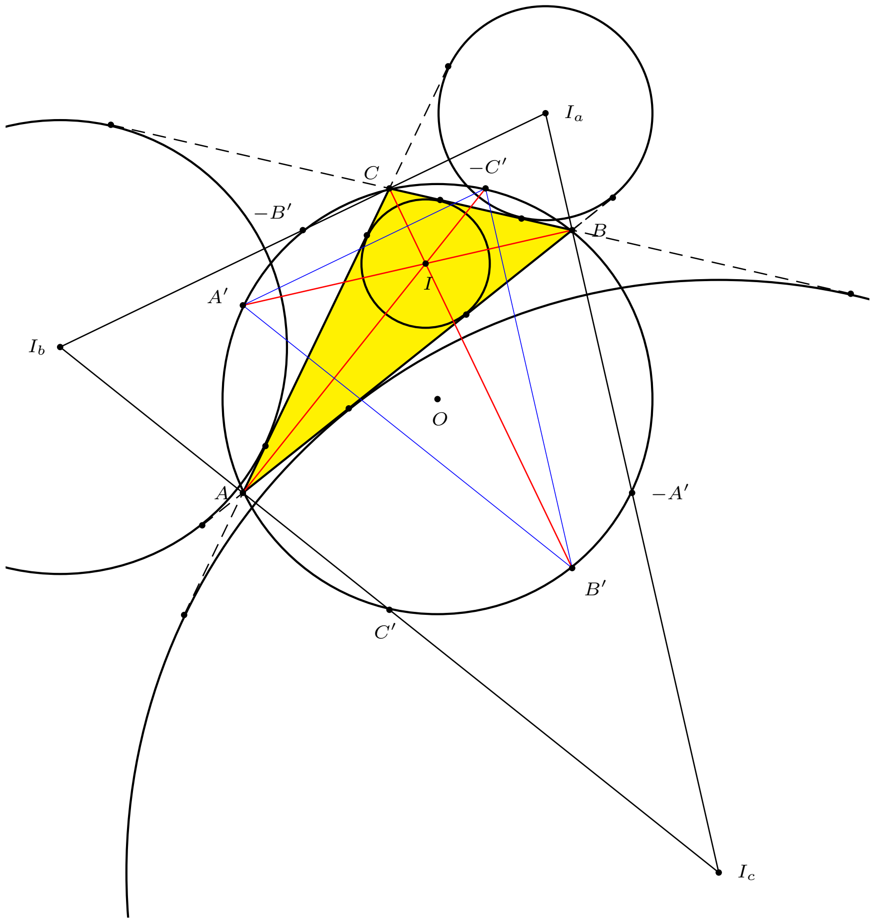

Figure 2. The tritangent centers

Center:	P
Residual vertex:	Q

Rotation	Vertices	Rotation	Vertices
ζ^4	$P+\zeta^4(Q-P)$	ζ^3	$P+\zeta^3(Q-P)$
ζ	$P+\zeta(Q-P)$	ζ^6	$P+\zeta^6(Q-P)$
ζ^2	$P+\zeta^2(Q-P)$	ζ^5	$P+\zeta^5(Q-P)$

2.3. While we shall mostly work in the cyclotomic field $\mathbb{Q}(\zeta)$, [1] the complex number coordinates of points we consider in this paper are *real* linear combinations of ζ^k for $0 \leq k \leq 6$, (the vertices of the regular heptagon on the circumcircle of

[1]See Corollary 23 for an exception.

$\mathbf{T}$). The real coefficients involved are rational combinations of

$$c_1 = \frac{\zeta + \zeta^6}{2} = \cos\frac{2\pi}{7}, \quad c_2 = \frac{\zeta^2 + \zeta^5}{2} = \cos\frac{4\pi}{7}, \quad c_3 = \frac{\zeta^3 + \zeta^4}{2} = \cos\frac{6\pi}{7}.$$

Note that $c_1 > 0$ and $c_2, c_3 < 0$. An expression of a complex number z as a real linear combination of ζ^4, ζ, ζ^2 (with sum of coefficients equal to 1) actually gives the absolute barycentric coordinate of the point z with reference to the heptagonal triangle $\mathbf{T}$. For example,

$$\begin{array}{rcrcrcr} \zeta^3 &=& -2c_2\cdot\zeta^4 &+& 2c_2\cdot\zeta &+& 1\cdot\zeta^2, \\ \zeta^5 &=& 2c_1\cdot\zeta^4 &+& 1\cdot\zeta &-& 2c_1\cdot\zeta^2, \\ \zeta^6 &=& 1\cdot\zeta^4 &-& 2c_3\cdot\zeta &+& 2c_3\cdot\zeta^2, \\ 1 &=& -2c_2\cdot\zeta^4 &-& 2c_3\cdot\zeta &-& 2c_1\cdot\zeta^2. \end{array}$$

We shall make frequent uses of the important result.

Lemma 1 (Gauss). $1 + 2(\zeta + \zeta^2 + \zeta^4) = \sqrt{7}i$.

Proof. Although this can be directly verified, it is actually a special case of Gauss' famous theorem that if $\zeta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ for an odd integer n, then

$$\sum_{k=0}^{n-1}\zeta^{k^2} = \begin{cases} \sqrt{n} & \text{if } n \equiv 1 \pmod 4, \\ \sqrt{n}i & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

For a proof, see [2, pp.75–76]. □

2.4. *Reflections and pedals.*

Lemma 2. *If α, β, γ are unit complex numbers, the reflection of γ in the line joining α and β is $\gamma' = \alpha + \beta - \alpha\beta\overline{\gamma}$.*

Proof. As points in the complex plane, γ' has equal distances from α and β as γ does. This is clear from

$$\gamma' - \alpha = \beta(1 - \alpha\overline{\gamma}) = \beta\overline{\gamma}(\gamma - \alpha),$$
$$\gamma' - \beta = \alpha(1 - \beta\overline{\gamma}) = \alpha\overline{\gamma}(\gamma - \beta).$$

□

Corollary 3. (1) *The reflection of ζ^k in the line joining ζ^i and ζ^j is $\zeta^i + \zeta^j - \zeta^{i+j-k}$.*
(2) *The pedal* (orthogonal projection) *of ζ^k on the line joining ζ^i and ζ^j is*

$$\frac{1}{2}(\zeta^i + \zeta^j + \zeta^k - \zeta^{i+j-k}).$$

(3) *The reflections of A in BC, B in CA, and C in AB are the points*

$$\begin{aligned} A^* &= \zeta + \zeta^2 - \zeta^6, \\ B^* &= \zeta^2 + \zeta^4 - \zeta^5, \\ C^* &= \zeta - \zeta^3 + \zeta^4. \end{aligned} \tag{4}$$

3. Concurrent Simson lines

The Simson line of a point on the circumcircle of a triangle is the line containing the pedals of the point on the sidelines of the triangle.

Theorem 4. *The Simson lines of A', B', C' with respect to the heptagonal triangle* $\mathbf{T}$ *are concurrent.*

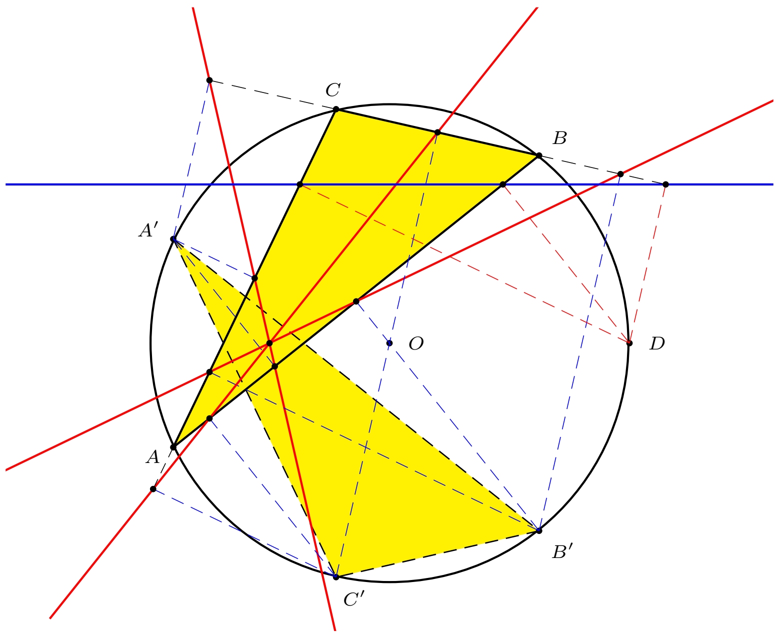

Figure 3. Simson lines

Proof. The pedals of A on BC is the midpoint A' of AA^*; similarly for those of B on CA and C on AB. We tabulate the coordinates of the pedals of A', B', C' on the sidelines BC, CA, AB respectively. These are easily calculated using Corollary 3.

	BC	CA	AB
A'	$\frac{1}{2}(-1+\zeta+\zeta^2+\zeta^3)$	$\frac{1}{2}(\zeta^2+\zeta^4)$	$\frac{1}{2}(\zeta-\zeta^2+\zeta^3+\zeta^4)$
B'	$\frac{1}{2}(\zeta+\zeta^2-\zeta^4+\zeta^6)$	$\frac{1}{2}(-1+\zeta^2+\zeta^4+\zeta^6)$	$\frac{1}{2}(\zeta+\zeta^4)$
C'	$\frac{1}{2}(\zeta+\zeta^2)$	$\frac{1}{2}(-\zeta+\zeta^2+\zeta^4+\zeta^5)$	$\frac{1}{2}(-1+\zeta+\zeta^4+\zeta^5)$

We check that the Simson lines of A', B', C' all contain the point $-\frac{1}{2}$. For these, it is enough to show that the complex numbers

$$(\zeta+\zeta^2+\zeta^3)(\overline{1+\zeta^2+\zeta^4}), \quad (\zeta^2+\zeta^4+\zeta^6)(\overline{1+\zeta+\zeta^4}), \quad (\zeta+\zeta^4+\zeta^5)(\overline{1+\zeta+\zeta^2})$$

are real. These are indeed $\zeta+\zeta^6$, $\zeta^2+\zeta^5$, $\zeta^3+\zeta^4$ respectively. $\square$

Remark. The Simson line of D, on the other hand, is parallel to OD (see Figure 3). This is because the complex number coordinates of the pedals of D, namely,

$$\frac{1+\zeta+\zeta^2-\zeta^3}{2}, \quad \frac{1+\zeta^2+\zeta^4-\zeta^6}{2}, \quad \frac{1+\zeta+\zeta^4-\zeta^5}{2},$$

all have the same imaginary part $\frac{1}{4}(\zeta-\zeta^6+\zeta^2-\zeta^5-\zeta^3+\zeta^4)$.

4. The nine-point circle

4.1. *A companion pair of heptagonal triangles on the nine-point circle.* As is well known, the nine-point circle is the circle through the vertices of the medial triangle and of the orthic triangle. The medial triangle of $\mathbf{T}$ clearly is heptagonal. It is known that $\mathbf{T}$ is the only obtuse triangle with orthic triangle similar to itself. [2] The medial and orthic triangles of $\mathbf{T}$ are therefore congruent. It turns out that they are companions.

Theorem 5. *The medial triangle and the orthic triangle of* $\mathbf{T}$ *are companion heptagonal triangles on the nine-point circle of* $\mathbf{T}$*. The residual vertex is the Euler reflection point* E (on the circumcircle of $\mathbf{T}$).

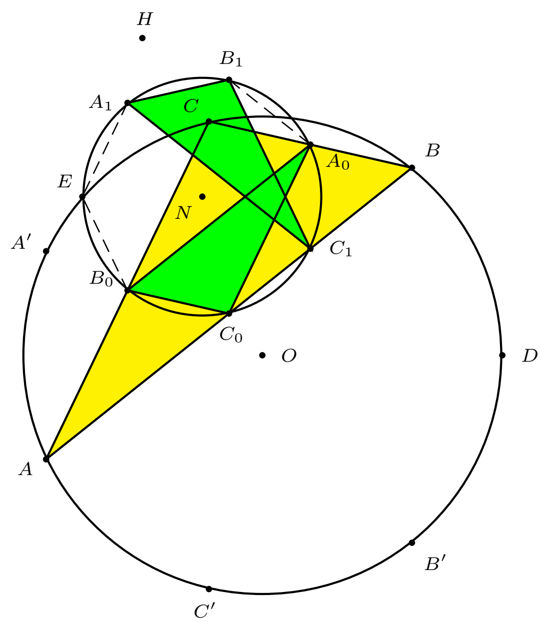

Figure 4. A companion pair on the nine-point circle

[2]If the angles of an obtuse angled triangle are $\alpha \leq \beta < \gamma$, those of its orthic triangle are 2α, 2β, and $2\gamma - \pi$. The two triangles are similar if and only if $\alpha = 2\gamma - \pi$, $\beta = 2\alpha$ and $\gamma = 2\beta$. From these, $\alpha = \frac{\pi}{7}$, $\beta = \frac{2\pi}{7}$ and $\gamma = \frac{4\pi}{7}$. This shows that the triangle is heptagonal. The equilateral triangle is the only acute angled triangle similar to its own orthic triangle.

Proof. (1) The companionship of the medial and orthic triangles on the nine-point circle is clear from the table below.

Center:	$N = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4)$
Residual vertex:	$E = \frac{1}{2}(-1 + \zeta + \zeta^2 + \zeta^4)$

Rotation	Medial triangle	Rotation	Orthic triangle
ζ^4	$A_0 = \frac{1}{2}(\zeta + \zeta^2)$	ζ^3	$C_1 = \frac{1}{2}(\zeta + \zeta^2 - \zeta^3 + \zeta^4)$
ζ	$B_0 = \frac{1}{2}(\zeta^2 + \zeta^4)$	ζ^6	$A_1 = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4 - \zeta^6)$
ζ^2	$C_0 = \frac{1}{2}(\zeta + \zeta^4)$	ζ^5	$B_1 = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4 - \zeta^5)$

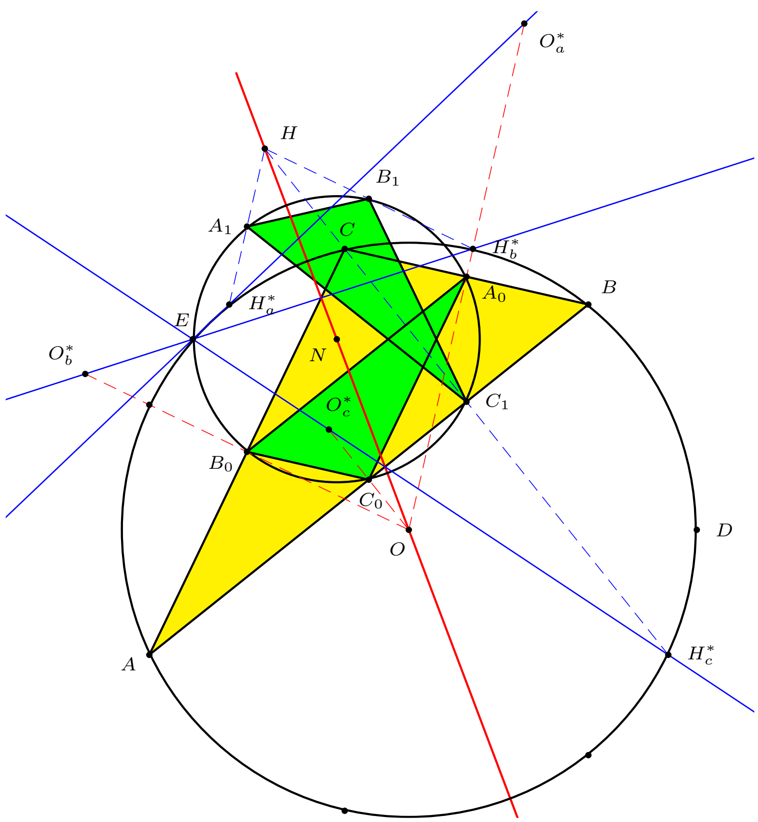

Figure 5. The Euler reflection point of $\mathbf{T}$

(2) We show that E is a point on the reflection of the Euler line in each of the sidelines of $\mathbf{T}$. In the table below, the reflections of O are computed from the simple fact that OBO_a^*C, OCO_b^*A, OAO_c^*B are rhombi. On the other hand, the reflections of H in the sidelines can be determined from the fact that HH_a^* and AA^* have the same midpoint, so do HH_b^* and BB^*, HH_c^* and CC^*. The various expressions for E given in the rightmost column can be routinely verified.

Line	Reflection of O	Reflection of H	$E =$
BC	$O_a^* = \zeta + \zeta^2$	$H_a^* = -\zeta^6$	$(-2c_1 - c_2 - c_3)O_a^* + (-c_2 - c_3)H_a^*$
CA	$O_b^* = \zeta^2 + \zeta^4$	$H_b^* = -\zeta^5$	$(-c_1 - 2c_2 - c_3)O_b^* + (-c_1 - c_3)H_b^*$
AB	$O_c^* = \zeta + \zeta^4$	$H_c^* = -\zeta^3$	$(-c_1 - c_2 - 2c_3)O_c^* + (-c_1 - c_2)H_c^*$

Thus, E, being the common point of the reflections of the Euler line of $\mathbf{T}$ in its sidelines, is the Euler reflection point of $\mathbf{T}$, and lies on the circumcircle of $\mathbf{T}$. □

4.2. *The second intersection of the nine-point circle and the circumcircle.*

Lemma 6. *The distance between the nine-point center N and the A-excenter I_a is equal to the circumradius of the heptagonal triangle $\mathbf{T}$.*

Proof. Note that $I_a - N = \frac{2+\zeta+\zeta^2+\zeta^4}{2} = \frac{3+1+2(\zeta+\zeta^2+\zeta^4)}{4} = \frac{3+\sqrt{7}i}{4}$ is a unit complex number. □

This simple result has a number of interesting consequences.

Proposition 7. (1) *The midpoint F_a of NI_a is the point of tangency of the nine-point circle and the A-excircle.*
(2) *The A-excircle is congruent to the nine-point circle.*
(3) *F_a lies on the circumcircle.*

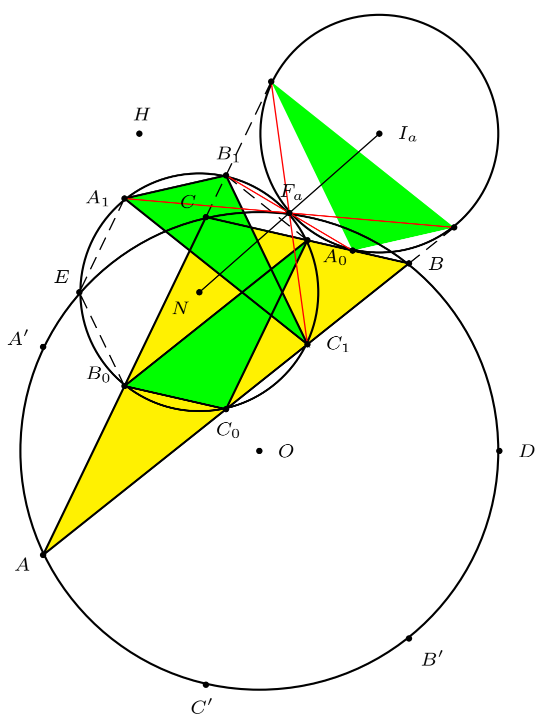

Figure 6. The A-Feuerbach point of $\mathbf{T}$

Proof. (1) By the Feuerbach theorem, the nine-point circle is tangent externally to each of the excircles. Since $NI_a = R$, the circumradius, and the nine-point circle has radius $\frac{1}{2}R$, the point of tangency with the A-excircle is the midpoint of NI_a, *i.e.*,

$$F_a = \frac{I_a + N}{2} = \frac{2 + 3(\zeta + \zeta^2 + \zeta^4)}{4}. \tag{5}$$

This proves (1).

(2) It also follows that the radius of the A-excircle is $\frac{1}{2}R$, and the A-excircle is congruent to the nine-point circle.

(3) Note that $F_a = \frac{1+3+6(\zeta+\zeta^2+\zeta^4)}{8} = \frac{1+3\sqrt{7}i}{8}$ is a unit complex number. □

Remark. The reflection of the orthic triangle in F_a is the A-extouch triangle, since the points of tangency are

$$-(\zeta^3 + \zeta^5 + \zeta^6) + \frac{\zeta^3}{2}, \qquad -(\zeta^3 + \zeta^5 + \zeta^6) + \frac{\zeta^5}{2}, \qquad -(\zeta^3 + \zeta^5 + \zeta^6) + \frac{\zeta^6}{2}$$

(see Figure 6).

4.3. *Another companion pair on the nine-point circle.*

Center:	$N = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4)$
Residual vertex:	$F_a = \frac{1}{4}(2 + 3(\zeta + \zeta^2 + \zeta^4))$

Rot.	Feuerbach triangle	Rot.	Companion
ζ^3	$F_b = \frac{1}{4}(\zeta + \zeta^2 + \zeta^3 + 2\zeta^4 - \zeta^6)$	ζ^4	$F'_a = \frac{1}{4}(3\zeta + 2\zeta^2 + 4\zeta^4 + \zeta^5 + \zeta^6)$
ζ^6	$F_e = \frac{1}{4}(2\zeta + \zeta^2 + \zeta^4 - \zeta^5 + \zeta^6)$	ζ	$F'_b = \frac{1}{4}(4\zeta + 3\zeta^2 + \zeta^3 + 2\zeta^4 + \zeta^5)$
ζ^5	$F_c = \frac{1}{4}(\zeta + 2\zeta^2 - \zeta^3 + \zeta^4 + \zeta^5)$	ζ^2	$F'_c = \frac{1}{4}(2\zeta + 4\zeta^2 + \zeta^3 + 3\zeta^4 + \zeta^6)$

Proposition 8. *F_e, F_a, F_b, F_c are the points of tangency of the nine-point circle with the incircle and the A-, B-, C-excircles respectively* (see Figure 7).

Proof. We have already seen that $F_a = \frac{1}{2} \cdot N + \frac{1}{2} \cdot I_a$. It is enough to show that the points F_e, F_b, F_c lie on the lines NI, NI_b, NI_c respectively:

$$F_e = -(c_1 - c_3) \cdot N + (-c_1 - 2c_2 - 3c_3) \cdot I,$$
$$F_b = (c_2 - c_3) \cdot N + (-2c_1 - 3c_2 - c_3) \cdot I_b,$$
$$F_c = (c_1 - c_2) \cdot N + (-3c_1 - c_2 - 2c_3) \cdot I_c.$$

□

Proposition 9. *The vertices F'_a, F'_b, F'_c of the companion of $F_bF_eF_c$ are the second intersections of the nine-point circle with the lines joining F_a to A, B, C respectively.*

Proof.

$$F'_a = -2c_2 \cdot F_a - 2(c_1 + c_3)A,$$
$$F'_b = -2c_3 \cdot F_a - 2(c_1 + c_2)B,$$
$$F'_c = -2c_1 \cdot F_a - 2(c_2 + c_3)C.$$

□

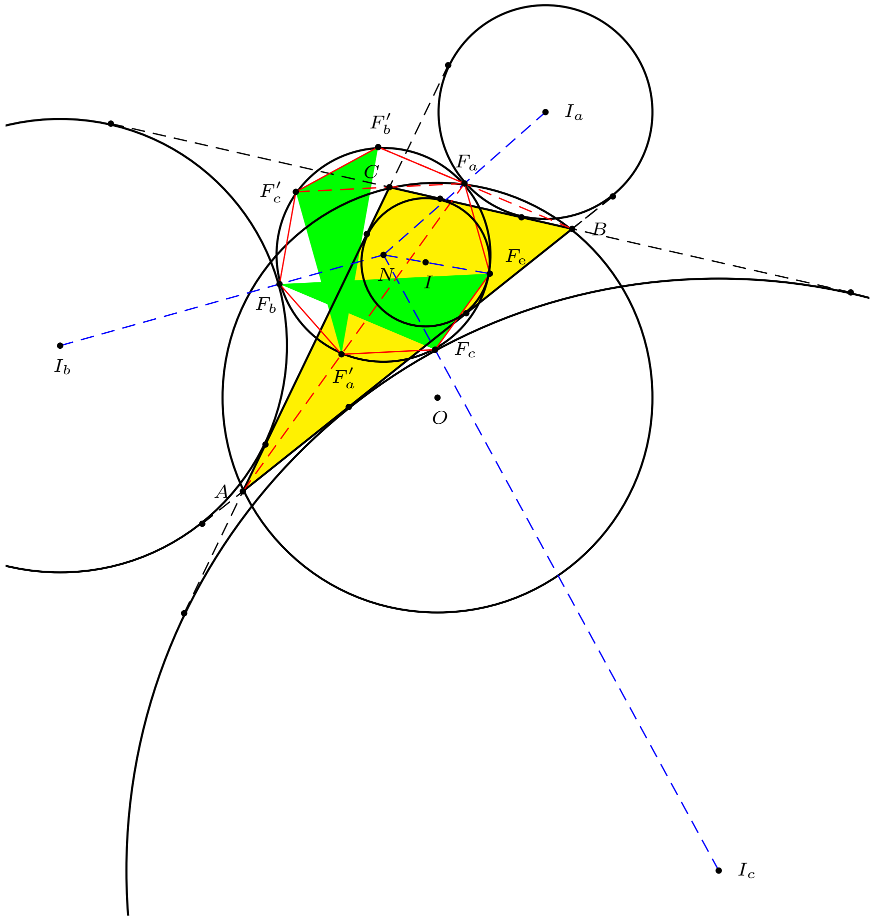

Figure 7. Another companion pair on the nine-point circle

5. The residual vertex as a Kiepert perspector

Theorem 10. *D is a Kiepert perspector of the heptagonal triangle ABC.*

Proof. What this means is that there are similar isosceles triangles $A''BC$, $B''CA$, $C''AB$ with the same orientation such that the lines AA'', BB'', CC'' all pass through the point D. Let A'' be the intersection of the lines AD and $A'B'$, B'' that of BD and $B'C'$, and C'' that of CD and $C'A'$ (see Figure 8). Note that $AC'B'A''$, $BAC'B''$, and $A'B'CC''$ are all parallelograms. From these,

$$A'' = \zeta^4 - \zeta^5 + \zeta^6,$$
$$B'' = \zeta - \zeta^3 + \zeta^5,$$
$$C'' = \zeta^2 + \zeta^3 - \zeta^6.$$

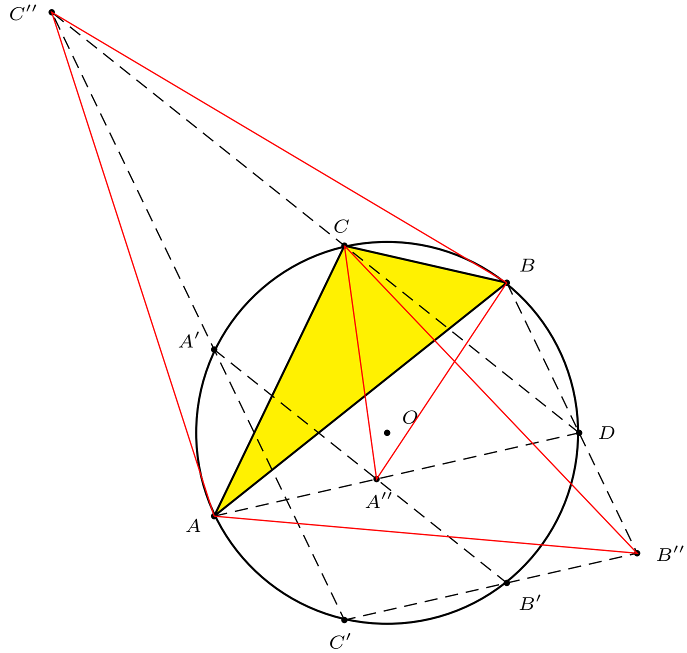

Figure 8. D as a Kiepert perspector of **T**

It is clear that the lines AA'', BB'' and CC'' all contain the point D. The coordinates of A'', B'', C'' can be rewritten as

$$A'' = \frac{\zeta^2+\zeta}{2} + \frac{\zeta^2-\zeta}{2}\cdot(1+2(\zeta+\zeta^2+\zeta^4)),$$
$$B'' = \frac{\zeta^4+\zeta^2}{2} + \frac{\zeta^4-\zeta^2}{2}\cdot(1+2(\zeta+\zeta^2+\zeta^4)),$$
$$C'' = \frac{\zeta+\zeta^4}{2} + \frac{\zeta-\zeta^4}{2}\cdot(1+2(\zeta+\zeta^2+\zeta^4)).$$

Since $1+2(\zeta+\zeta^2+\zeta^4) = \sqrt{7}i$ (Gauss sum), these expressions show that the three isosceles triangles all have base angles $\arctan\sqrt{7}$. Thus, the triangles $A''BC$, $B''CA$, $C''AB$ are similar isosceles triangles of the same orientation. From these we conclude that D is a point on the Kiepert hyperbola. □

Corollary 11. *The center of the Kiepert hyperbola is the point*

$$K_{\rm i} = -\frac{1}{2}(\zeta^3+\zeta^5+\zeta^6). \tag{6}$$

Proof. Since D is the intersection of the Kiepert hyperbola and the circumcircle, the center of the Kiepert hyperbola is the midpoint of DH, where H is the orthocenter of triangle ABC (see Figure 9). This has coordinate as given in (6) above. □

Remark. $K_{\rm i}$ is also the midpoint of OI_a.

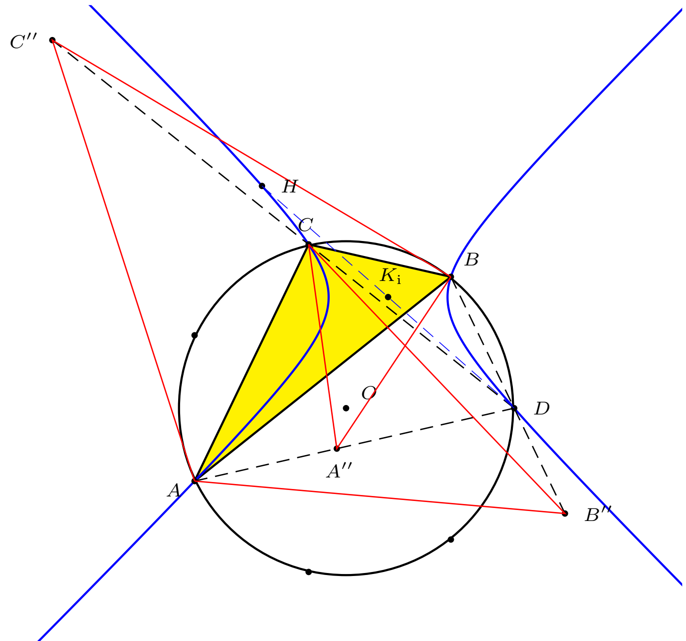

Figure 9. The Kiepert hyperbola of $\mathbf{T}$

Since $X = -1$ is antipodal to the Kiepert perspector $D = 1$ on the circumcircle, it is the Steiner point of $\mathbf{T}$, which is the fourth intersection of the Steiner ellipse with the circumcircle. The Steiner ellipse also passes through the circumcenter, the A-excenter, and the midpoint of HG. The tangents at I_a and X pass through H, and that at O passes through $Y = \frac{1}{2}(1 - (\zeta^3 + \zeta^5 + \zeta^6))$ on the circumcircle such that $OXNY$ is a parallelogram (see Lemma 21).

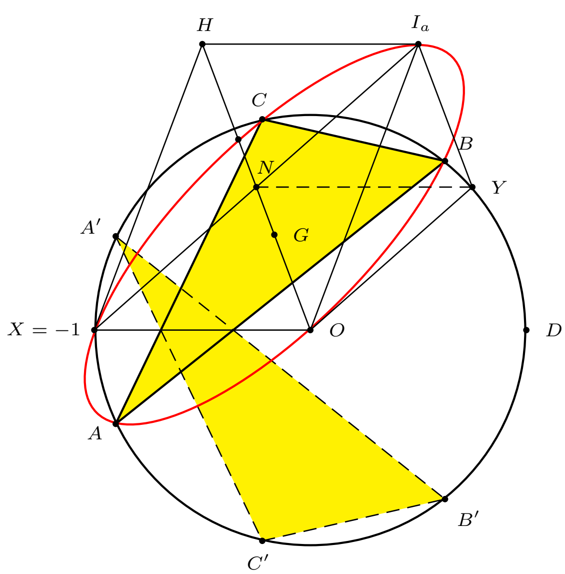

Figure 10. The Steiner ellipse of $\mathbf{T}$

6. The Brocard circle

6.1. *The Brocard points.*

Proposition 12 (Bankoff and Garfunkel)**.** *The nine-point center N is the first Brocard point.*

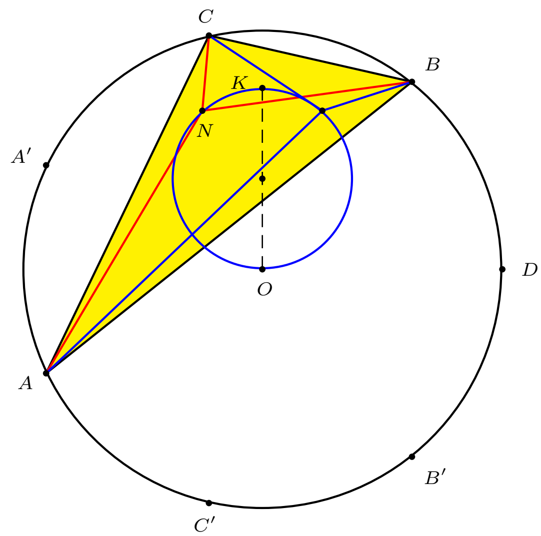

Figure 11. The Brocard points of the heptagonal triangle $\mathbf{T}$

Proof. The relations

$$\frac{1}{2}(\zeta+\zeta^2+\zeta^4)-\zeta^4 = \frac{(-2c_1-3c_2-2c_3)(4+\zeta+\zeta^2+\zeta^4)}{7}\cdot(\zeta-\zeta^4),$$
$$\frac{1}{2}(\zeta+\zeta^2+\zeta^4)-\zeta = \frac{(-2c_1-2c_2-3c_3)(4+\zeta+\zeta^2+\zeta^4)}{7}\cdot(\zeta^2-\zeta),$$
$$\frac{1}{2}(\zeta+\zeta^2+\zeta^4)-\zeta^2 = \frac{(-3c_1-2c_2-2c_3)(4+\zeta+\zeta^2+\zeta^4)}{7}\cdot(\zeta^4-\zeta^2)$$

show that the lines NA, NB, NC are obtained by rotations of BA, CB, AC through the same angle (which is necessarily the Brocard angle ω). This shows that the nine-point center N is the first Brocard point of the heptagonal triangle $\mathbf{T}$. □

Remark. It follows that $4+\zeta+\zeta^2+\zeta^4 = \sqrt{14}(\cos\omega + i\sin\omega)$.

Proposition 13. *The symmedian point K has coordinate* $\frac{2(1+2(\zeta+\zeta^2+\zeta^4))}{7} = \frac{2i}{\sqrt{7}}$.

Proof. It is known that on the Brocard circle with diameter OK, $\angle NOK = -\omega$. From this,

$$\begin{aligned}K &= \frac{1}{\cos\omega}(\cos\omega - i\sin\omega)\cdot N\\ &= \left(1-\frac{i}{\sqrt{7}}\right)\cdot N\\ &= \frac{2(4+\zeta^3+\zeta^5+\zeta^6)}{7}\cdot\frac{\zeta+\zeta^2+\zeta^4}{2}\\ &= \frac{2}{7}(1+2(\zeta+\zeta^2+\zeta^4))\\ &= \frac{2i}{\sqrt{7}}\end{aligned}$$

by Lemma 1. □

Corollary 14. *The second Brocard point is the Kiepert center* $K_{\rm i}$.

Proof. By Proposition 13, the Brocard axis OK is along the imaginary axis. Now, the second Brocard point, being the reflection of N in OK, is simply $-\frac{1}{2}(\zeta^3+\zeta^5+\zeta^6)$. This, according to Corollary 11, is the Kiepert center $K_{\rm i}$. □

Since OD is along the real axis, it is tangent to the Brocard circle.

6.2. *A companion pair on the Brocard circle.*

Center:	$\frac{1}{7}(1+2(\zeta+\zeta^2+\zeta^4))$
Residual vertex:	$O=0$

Rot.	First Brocard triangle	Rot.	Companion
ζ^3	$A_{-\omega}=\frac{1}{7}(-4c_1-2c_2-8c_3)\cdot(-\zeta^5)$	ζ^4	$\frac{1}{7}(-4c_1-2c_2-8c_3)\cdot\zeta^2$
ζ^6	$B_{-\omega}=\frac{1}{7}(-8c_1-4c_2-2c_3)\cdot(-\zeta^3)$	ζ	$\frac{1}{7}(-8c_1-4c_2-2c_3)\cdot\zeta^4$
ζ^5	$C_{-\omega}=\frac{1}{7}(-2c_1-8c_2-4c_3)\cdot(-\zeta^6)$	ζ^2	$\frac{1}{7}(-2c_1-8c_2-4c_3)\cdot\zeta$

Since $-\zeta^5$ is the midpoint of the minor arc joining ζ and ζ^2, the coordinate of the point labeled $A_{-\omega}$ shows that this point lies on the perpendicular bisector of BC. Similarly, $B_{-\omega}$ and $C_{-\omega}$ lie on the perpendicular bisectors of CA and AB respectively. Since these points on the Brocard circle, they are the vertices of the first Brocard triangle.

The vertices of the companion are the second intersections of the Brocard circle with and the lines joining O to C, A, B respectively.

Proposition 15. *The first Brocard triangle is perspective with* ABC *at the point* $-\frac{1}{2}$ (see Figure 12).

Proof.

$$\begin{aligned}-\frac{1}{2} &= (-3c_1-2c_2-2c_3)\cdot A_{-\omega}+c_1\cdot\zeta^4,\\ &= (-2c_1-3c_2-2c_3)\cdot B_{-\omega}+c_2\cdot\zeta,\\ &= (-2c_1-2c_2-3c_3)\cdot C_{-\omega}+c_3\cdot\zeta^2.\end{aligned}$$

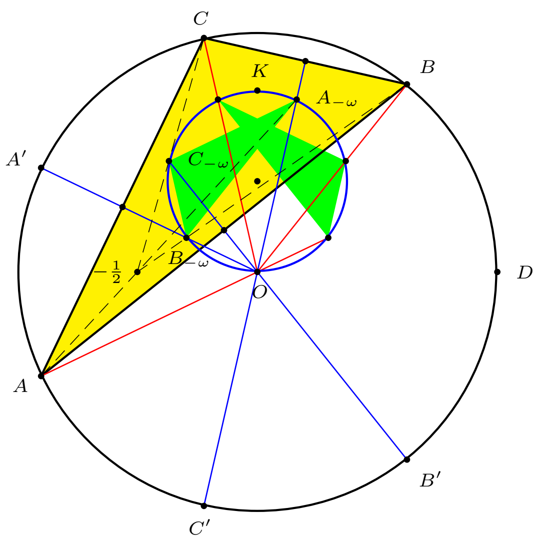

Figure 12. A regular heptagon on the Brocard circle

□

7. A companion of the triangle of reflections

We have computed the coordinates of the vertices of the triangle of reflections $A^*B^*C^*$ in (4). It is interesting to note that this is also a heptagonal triangle, and its circumcenter coincides with I_a. The residual vertex is the reflection of O in I_a.

Center:	$I_a = -(\zeta^3+\zeta^5+\zeta^6)$
Residual vertex:	$\overline{D} = -2(\zeta^3+\zeta^5+\zeta^6)$

Rotation	Triangle of reflections	Rotation	Companion
ζ^4	$A^* = \zeta+\zeta^2-\zeta^6$	ζ^3	$\overline{B} = 1+\zeta^4-\zeta^6$
ζ	$B^* = \zeta^2+\zeta^4-\zeta^5$	ζ^6	$\overline{C} = 1+\zeta-\zeta^5$
ζ^2	$C^* = \zeta-\zeta^3+\zeta^4$	ζ^5	$\overline{A} = 1+\zeta^2-\zeta^3$

The companion has vertices on the sides of triangle ABC,

$$\overline{A} = (1+2c_1)\zeta - 2c_1\cdot\zeta^2;$$
$$\overline{B} = (1+2c_2)\zeta^2 - 2c_2\cdot\zeta^4;$$
$$\overline{C} = (1+2c_3)\zeta^4 - 2c_3\cdot\zeta.$$

It is also perspective with $\mathbf{T}$. Indeed, the lines $A\overline{A}$, $B\overline{B}$, $C\overline{C}$ are all perpendicular to the Euler line, since the complex numbers

$$\frac{1+\zeta^2-\zeta^3-\zeta^4}{\zeta+\zeta^2+\zeta^4},\quad \frac{1+\zeta^4-\zeta^6-\zeta}{\zeta+\zeta^2+\zeta^4},\quad \frac{1+\zeta-\zeta^5-\zeta^2}{\zeta+\zeta^2+\zeta^4}$$

are all imaginary, being respectively $-\sqrt{2}(\zeta^2-\zeta^5)$, $\sqrt{2}(\zeta^3-\zeta^4)$, $-\sqrt{2}(\zeta-\zeta^6)$.

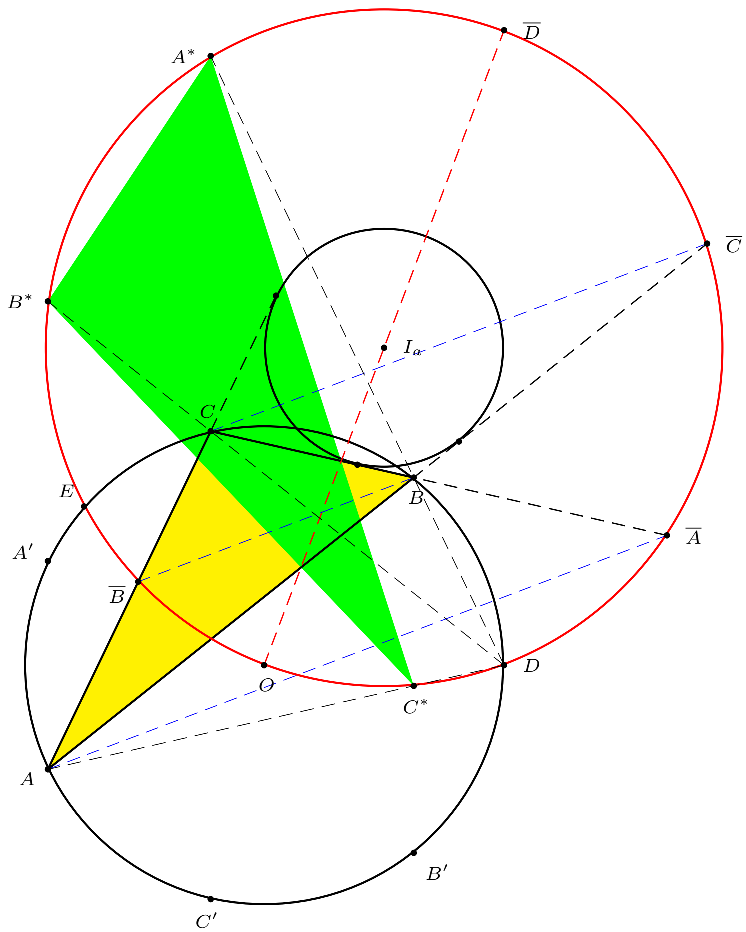

Figure 13. The triangle of reflections of $\mathbf{T}$

Proposition 16. *The triangle of reflections $A^*B^*C^*$ is triply perspective with* $\mathbf{T}$.

Proof. The triangle of reflection $A^*B^*C^*$ is clearly perspective with ABC at the orthocenter H. Since A^*C, B^*A, C^*B are all parallel (to the imaginary axis), the two triangles are triply perspective ([3, Theorem 381]). In other words, $A^*B^*C^*$ is also perspective with BCA. In fact, the perspector is the residual vertex D:

$$A^* = -(1+2c_1)\cdot 1 + (2+2c_1)\zeta,$$
$$B^* = -(1+2c_2)\cdot 1 + (2+2c_2)\zeta^2,$$
$$C^* = -(1+2c_3)\cdot 1 + (2+2c_3)\zeta^4.$$

□

Remark. The circumcircle of the triangle of reflections also contains the circumcenter O, the Euler reflection point E, and the residual vertex D.

8. A partition of T by the bisectors

Let $A_IB_IC_I$ be the cevian triangle of the incenter I of the heptagonal triangle $\mathbf{T} = ABC$. It is easy to see that triangles BCI, ACC_I and BB_IC are also heptagonal. Each of these is the image of the heptagonal triangle ABC under an affine mapping of the form $w = \alpha z + \beta$ or $w = \alpha\overline{z} + \beta$, according as the triangles have the same or different orientations. Note that the image triangle has circumcenter β and circumradius $|\alpha|$.

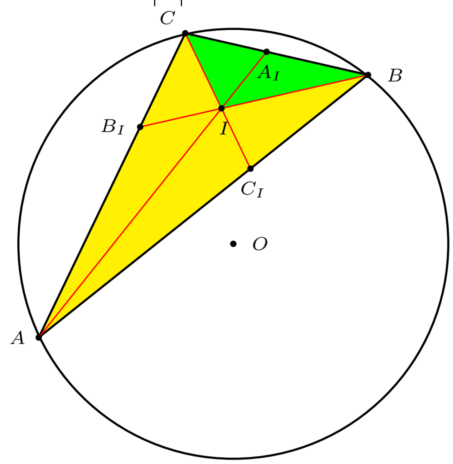

Figure 14. Partition of $\mathbf{T}$ by angle bisectors

Each of these mappings is determined by the images of two vertices. For example, since ABC and BCI have the same orientation, the mapping $f_1(z) = \alpha z + \beta$ is determined by the images $f_1(A) = B$ and $f_1(B) = C$; similarly for the mappings f_2 and f_3.

Affine mapping	A	B	C
$f_1(z) = (\zeta + \zeta^4)z - \zeta^5$	B	C	I
$f_2(z) = (1 + \zeta + \zeta^3 + \zeta^4)\overline{z} - (1 + \zeta^3 + \zeta^6)$	A	C	C_I
$f_3(z) = (1 + \zeta^2 + \zeta^4 + \zeta^5)\overline{z} - (1 + \zeta^3 + \zeta^5)$	B	B_I	C

Thus, we have

$$\begin{aligned} I =&\ f_1(C) = \zeta^3 - \zeta^5 + \zeta^6, \\ C_I =&\ f_2(C) = -1 + \zeta + \zeta^2 - \zeta^3 + \zeta^5, \\ B_I =&\ f_3(B) = -1 + \zeta + \zeta^4 - \zeta^5 + \zeta^6. \end{aligned}$$

Note also that from $f_2(A_I) = I$, it follows that

$$A_I = 1 + \zeta^2 - \zeta^3 + \zeta^4 - \zeta^6.$$

Remark. The affine mapping that associates a heptagonal triangle with circumcenter c and residual vertex d to its companion is given by

$$w = \frac{d - c}{\overline{d} - \overline{c}} \cdot \overline{z} + \frac{\overline{d}c - \overline{c}d}{\overline{d} - \overline{c}}.$$

8.1. *Four concurrent lines.* A simple application of the mapping f_1 yields the following result on the concurrency of four lines.

Proposition 17. *The orthocenter of the heptagonal triangle BCI lies on the line OC and the perpendicular from C_I to AC.*

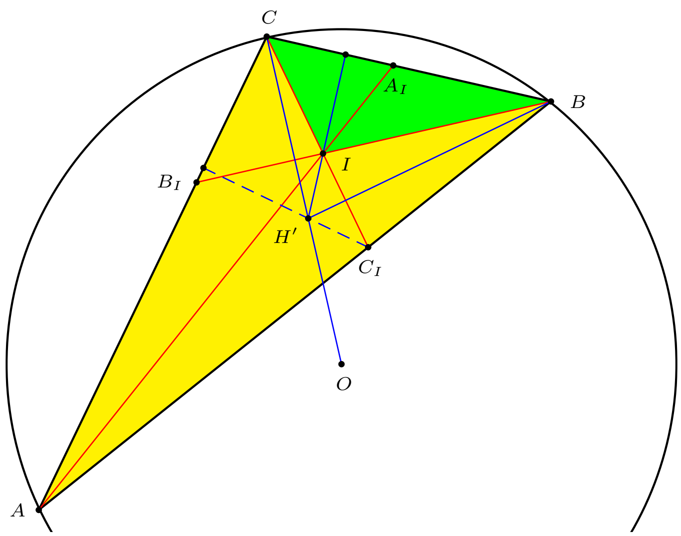

Figure 15. Four concurrent altitudes

Proof. Since ABC has orthocenter $H = \zeta + \zeta^2 + \zeta^4$, the orthocenter of triangle BCI is the point

$$H' = f_1(H) = -(1 + \zeta^4) = -(\zeta^2 + \zeta^5)\zeta^2.$$

This expression shows that H' lies on the radius OC. Now, the vector $H'C_I$ is given by

$$\begin{aligned} C_I - H' = & (-1 + \zeta + \zeta^2 - \zeta^3 + \zeta^5) + (1 + \zeta^4) \\ = & \zeta + \zeta^2 - \zeta^3 + \zeta^4 + \zeta^5. \end{aligned}$$

On the other hand, the vector AC is given by $\zeta^2 - \zeta^4$. To check that $H'C_I$ is perpendicular to AC, we need only note that

$$(\zeta + \zeta^2 - \zeta^3 + \zeta^4 + \zeta^5)\overline{(\zeta^2 - \zeta^4)} = -2(\zeta - \zeta^6) + (\zeta^2 - \zeta^5) + (\zeta^3 - \zeta^4)$$

is purely imaginary. □

Remark. Similarly, the orthocenter of ACC_I lies on the C-altitude of ABC, and that of BB_IC on the B-altitude.

8.2. *Systems of concurrent circles.*

Proposition 18. *The nine-point circles of* ACC_I *and* (the isosceles triangle) $B'A'C$ *are tangent internally at the midpoint of* $B'C$.

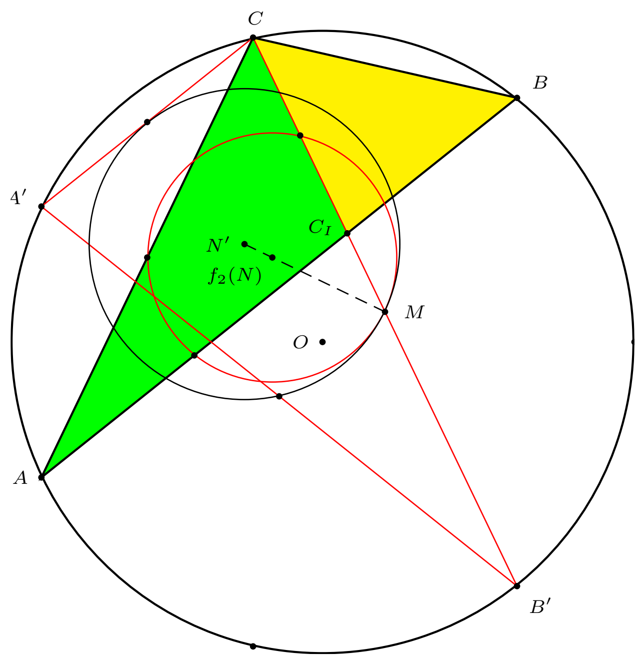

Figure 16. Two tangent nine-point circles

Proof. The nine-point circle of the isosceles triangle $B'A'C$ clearly contains the midpoint M of $B'C$. Since triangle $AB'C$ is also isosceles, the perpendicular from A to $B'C$ passes through M. This means that M lies on the nine-point circle of triangle ACC_I. We show that the two circles are indeed tangent at M.

The nine-point center of ACC_I is the point

$$f_2(N) = \frac{1}{2}(2\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^6).$$

On the other hand, the nine-point center of the isosceles triangle $B'A'C$ is the point

$$N' = \frac{1}{2}(\zeta^2 + \zeta^3 + \zeta^6).$$

Since

$$M = \frac{\zeta^2 + \zeta^6}{2} = (1 - 2c_2 - 4c_3)f_2(N) + (2c_2 + 4c_3)N'$$

as can be verified directly, we conclude that the two circles are tangent internally. □

Theorem 19. *The following circles have a common point.*
(i) *the circumcircle of* ACC_I,
(ii) *the nine-point circle of* ACC_I,
(iii) *the* A*-excircle of* ACC_I,
(iv) *the nine-point circle of* BB_IC.

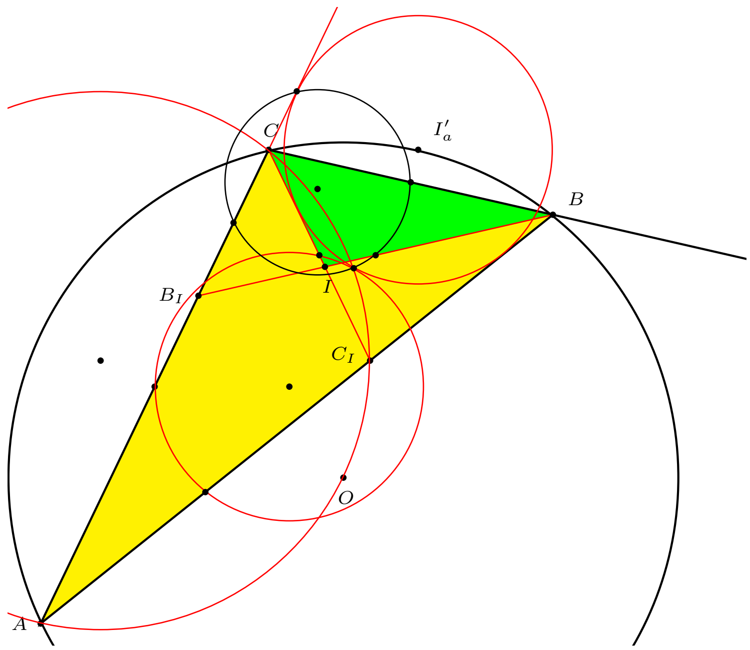

Figure 17. Four concurrent circles

Proof. By Proposition 7(3), the first three circles concur at the A-Feuerbach point of triangle ACC_I, which is the point

$$f_2(F_a) = \frac{1}{4}(\zeta + 2\zeta^2 + \zeta^4 - \zeta^5 + \zeta^6).$$

It is enough to verify that this point lies on the nine-point circle of BB_IC, which has center

$$f_3\left(\frac{\zeta + \zeta^2 + \zeta^4}{2}\right) = \frac{2\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^6}{2},$$

and square radius

$$\frac{1}{4}|1 + \zeta^2 + \zeta^4 + \zeta^5|^2 = -\frac{1}{4}\left(3(\zeta + \zeta^6) + (\zeta^2 + \zeta^5) + 2(\zeta^3 + \zeta^4)\right).$$

This is exactly the square distance between $f_2(F_a)$ and the center, as is directly verified. This shows that $f_2(F_a)$ indeed lies on the nine-point circle of BB_IC. □

Theorem 20. *Each of the following circles contains the Feuerbach point $F_{\rm e}$ of* $\mathbf{T}$:
(i) *the nine-point circle of* $\mathbf{T}$,
(ii) *the incircle of* $\mathbf{T}$,
(iii) *the nine-point circle of the heptagonal triangle* BCI,
(iv) *the* C*-excircle of* BCI,
(v) *the* A*-excircle of the heptagonal triangle* ACC_I,
(vi) *the incircle of the isosceles triangle* BIC_I.

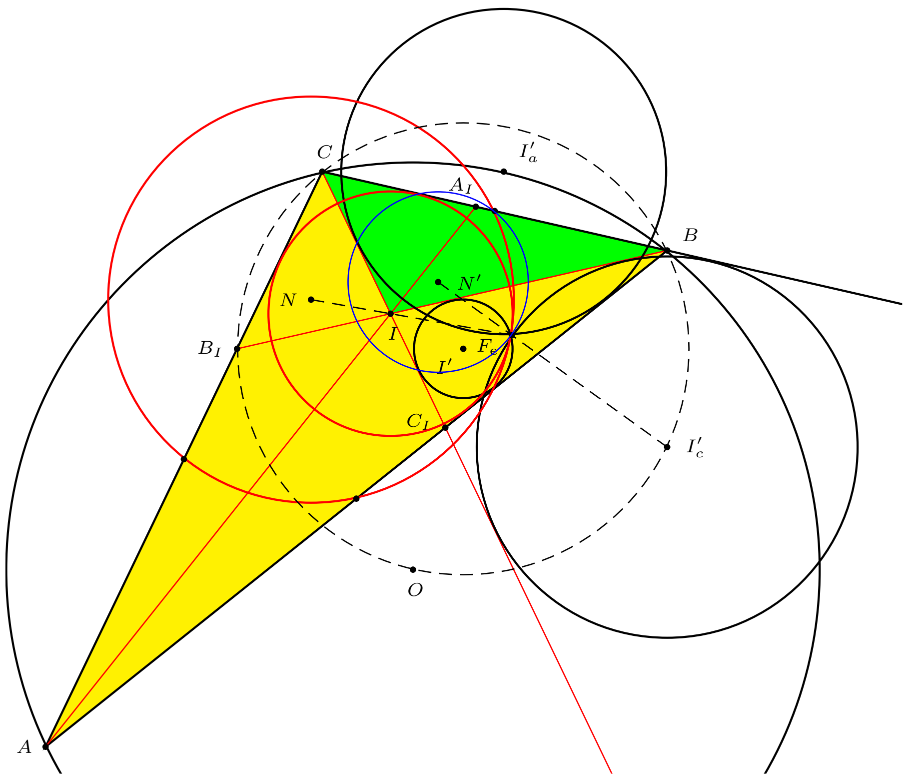

Figure 18. Six circles concurrent at the Feuerbach point of $\mathbf{T}$

Proof. It is well known that the nine-point circle and the incircle of $\mathbf{T}$ are tangent to each other internally at the Feuerbach point $F_{\rm e}$. It is enough to verify that this point lies on each of the remaining four circles.

(iii) and (iv) The C-excircle of BCI is the image of the B-excircle of ABC under the affine mapping f_1. It is therefore enough to check that $f_1(F_b) = F_{\rm e}$:

$$\begin{aligned} f_1(F_b) = &\ \frac{1}{4}(\zeta+\zeta^4)(\zeta+\zeta^2+\zeta^3+2\zeta^4-\zeta^6)-\zeta^5 \\ = &\ \frac{1}{4}(2\zeta+\zeta^2+\zeta^4-\zeta^5+\zeta^6) = F_{\rm e}. \end{aligned}$$

(v) The heptagonal triangle ACC_I is the image of ABC under the mapping f_2. It can be verified directly that $W = -\frac{1}{4}(\zeta - \zeta^2 + 3\zeta^3 + 3\zeta^5) - \zeta^6$ is the point for which $f_2(W) = F_{\rm e}$. The square distance of W from the A-excenter $I_a = -(\zeta^3 + \zeta^5 + \zeta^6)$ is the square norm of $W - I_a = \frac{1}{4}(-\zeta + \zeta^2 + \zeta^3 + \zeta^5)$. An easy calculation shows that this is

$$\frac{1}{16}(-\zeta + \zeta^2 + \zeta^3 + \zeta^5)(\zeta^2 + \zeta^4 + \zeta^5 - \zeta^6) = \frac{1}{4} = r_a^2.$$

It follows that, under the mapping f_2, $F_{\rm e}$ lies on the A-excircle of ACC_I.

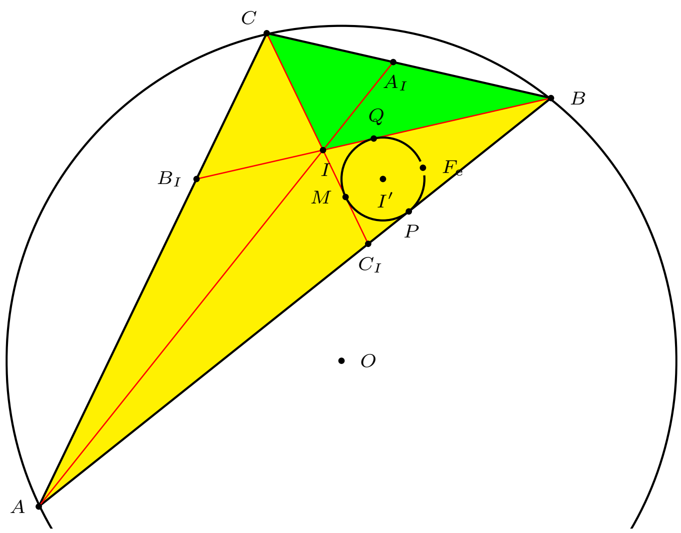

Figure 19. The incircle of an isosceles triangle

(vi) Since C_IBC and ICB_I are isosceles triangles, the perpendicular bisectors of BC and CB_I are the bisectors of angles IC_IB and C_IIB respectively. It follows that the incenter of the isosceles triangle BIC_I coincides with the circumcenter of triangle BB_IC, which is the point $I' = -(1 + \zeta^3 + \zeta^5)$ from the affine mapping f_3. This incircle touches the side IC_I at its midpoint M, the side IB at the midpoint Q of BB_I, and the side BC_I at the orthogonal projection P of C on AB (see Figure 19). A simple calculation shows that $\angle PMQ = \frac{3\pi}{7}$. To show that $F_{\rm e}$ lies on the same circle, we need only verify that $\angle PF_{\rm e}Q = \frac{4\pi}{7}$. To this end, we first determine some complex number coordinates:

$$P = \frac{1}{2}(\zeta + \zeta^2 - \zeta^3 + \zeta^4),$$

$$Q = \frac{1}{2}(-1 + 2\zeta + \zeta^4 - \zeta^5 + \zeta^6).$$

Now, with $F_{\rm e} = \frac{1}{4}(2\zeta + \zeta^2 + \zeta^4 - \zeta^5 + \zeta^6)$, we have

$$Q - F_{\rm e} = (\zeta^4 + \zeta^6)(P - F_{\rm e}).$$

From the expression $\zeta^4 + \zeta^6 = \zeta^{-2}(\zeta + \zeta^6)$, we conclude that indeed $\angle PF_{\rm e}Q = \frac{4\pi}{7}$. □

9. A theorem on the Fermat points

Lemma 21. *The perpendicular bisector of the segment ON is the line containing $X = -1$ and $Y = \frac{1}{2}(1 - (\zeta^3 + \zeta^5 + \zeta^6))$.*

Proof. (1) Complete the parallelogram OI_aHX, then

$$X = O + H - I_a = (\zeta + \zeta^2 + \zeta^4) + (\zeta^3 + \zeta^5 + \zeta^6) = -1$$

is a point on the circumcircle. Note that N is the midpoint of I_aX. Thus, $NX = NI_a = R = OX$. This shows that X is on the bisector of ON.

(2) Complete the parallelogram ONI_aY, with $Y = O + I_a - N$. Explicitly, $Y = \frac{1}{2}(1 - (\zeta^3 + \zeta^5 + \zeta^6))$. But we also have

$$X + Y = (O + H - I_a) + (O + I_a - N) = (2 \cdot N - I_a) + (O + I_a - N) = O + N.$$

This means that $OXNY$ is a rhombus, and $NY = OY$.

From (1) and (2), XY is the perpendicular bisector of ON. □

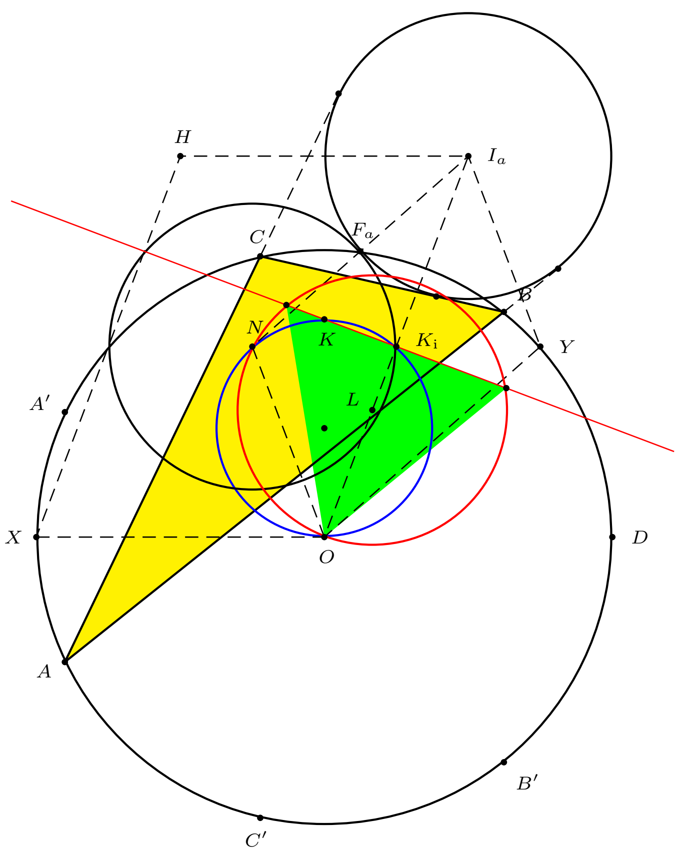

Figure 20. The circumcenter and the Fermat points form an equilateral triangle

Theorem 22. *The circumcenter and the Fermat points of the heptagonal triangle* **T** *form an equilateral triangle.*

Proof. (1) Consider the circle through O, with center at the point

$$L := -\frac{1}{3}(\zeta^3 + \zeta^5 + \zeta^6).$$

This is the center of the equilateral triangle with O as a vertex and $K_{\rm i} = -\frac{1}{2}(\zeta^3 + \zeta^5 + \zeta^6)$ the midpoint of the opposite side. See Figure 20.

(2) With X and Y in Lemma 21, it is easy to check that $L = \frac{1}{3}(X + 2Y)$. This means that L lies on the perpendicular bisector of ON.

(3) Since $K_{\rm i}$ is on the Brocard circle (with diameter OK), $OK_{\rm i}$ is perpendicular to the line $KK_{\rm i}$. It is well known that the line $KK_{\rm i}$ contains the Fermat points. [3] Indeed, $K_{\rm i}$ is the midpoint of the Fermat points. This means that L is lies on the perpendicular bisector of the Fermat points.

(4) By a well known theorem of Lester (see, for example, [5]), the Fermat points, the circumcenter, and the nine-point center are concyclic. The center of the circle containing them is necessarily L, and this circle coincides with the circle constructed in (1). The side of the equilateral triangle opposite to O is the segment joining the Fermat points. □

Corollary 23. *The Fermat points of the heptagonal triangle* $\mathbf{T}$ *are the points*

$$F_+ = \frac{1}{3}(\lambda + 2\lambda^2)(\zeta^3 + \zeta^5 + \zeta^6),$$
$$F_- = \frac{1}{3}(\lambda^2 + 2\lambda)(\zeta^3 + \zeta^5 + \zeta^6),$$

where $\lambda = \frac{1}{2}(-1 + \sqrt{3}i)$ *and* $\lambda^2 = \frac{1}{2}(-1 - \sqrt{3}i)$ *are the imaginary cube roots of unity.*

Remarks. (1) The triangle with vertices I_a and the Fermat points is also equilateral.

(2) Since $OI_a = \sqrt{2}R$, each side of the equilateral triangle has length $\sqrt{\frac{2}{3}}R$.

(3) The Lester circle is congruent to the orthocentroidal circle, which has HG as a diameter.

(4) The Brocard axis OK is tangent to the A-excircle at the midpoint of I_aH.

References

[1] L. Bankoff and J. Garfunkel, The heptagonal triangle, *Math. Mag.*, 46 (1973) 7–19.

[2] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, 2nd edition, 1990, Springer-Verlag.

[3] R. A. Johnson, *Advanced Euclidean Geometry*, 1929, Dover reprint 2007.

[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[5] P. Yiu, The circles of Lester, Evans, Parry, and their generalizations, preprint.

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, 777 Glades Road, Boca Raton, Florida 33431-0991, USA

E-mail address: `yiu@fau.edu`

[3] The line joining the Fermat points contains K and $K_{\rm i}$.