

## On Trigonometric Proofs of the Steiner-Lehmus Theorem

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*Dedicated to the memory of Professor Ferenc Radó (1921-1990)*

**Abstract.** We offer a survey of some lesser known or new trigonometric proofs of the Steiner-Lehmus theorem. A new proof of a recent refined variant is also given.

### 1. Introduction

The famous Steiner-Lehmus theorem states that if the internal angle bisectors of two angles of a triangle are equal, then the triangle is isosceles. For a recent survey of the Steiner-Lehmus theorem, see M. Hajja [8]. From the bibliography of [8] one can find many methods of proof, purely geometric, or trigonometric, of this theorem. Our aim in this note is to add some new references, and to draw attention to some little or unknown proofs, especially trigonometric ones. We shall also include a new trigonometric proof of a refined version of the Steiner-Lehmus theorem, published recently [9].

First, we want to point out some classical geometric proofs published in 1967 by A. Froda [4], attributed to W. T. Williams and G. T. Savage. Another interesting proof by A. Froda appears in his book [5] (see also the book of the second author [15]). Another purely geometric proof was published in 1973 by M. K. Sathya Narayana [16]. Other papers are by K. Seydel and C. Newman [17], or the more recent papers by D. Beran [1] or D. Rüthing [13]. None of the recent extensive surveys connected with the Steiner-Lehmus theorem mentions the use of complex numbers in the proof. Such a method appears in the paper by C. I. Lubin [11] in 1959.

Trigonometric proofs of Euclidean theorems have gained additional importance after the appearance of Ungar's book [18]. In this book, the author develops a kind of trigonometry that serves Hyperbolic Geometry in the same way our ordinary trigonometry does Euclidean Geometry. He calls it Gyrotrigonometry and proves that the ordinary trigonometric identities have counterparts in that trigonometry.

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The authors thank Professor D. Plachky for a reprint of [9] and Professor A. Furdek for providing a copy of [11]. They are also indebted to the referee for his many remarks and suggestions which helped improve the presentation of the paper, and for pointing out the new references [6, 7, 10, 18].

Consequently, he takes certain trigonometrical proofs of Euclidean theorems and shows that these proofs, hence also the corresponding theorems, remain valid in Hyperbolic Geometry. In this context, he includes the trigonometric proofs of the Urquhart and the Steiner-Lehmus theorems that appeared in [7] and [8]. Related to the question, first posed by Sylvester (also mentioned in [8]), whether there is a direct proof of the Steiner-Lehmus theorem, recently J. H. Conway (see [2]) has given an intriguing argument that there is no such proof. However, the validity of Conway's argument is debatable since a claim of the non-existence of a direct proof should be formulated in a more precise manner using, for example, the language of intuitionistic logic.

## 2. Trigonometric proofs of the Steiner-Lehmus theorem

2.1. Perhaps one of the shortest trigonometric proofs of the Steiner-Lehmus theorem one can find in a forgotten paper (written in Romanian) in 1916 by V. Cristescu [3]. Let  $BB'$  and  $CC'$  denote two angle bisectors of the triangle  $ABC$  (see Fig. 1). By using the law of sines in triangle  $BB'C$ , one gets

$$\frac{BB'}{\sin C} = \frac{BC}{\sin\left(C + \frac{B}{2}\right)}.$$

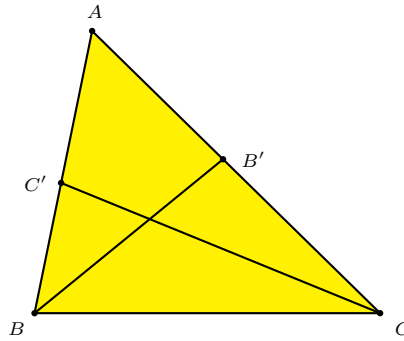


Figure 1.

As  $C + \frac{B}{2} = C + \frac{180^\circ - C - A}{2} = 90^\circ - \frac{A - C}{2}$ , one has

$$BB' = a \cdot \frac{\sin C}{\cos \frac{A - C}{2}}.$$

Similarly,

$$CC' = a \cdot \frac{\sin B}{\cos \frac{A - B}{2}}.$$

Assuming  $BB' = CC'$ , and using the identities  $\sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2}$ , and  $\sin \frac{C}{2} = \cos \frac{A+B}{2}$ ,  $\sin \frac{B}{2} = \cos \frac{A+C}{2}$ , we have

$$\cos \frac{C}{2} \cdot \cos \frac{A+B}{2} \cos \frac{A-B}{2} = \cos \frac{B}{2} \cos \frac{A+C}{2} \cos \frac{A-C}{2}. \quad (1)$$

Now from the identity

$$\cos(x + y) \cdot \cos(x - y) = \cos^2 x + \cos^2 y - 1,$$

relation (1) becomes

$$\cos \frac{C}{2} \left( \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - 1 \right) = \cos \frac{B}{2} \left( \cos^2 \frac{A}{2} + \cos^2 \frac{C}{2} - 1 \right).$$

This simplifies into

$$\left( \cos \frac{B}{2} - \cos \frac{C}{2} \right) \left( \sin^2 \frac{A}{2} + \cos \frac{B}{2} \cos \frac{C}{2} \right) = 0.$$

As the second paranthesis of (6) is strictly positive, this implies  $\cos \frac{B}{2} - \cos \frac{C}{2} = 0$ , so  $B = C$ .

2.2. In 2000, respectively 2001, the German mathematicians D. Plachky [12] and D. R uthing [14] have given other trigonometric proofs of the Steiner-Lehmus theorem, based on area considerations. We present here the method by Plachky. Denote the angles at  $B$  and  $C$  respectively by  $\beta$  and  $\gamma$ , and the angle bisectors  $BB'$  and  $AA'$  by  $w_b$  and  $w_a$  (see Figure 2).

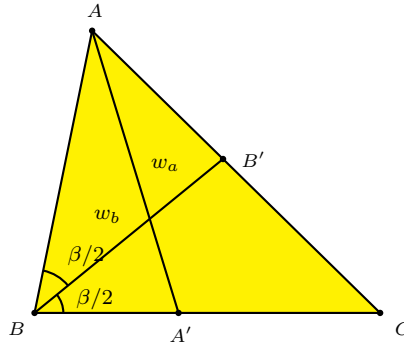


Figure 2.

By using the trigonometric form  $\frac{1}{2}ab \sin \gamma$  of the area of triangle  $ABC$ , and decomposing the initial triangle in two triangles, we get

$$\frac{1}{2}aw_b \sin \frac{\beta}{2} + \frac{1}{2}cw_b \sin \frac{\beta}{2} = \frac{1}{2}bw_a \sin \frac{\alpha}{2} + \frac{1}{2}cw_a \sin \frac{\alpha}{2}.$$

By the law of sines we have

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin(\pi - (\alpha + \beta))}{c},$$

so assuming  $w_\alpha = w_\beta$ , we obtain

$$\frac{c \sin \alpha}{\sin(\alpha + \beta)} \sin \frac{\beta}{2} + c \sin \frac{\beta}{2} = \frac{c \sin \beta}{\sin(\alpha + \beta)} \sin \frac{\alpha}{2} + c \sin \frac{\alpha}{2},$$

or

$$\sin(\alpha + \beta) \left( \sin \frac{\alpha}{2} - \sin \frac{\beta}{2} \right) + \sin \frac{\alpha}{2} \sin \beta - \sin \alpha \sin \frac{\beta}{2} = 0. \quad (2)$$

Writing  $\sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$  etc, and using the formulae

$$\sin u - \sin v = 2 \sin \frac{u-v}{2} \cos \frac{u+v}{2}, \quad (3)$$

$$\cos u - \cos v = -2 \sin \frac{u-v}{2} \sin \frac{u+v}{2}, \quad (4)$$

we rewrite (2) as

$$2 \sin \frac{\alpha - \beta}{4} \left( \sin(\alpha + \beta) \cos \frac{\alpha + \beta}{4} + 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\alpha + \beta}{2} \right) = 0.$$

Since  $\alpha + \beta < \pi$ , the expression inside the parenthesis is strictly positive. It follows that  $\alpha = \beta$ .

2.3. The following trigonometric proof seems to be much simpler. It can be found in [10, pp. 194-196]. According to Honsberger, this proof was rediscovered by M. Hajja who later came across it in some obscure Russian book. The authors rediscovered again this proof, and wish to thank the referee for this information. Writing the area of triangle  $ABC$  in two different ways (using triangles  $ABB'$  and  $BB'C$ ) we get immediately

$$w_b = \frac{2ac}{a+c} \cos \frac{\beta}{2}. \quad (5)$$

Similarly,

$$w_a = \frac{2bc}{b+c} \cos \frac{\alpha}{2}. \quad (6)$$

Suppose now that,  $a > b$ . Then  $\alpha > \beta$ , so  $\frac{\alpha}{2} > \frac{\beta}{2}$ . As  $\frac{\alpha}{2}, \frac{\beta}{2} \in (0, \frac{\pi}{2})$ , one gets  $\cos \frac{\alpha}{2} < \cos \frac{\beta}{2}$ . Also,  $\frac{bc}{b+c} < \frac{ac}{a+c}$  is equivalent to  $b < a$ . Thus (5) and (6) imply  $w_a > w_b$ . This is indeed a proof of the Steiner-Lehmus theorem, as supposing  $w_a = w_b$  and letting  $a > b$ , we would lead to the contradiction  $w_a > w_b$ , a contradiction; similarly with  $a < b$ .

For another trigonometric proof of a generalized form of the theorem, we refer the reader to [6].

### 3. A new trigonometric proof of a refined version

Recently, M. Hajja [9] proved the following stronger version of the Steiner-Lehmus theorem. Let  $BY$  and  $CZ$  be the angle bisectors and let  $BY = y$ ,  $CZ = z$ ,  $YC = v$ ,  $BZ = V$  (see Figure 3).

Then

$$c > b \Rightarrow y + v > z + V. \quad (7)$$

As  $V = \frac{ac}{a+b}$ ,  $v = \frac{ab}{a+c}$ , it is immediate that  $c > b \Rightarrow V > v$ . Thus, assuming  $c > b$ , and using (7) we get  $y > z$ , i.e. the Steiner-Lehmus theorem (see §2.3). In [9], the proof of (7) made use of a nice lemma by R. Breusch. We offer here a new trigonometric proof of (7), based only on the law of sines, and simple trigonometric facts.

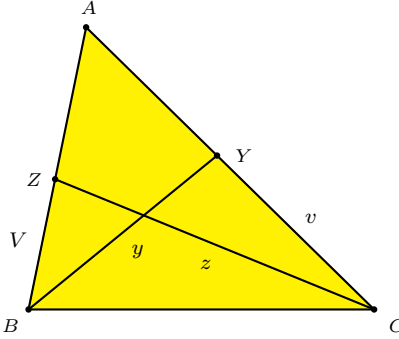


Figure 3.

In triangle  $BCY$  one can write

$$\frac{a}{\sin\left(C + \frac{B}{2}\right)} = \frac{CY}{\sin\frac{B}{2}} = \frac{BY}{\sin C},$$

so

$$\frac{y + v}{\sin C + \sin\frac{B}{2}} = \frac{a}{\sin\left(C + \frac{B}{2}\right)},$$

implying

$$y + v = \frac{a\left(\sin C + \sin\frac{B}{2}\right)}{\sin\left(C + \frac{B}{2}\right)}. \quad (8)$$

Similarly,

$$z + V = \frac{a\left(\sin B + \sin\frac{C}{2}\right)}{\sin\left(B + \frac{C}{2}\right)}. \quad (9)$$

Assume now that  $y + v > z + V$ . Applying  $\sin u + \sin v = 2 \sin \frac{u+v}{2} \cos \frac{u-v}{2}$  and using the facts that  $\cos\left(\frac{C}{2} + \frac{B}{4}\right) > 0$ ,  $\cos\left(\frac{B}{2} + \frac{C}{4}\right) > 0$ , after simplification, from (8) and (9) we get the inequality

$$\cos\left(\frac{C}{2} - \frac{B}{4}\right) \cos\left(\frac{B}{2} + \frac{C}{4}\right) > \cos\left(\frac{B}{2} - \frac{C}{4}\right) \cos\left(\frac{C}{2} + \frac{B}{4}\right).$$

Using  $2 \cos u \cos v = \cos \frac{u+v}{2} + \cos \frac{u-v}{2}$ , this implies

$$\cos\left(\frac{3C}{4} + \frac{B}{4}\right) + \cos\left(\frac{C}{4} - \frac{3B}{4}\right) > \cos\left(\frac{3B}{4} + \frac{C}{4}\right) + \cos\left(\frac{B}{4} - \frac{3C}{4}\right),$$

or

$$\cos\left(\frac{3C}{4} + \frac{B}{4}\right) - \cos\left(\frac{3B}{4} + \frac{C}{4}\right) > \cos\left(\frac{B}{4} - \frac{3C}{4}\right) - \cos\left(\frac{C}{4} - \frac{3B}{4}\right).$$

Now applying (4), we get

$$-\sin\frac{B}{2} \sin\frac{3C}{2} > -\sin\frac{C}{2} \sin\frac{3B}{2}. \quad (10)$$

By  $\sin 3u = 3 \sin u - 4 \sin^3 u$  we get immediately from (10) that

$$-3 + 4 \sin^2 \frac{C}{2} > -3 + 4 \sin^2 \frac{B}{2}. \quad (11)$$

Since the function  $x \mapsto \sin^2 x$  is strictly increasing in  $x \in (0, \frac{\pi}{2})$ , the inequality (11) is equivalent to  $C > B$ . We have actually shown that  $y+v > z+V \Leftrightarrow C > B$ , as desired.

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