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An Angle Bisector Parallel Applied to Triangle Construction

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Abstract. We prove a theorem describing a line parallel to an angle bisector of a triangle and passing through the midpoint of the opposite side. We then apply this theorem to the solution of four triangle construction problems.

Consider the triangle ABC with angle bisector AT_a, altitude AH_a, midpoint M_a of side BC and Euler point E_a (see Figure 1). Let the circle with center at E_a and passing through M_a intersect AH_a at P. Draw the line M_aP. We prove the following theorem.

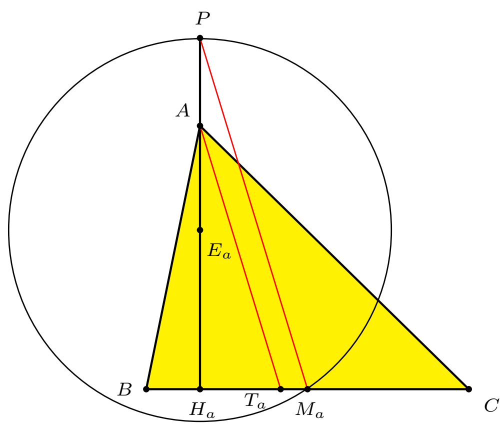

Figure 1.

Theorem 1. *In any triangle ABC with H_a not coinciding with M_a, the line M_aP is parallel to the angle bisector AT_a.*

Proof. Let O be the circumcenter of ABC (see Figure 2). The perpendicular bisector M_aO and the angle bisector AT_a intersect the circumcircle (O) at S. Let the midpoint of E_aO be R, and reflect the entire figure through R. Let the reflection of ABC be $A'B'C'$. Since E_aM_a is equal to the circumradius, the circle $E_a(M_a)$ is the reflection of (O) and is the circumcircle of $A'B'C'$. Since segments AE_a and M_aO are equal and parallel, A is the reflection of M_a and is therefore the midpoint of $B'C'$. Thus, AH_a is the perpendicular bisector of $B'C'$. Finally, AH_a intersects circle $E_a(M_a)$ at P, therefore M_aP is the bisector of angle $B'A'C'$ and parallel to AT_a. □

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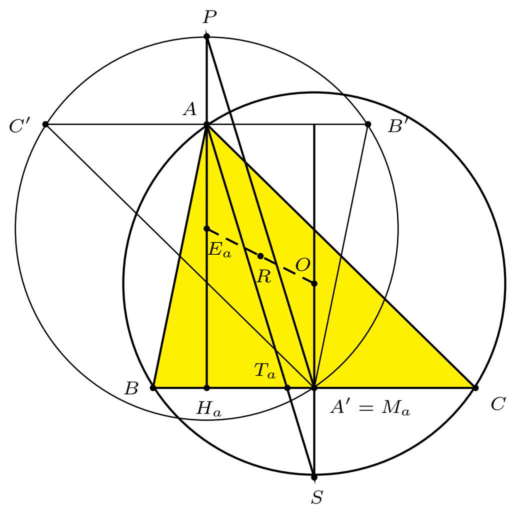

Figure 2.

Remark. For the case where H_a and M_a coincide, triangle ABC is isosceles (with apex at A) or equilateral. The lines M_aP and AT_a will coincide.

Wernick [3] presented 139 problems of constructing a triangle with ruler and compass given the location of three points associated with the triangle and chosen from a list of sixteen points. See Meyers [2] for updates on the status of the problems from this list. Connelly [1] extended this work by adding four more points to the list and 140 additional problems, many of which were designated as unresolved. We now apply Theorem 1 to solve four of these previously unresolved problems. The problem numbers are those given by Connelly.

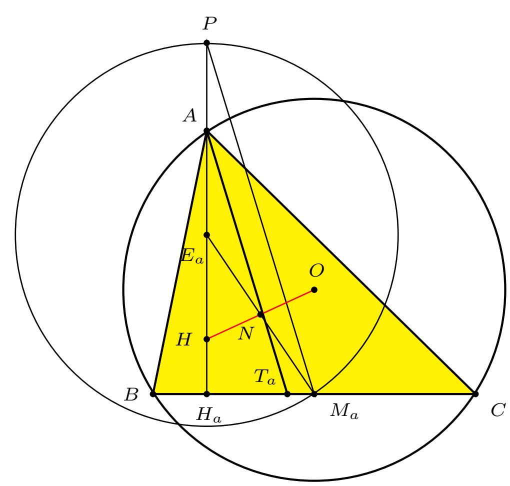

Figure 3.

Problem 99. Given E_a, M_a and T_a construct triangle ABC.
Solution. Draw line M_aT_a containing the side BC and then the altitude E_aH_a

to this side (see Figure 3). The circle with center E_a and passing through M_a intersects the altitude at P. Draw M_aP. By Theorem 1, the line through T_a parallel to M_aP intersects the altitude at A. Reflect A through E_a to get the orthocenter H. The midpoint of E_aM_a is the nine-point center N. Reflect H through N to obtain O. Draw the circumcircle through A, intersecting M_aT_a at B and C.

Number of Solutions. Depending on the relative positions of the three points, there are two solutions, no solution or an infinite number of solutions. We start by locating E_a and M_a. Then the segment E_aM_a is a diameter of the nine-point circle (N). Since, for any triangle, angle $E_aT_aM_a$ must be greater than $90°$, T_a must be inside (N), or coincide with M_a, to have a valid solution. For the case with T_a inside (N), we have two solutions since the circle $E_a(M_a)$ intersects the altitude twice and each intersection leads to a distinct solution. If the three points are collinear, the two triangles are congruent reflections of each other through the line. If T_a is outside or on (N), except at M_a, there is no solution. If T_a coincides with M_a, there are an infinite number of solutions. In this case, the vertex A can be chosen anywhere on the open segment M_aM_a' (where M_a' is the reflection of M_a in E_a), and there is a resultant isosceles triangle.

Problem 108. Given E_a, N and T_a construct triangle ABC.

Problem 137. Given M_a, N and T_a construct triangle ABC.
Solution. Since N is the midpoint of E_aM_a, both of these problems reduce to Problem 99.

Problem 130. Given H_a, N and T_a construct triangle ABC.
Solution. The nine-point circle, with center N and passing through H_a, intersects line H_aT_a again at M_a, also reducing this problem to Problem 99.

Related to these, the solutions of the following two problems are locus restricted:
(i) Problem 78: given E_a, H_a, T_a;
(ii) Problem 99 in Wernick's list [3]: given M_a, H_a, T_a.

References

[1] H. Connelly, An extension of triangle construction from located points, *Forum Geom.*, 9 (2009) 109–112.

[2] L. F. Meyers, Update on William Wernick's "Triangle constructions with three located points", *Math. Mag.*, 69 (1996) 46–49.

[3] W. Wernick, Triangle constructions with three located points, *Math. Mag.*, 55 (1982) 227–230.

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