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A Family of Quartics Associated with a Triangle

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Abstract. It is known [1, p.115] that the envelope of the family of pedal lines (Simson or Wallace lines) of a triangle ABC is Steiner's deltoid, a three-cusped hypocycloid that is concentric with the nine-point circle of ABC and touches it at three points. Also known [2, p.249] is that the nine-point circle is the locus of the intersection point of two perpendicular pedal lines. This paper considers a generalization in which two pedal lines form any acute angle θ. It is found that the locus of their intersection point, for any value of θ, is a quartic curve with the same axes of symmetry as the deltoid. Moreover, the deltoid is the envelope of the family of quartics. Finally, it is shown that all of these quartics, as well as the deltoid and the nine-point circle, may be simultaneously generated by points on a circular disk rolling on the inside of a fixed circle.

1. Sketching the loci

Consider two pedal lines of triangle ABC which intersect and form an angle θ. It is required to find the locus of the intersection point for all such pairs of pedal lines for any fixed value of θ. There are infinitely many loci as θ varies between 0 and $\frac{\pi}{2}$. By plotting points, some of the loci are sketched in Figure 1. These include the cases $\theta = \frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{5\pi}{12}$, and $\frac{\pi}{2}$, the curves have been colored. As $\theta \to 0$, the locus approaches Steiner's deltoid. It will be shown later that in general the locus is a quartic curve. As $\theta \to \frac{\pi}{2}$, the quartic merges into two coincident circles (the nine-point circle). Otherwise each curve has three double points, which seem to merge into a triple point when $\theta = \frac{\pi}{3}$. This case resembles the familiar trefoil, or "three-leaved rose" of polar coordinates.

2. A conjecture

Figure 1 seems to suggest that all of the loci might be generated simultaneously by points on a circular disk that rolls inside a fixed circle concentric with the nine-point circle. For example, the deltoid could be generated by a point on the circumference of the disk, provided that the radius of the disk is one third that of the circle. The other curves might be hypotrochoids generated by interior points of the disk. However, this fails because, for example, there is no generating point for the nine-point circle.

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Another possible approach is given by Zwikker [2, pp.248–249], who shows that the same hypocycloid of three cusps may be generated when the radius of the rolling circle is two thirds of the radius of the fixed circle. In this case the deltoid is generated in the opposite sense, and two circuits of the rolling circle are required to generate the entire curve. Simultaneously the nine-point circle is generated by the center of the rolling disk. It is now necessary to prove that every locus in the family is generated by a point on the rolling disk.

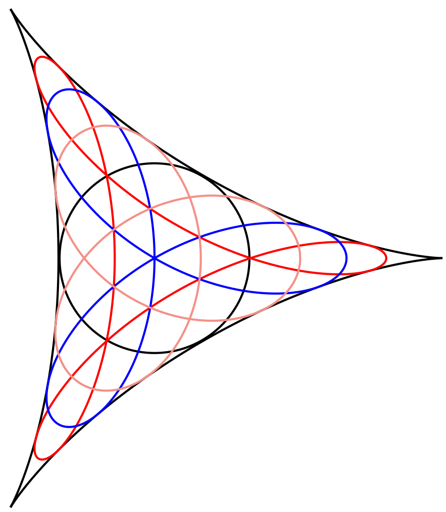

Figure 1

3. Partial proof of the conjecture

In Figure 2 the nine-point circle is placed with its center at the origin of an xy-plane. Its radius is $\frac{R}{2}$, R being the radius of the circumcircle of ABC. The radius of the fixed circle, also with center at the origin, is $\frac{3R}{2}$. The rolling disk has radius R, and initially it is placed so that it is touching the fixed circle at a cusp of the deltoid. Let the x-axis pass through this point of tangency. The center of the rolling disk is designated by Q, so that $OQ = \frac{R}{2}$. ST is a diameter of the rolling disk, with T initially at its starting point $\left(\frac{3R}{2}, 0\right)$. Let $P = \left(\frac{R}{2} + u,\ 0\right)$ be any point on the radius $QT (0 \leq u \leq R)$. Then, as the disk rotates clockwise about its center, it rolls counterclockwise along the circumference of the fixed circle, and the locus of P is represented parametrically by

$$\begin{aligned} x &= \frac{R}{2}\cos t + u\cos\frac{t}{2}, \\ y &= \frac{R}{2}\sin t - u\sin\frac{t}{2}. \end{aligned} \tag{1}$$

In these equations u is the parameter of the family of hypotrochoids, while t is the running parameter on each curve. When $u = 0$, the locus is the nine-point circle $x^2 + y^2 = \frac{R^2}{4}$. When $u = R$, the parametric equations become

$$x = \frac{R}{2}\left(\cos t + 2\cos\frac{t}{2}\right),$$
$$y = \frac{R}{2}\left(\sin t - 2\sin\frac{t}{2}\right), \qquad (2)$$

which are well known to represent a deltoid.

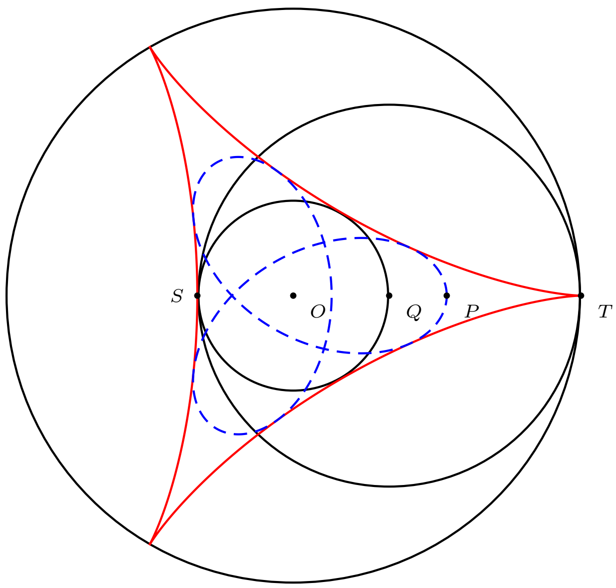

Figure 2

One further example is the case $u = \frac{R}{2}$, for which

$$x = \frac{R}{2}\left(\cos t + \cos\frac{t}{2}\right) = R\cos\frac{3t}{4}\cos\frac{t}{4},$$
$$y = \frac{R}{2}\left(\sin t - \sin\frac{t}{2}\right) = R\cos\frac{3t}{4}\sin\frac{t}{4}. \qquad (3)$$

These equations represent a trefoil, for which the standard equation in polar coordinates is

$$r = a\cos 3\theta,$$

from which $x = a\cos 3\theta\cos\theta$ and $y = a\cos 3\theta\sin\theta$. This result is identical with (3) when $t = 4R$ and $R = a$. Hence $u = \frac{R}{2}$ gives a trefoil (see Figure 3).

The foregoing is not a complete proof of the conjecture, because it is necessary to establish a connection with the loci of Figure 1. These are the curves generated by the intersection points of pedals lines that form a constant angle.

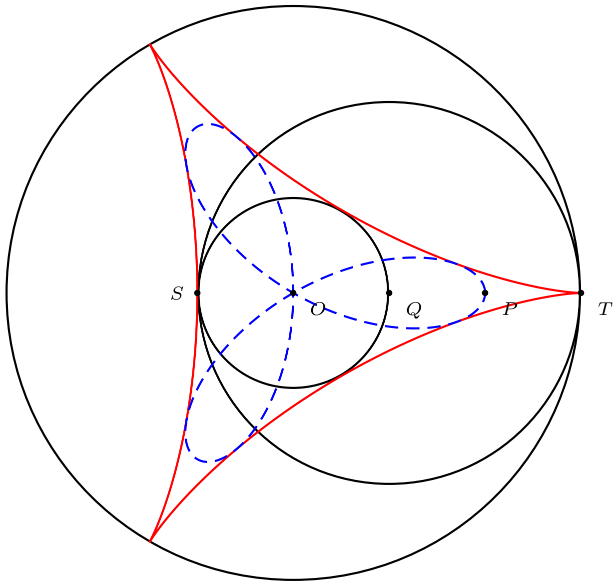

Figure 3

4. A family of quartics

By means of elementary algebra and trigonometric identities, the parameter t may be eliminated from equations (1) to obtain

$$(4R^2(x^2+y^2)+24u^2Rx+8u^4+2u^2R62-T^4)^2$$
$$=4u^2(4Rx+4u^2-R^2)^2(4Rx+u^2+2R^2). \quad (4)$$

Thus, (1) is transformed into an equation of degree 4 in x and y. The only exceptional case is $u=0$, which reduces to $(4x^2+4y^2-R^2)^2=0$. This represents the nine-point circle, taken twice.

5. Envelope of the family

In order to find the envelope of (4), it is more practical to use the parametric form (1). The parameter u will be eliminated by using the partial differential equation

$$\frac{\partial x}{\partial t}\frac{\partial y}{\partial u}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial t},$$

or

$$-\left(\frac{R}{2}\sin t+\frac{u}{2}\sin\frac{t}{2}\right)\left(-\sin\frac{t}{2}\right)=\left(\cos\frac{t}{2}\right)\left(\frac{R}{2}\cos t-\frac{u}{2}\cos\frac{t}{2}\right).$$

This reduces to $u=R\cos\frac{3t}{2}$, and substitution in (1) results in the equations

$$x=\frac{R}{2}\left(2\cos t+\cos 2t\right),$$
$$y=\frac{R}{2}\left(2\sin t-\sin 2t\right). \quad (5)$$

Replacing t by $-\frac{t}{2}$ transforms (5) to (2), showing that the envelope of (1) or (4) is the deltoid, which is itself a member of the family.

6. A "rolling" diameter

At the point given by (2) the slope of the deltoid is easily found to be $\tan\frac{t}{4}$. Hence the equation of the tangent line may be calculated to be

$$y - \frac{R}{2}\left(\sin t - 2\sin\frac{t}{2}\right) = \tan\frac{t}{4}\left(x - \frac{R}{2}\left(\cos t + 2\cos\frac{t}{2}\right)\right),$$

or

$$y = \tan\frac{t}{4}\left(x - \frac{R}{2}\left(1 + 2\cos\frac{t}{2}\right)\right). \tag{6}$$

Since the deltoid is a quartic curve, and since the point of tangency may be regarded as a double intersection with the tangent line (6), the tangent must meet the curve at two other points. Let $\left(\frac{R}{2}\left(\cos v + 2\cos\frac{v}{2}\right), \frac{R}{2}\left(\sin v - 2\sin\frac{v}{2}\right)\right)$ be any point on the curve, and substitute this for (x, y) in (6). The result is

$$\frac{R}{2}\left(\sin v - 2\sin\frac{v}{2}\right) = \tan\frac{t}{4}\left(\frac{R}{2}\left(\cos v + 2\cos\frac{v}{2}\right) - \frac{R}{2}\left(1 + 2\cos\frac{t}{2}\right)\right),$$

which becomes

$$\begin{aligned} &2\sin\frac{v}{2}\cos\frac{v}{2} - 2\sin\frac{v}{2} \\ = \; &\tan\frac{t}{4}\left(\cos^2\frac{v}{2} - \sin^2\frac{v}{2} + 2\cos\frac{v}{2} - 1 - 2\cos\frac{t}{2}\right). \end{aligned} \tag{7}$$

In order to rewrite this as a homogeneous quartic equation, we make use of the identities

$$\begin{aligned} \sin\frac{v}{2} &= 2\sin\frac{v}{4}\cos\frac{v}{4}, \\ \cos\frac{v}{2} &= \cos^2\frac{v}{4} - \sin^2\frac{v}{4}, \\ 1 &= \sin^2\frac{v}{4} + \cos^2\frac{v}{4}. \end{aligned}$$

Then (7) becomes

$$\begin{aligned} &4\sin\frac{v}{4}\cos\frac{v}{4}\left(\cos^2\frac{v}{4} - \sin^2\frac{v}{4}\right) - 4\sin\frac{v}{4}\cos\frac{v}{4}\left(\sin^2\frac{v}{4} + \cos^2\frac{v}{4}\right) \\ = \; &\tan\frac{t}{4}\left[\left(\cos^2\frac{v}{4} - \sin^2\frac{v}{4}\right)^2 - \left(2\sin\frac{v}{4}\cos\frac{v}{4}\right)^2\right. \\ &+ 2\left(\cos^2\frac{v}{4} - \sin^2\frac{v}{4}\right)\left(\sin^2\frac{v}{4} + \cos^2\frac{v}{4}\right) - \left(\sin^2\frac{v}{4} + \cos^2\frac{v}{4}\right)^2 \\ &\left.-2\cos\frac{t}{2}\left(\sin^2\frac{v}{4} + \cos^2\frac{v}{4}\right)^2\right]. \end{aligned}$$

The terms are then arranged according to descending powers of $\sin\frac{v}{4}$ to obtain

$$2\tan\frac{t}{4}\left(1+\cos\frac{t}{2}\right)\sin^4\frac{v}{4}-8\sin^3\frac{v}{4}\cos\frac{v}{4}$$
$$+4\tan\frac{t}{4}\left(2+\cos\frac{t}{2}\right)\sin^2\frac{v}{4}\cos^2\frac{v}{4}-2\tan\frac{t}{4}\left(1-\cos\frac{t}{2}\right)\cos^4\frac{v}{4}=0.$$

Dividing by $2\tan\frac{t}{4}\cos^4\frac{v}{4}$ and letting $V:=\tan\frac{v}{4}$ simplifies this to

$$\left(1+\cos\frac{t}{2}\right)V^4-4\cot\frac{t}{4}\cdot V^3+2\left(2+\cos\frac{t}{2}\right)V^2-\left(1-\cos\frac{t}{2}\right)=0.$$

Since the tangent line touches the deltoid where $v=t$, the quartic expression must contain the double factor $\left(V-\tan\frac{t}{4}\right)^2$. The factored result is

$$\left(1+\cos\frac{t}{2}\right)\left(V-\tan\frac{t}{4}\right)^2\left[V^2-2\cot\frac{t}{4}\cdot V-1\right]=0.$$

Hence the other solutions are found by solving

$$V^2-2\cot\frac{t}{4}\cdot V-1=0,$$

which yields $V=\cot\frac{t}{4}\pm\csc\frac{t}{4}=\cot\frac{t}{8}$ or $-\tan\frac{t}{8}$. Since $V=\tan\frac{v}{4}$, these may be expressed as $v=2\pi-\frac{t}{2}$ and $v=-\frac{t}{2}$ respectively. Because of periodicity there are other solutions to the quadratic equation, but geometrically there are only two, and the ones found here are distinct. The first one, substituted in (2), gives

$$(x,y)=\left(\frac{R}{2}\left(\cos\frac{t}{2}-2\cos\frac{t}{4}\right),\ \frac{R}{2}\left(-\sin\frac{t}{2}-2\sin\frac{t}{4}\right)\right).$$

Let this be the point T, shown in Figures 2 and 3. The point S at the other end of the diameter is given by the second solution $v=-\frac{t}{2}$:

$$S=\left(\frac{R}{2}\left(\cos\frac{t}{2}+2\cos\frac{t}{4}\right),\ \frac{R}{2}\left(-\sin\frac{t}{2}+2\sin\frac{t}{4}\right)\right).$$

The usual distance formula shows that the length of ST is $2R$. Moreover, the midpoint of ST is $\left(\frac{R}{2}\cos\frac{t}{2},\ -\frac{R}{2}\sin\frac{t}{2}\right)$, which is on the nine-point circle. Therefore it is the center of the rolling disk, and ST is a diameter. Since both S and T generate the deltoid, this confirms the fact that, for any line tangent to the deltoid, the segment within the curve is of constant length. See [2, p.249].

In order for the point T to trace one arch of the deltoid, the rolling disk travels through $\frac{4\pi}{3}$ radians on the fixed circle. Simultaneously the diameter ST rolls end over end to generate (as a tangent) the other two arches of the deltoid.

7. Proof of the conjecture

It remains to be shown that every locus defined by the intersection point of two pedal lines meeting at a fixed angle is a hypotrochoid defined by (1). Let one pedal line be given by (6), with slope $\tan\frac{t}{4}$. A second pedal line, forming the angle θ with the first, is obtained by replacing $\frac{t}{4}$ by $\frac{t}{4}+\theta$. (There is no need to include

$-\theta$, because this will be taken care of while t ranges over all of its values). The equation of the second pedal line will therefore be

$$y = \tan\left(\frac{t}{4} + \theta\right)\left[x - \frac{R}{2}\left(1 + 2\cos\left(\frac{t}{2} + 2\theta\right)\right)\right]. \tag{8}$$

Simultaneous solution of (6) and (7), after manipulation with trigonometrical identities, gives the result

$$\begin{aligned} x &= \frac{R}{2}\left[\cos(t + 2\theta) + 2\cos\theta\cos\left(\frac{t}{2} + \theta\right)\right], \\ y &= \frac{R}{2}\left[\sin(t + 2\theta) - 2\cos\theta\sin\left(\frac{t}{2} + \theta\right)\right]. \end{aligned} \tag{9}$$

Finally, replacing $t + 2\theta$ by t and $R\cos\theta$ by u, we transform (9) into

$$\begin{aligned} x &= \frac{R}{2}\cos t + u\cos\frac{t}{2}, \\ y &= \frac{R}{2}\sin t - u\sin\frac{t}{2}, \end{aligned}$$

precisely equal to (1), the parametric equations of the family of hypotrochoids. Thus the result is established.

Remark. The family of quartics contains loci which are outside the deltoid, but these correspond to values of $u > R$, in which case θ would be imaginary.

References

[1] H. S. M. Coxeter, *Introduction to Geoemtry*, Wiley, New York, 1951.
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