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Circle Chains Inside a Circular Segment

Giovanni Lucca

Abstract. We consider a generic circles chain that can be drawn inside a circular segment and we show some geometric properties related to the chain itself. We also give recursive and non recursive formulas for calculating the centers coordinates and the radius of the circles.

1. Introduction

Consider a circle with diameter AB, center C, and a chord GH perpendicular to AB (see Figure 1). Point O is the intersection between the diameter and the chord. Inside the circular segment bounded by the chord GH and the arc GBH, it is possible to construct a doubly infinite chain of circles each tangent to the chord, and to its two immediate neighbors.

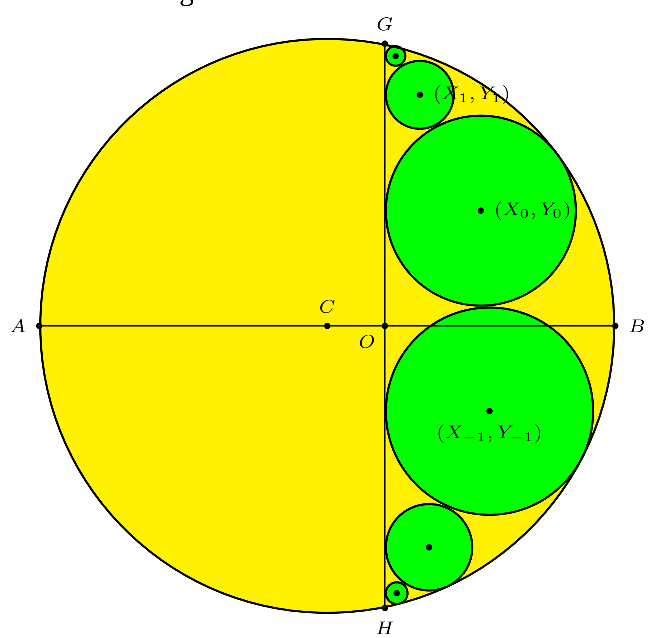

Figure 1. Circle chain inside a circular segment

Let $2(a + b)$ be the diameter of the circle and $2b$ the length of the segment OB. We set up a cartesian coordinate system with origin at O. Beginning with a circle with center (X_0, Y_0) and radius r_0 tangent to the chord GH and the arc GBH, we construct a doubly infinite chain of tangent circles, with centers (X_i, Y_i) and radius r_i for integer values of i, positive and negative.

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2. Some geometric properties of the chain

We first demonstrate some basic properties of the doubly infinite chain of circles.

Proposition 1. *The centers of the circles lie on the parabola with axis along AB, focus at C, and vertex the midpoint of OB.*

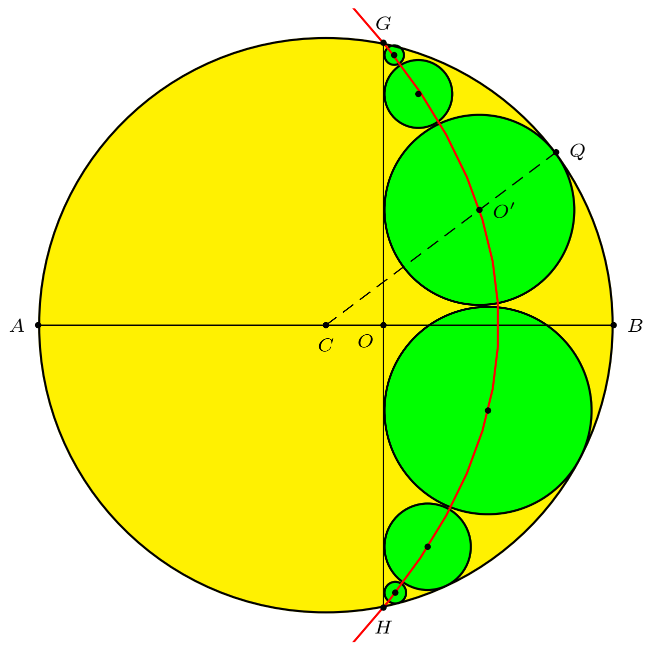

Figure 2. Centers of circles in chain on a parabola

Proof. Consider a circle of the chain with center $O'(x, y)$, radius r, tangent to the arc GBH at Q. Since the segment CQ contains O' (see Figure 2), we have, by taking into account that C has coordinates $(b - a, 0)$ and

$$\begin{aligned} CQ &= a + b, \\ CO' &= \sqrt{(x - b + a)^2 + y^2}, \\ O'Q &= r = x, \\ CO' &= CQ - O'Q. \end{aligned}$$

From these, we have

$$\sqrt{(x - b + a)^2 + y^2} = a + b - x,$$

which simplifies into

$$y^2 = -4a(x - b). \tag{1}$$

This clearly represents the parabola symmetric with respect to the x-axis, vertex $(b, 0)$, the midpoint of OB, and focus $(b - a, 0)$, which is the center C of the given circle. □

Proposition 2. *The points of tangency between consecutive circles of the chain lie on the circle with center A and radius AG.*

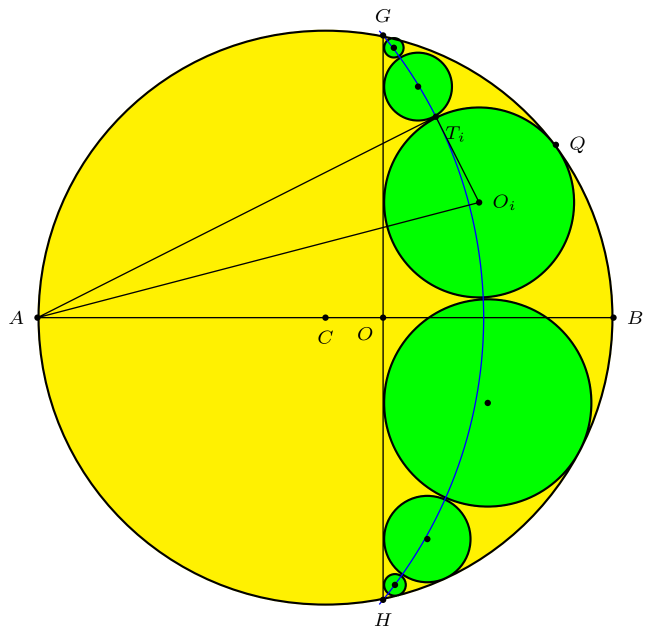

Figure 3. Points of tangency on a circular arc

Proof. Consider two neighboring circles with centers (X_i, Y_i), (X_{i+1}, Y_{i+1}), radii r_i, r_{i+1} respectively, tangent to each other at T_i (see Figure 3). By using Proposition 1 and noting that A has coordinates $(-2a, 0)$, we have

$$AO_i^2 = (X_i + 2a)^2 + Y_i^2 = \left(-\frac{Y_i^2}{4a} + b + 2a\right)^2 + Y_i^2,$$

$$r_i^2 = X_i^2 = \left(-\frac{Y_i^2}{4a} + b\right)^2.$$

Applying the Pythagorean theorem to the right triangle AO_iT_i, we have

$$AT_i^2 = AO_i^2 - r_i^2 = 4a(a+b) = AO \cdot AB = AG^2.$$

It follows that T_i lies on the circle with center A and radius AG. □

Proposition 3. *If a circle of the chain touches the chord GH at P and the arc GBH at Q, then the points A, P, Q are collinear.*

Proof. Suppose the circle has center O'. It touches GH at P and the arc GBH at Q (see Figure 4). Note that triangles CAQ and $O'PQ$ are isosceles triangles with $\angle ACQ = \angle PO'Q$. It follows that $\angle CQA = \angle O'QP$, and the points A, P, Q are collinear. □

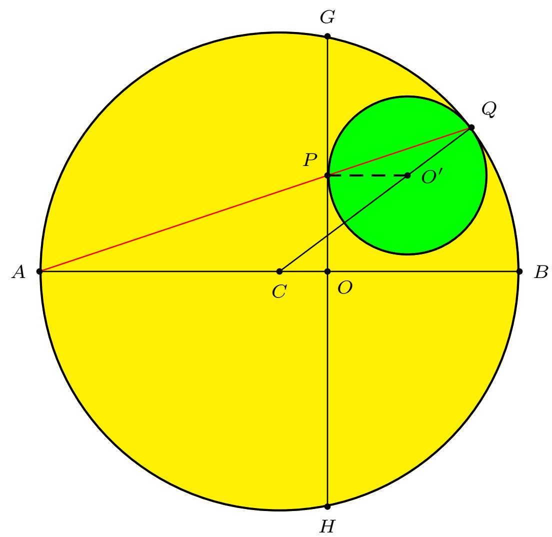

Figure 4. Line joining points of tangency

Remark. Proposition 3 gives an easy construction of the circle given any one of the points of tangency. The center of the circle is the intersection of the line CQ and the perpendicular to GH at P.

3. Coordinates of centers and radii

Figure 5 shows a right triangle $O_iO_{i-1}K_i$ with the centers O_{i-1} and O_i of two neighboring circles of the chain. Since these circles have radii $r_{i-1} = X_{i-1}$ and $r_i = X_i$ respectively, we have

$$\begin{aligned}(X_{i-1} - X_i)^2 + (Y_i - Y_{i-1})^2 &= (r_i + r_{i-1})^2 = (X_i + X_{i-1})^2, \\ (Y_i - Y_{i-1})^2 &= 4X_iX_{i-1}.\end{aligned}$$

Making use of (1), we rewrite this as

$$(Y_i - Y_{i-1})^2 = 4\left(b - \frac{Y_i^2}{4a}\right)\left(b - \frac{Y_{i-1}^2}{4a}\right),$$

or

$$\frac{4a(a+b) - Y_{i-1}^2}{4a^2} \cdot Y_i^2 - 2Y_{i-1}Y_i + \frac{(a+b)Y_{i-1}^2 - 4ab^2}{a} = 0. \qquad (2)$$

If we index the circles in the chain in such a way that the ordinate Y_i increases with the index i, then from (2) we have

$$Y_i = \frac{2Y_{i-1} - \left(\frac{Y_{i-1}^2}{a} - 4b\right)\sqrt{1 + \frac{b}{a}}}{2\left(1 + \frac{b}{a} - \frac{Y_{i-1}^2}{4a^2}\right)}. \qquad (3a)$$

This is a recursive formula that can be applied provided that the ordinate Y_0 of the first circle is known. Note that Y_0 must be chosen in the interval $(-2\sqrt{ab},\ 2\sqrt{ab})$.

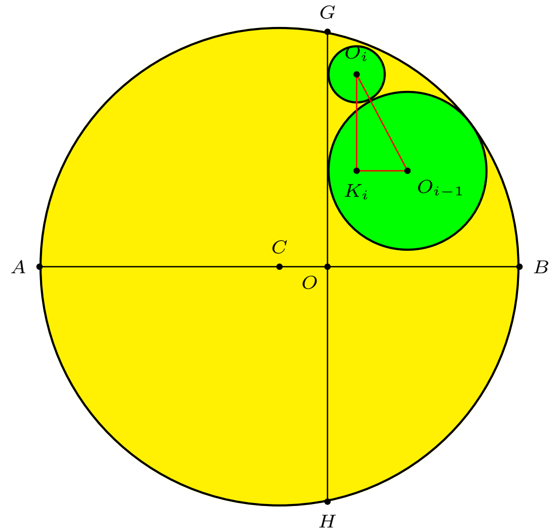

Figure 5. Construction for determination of recursive formula

Formulas for the abscissa of the centers and radii are immediately derived from (1), *i.e.*:

$$X_i = r_i = -\frac{Y_i^2}{4a} + b. \tag{4}$$

Now, it is possible to transform the recursion formula (**??**) into a continued fraction. In fact, after some simple algebraic steps (which we omit for brevity), we have

$$Y_i = 2a\left(\sqrt{1+\frac{b}{a}} - \frac{1}{\frac{Y_{i-1}}{2a} + \sqrt{1+\frac{b}{a}}}\right). \tag{5}$$

Defining

$$\alpha = 2\sqrt{1+\frac{b}{a}}, \quad \text{and} \quad Z_i = \frac{Y_i}{2a} - \sqrt{1+\frac{b}{a}} \tag{6}$$

for $i = 1,\ 2,\ \dots,$ we have

$$Z_i = -\frac{1}{\alpha + Z_{i-1}}.$$

Thus, for positive integer values of i,

$$Z_i = -\cfrac{1}{\alpha - \cfrac{1}{\alpha - \cfrac{1}{\ddots - \frac{1}{\alpha + Z_{0+}}}}},$$

where we have used Z_{0+} in place of Z_0

$$Z_{0+} = \frac{Y_0}{2a} - \sqrt{1+\frac{b}{a}}.$$

This is to distinguish from the extension of the chain by working the recursion (3a) backward: [1]

$$Y_{i-1} = \frac{2Y_i + \left(\frac{Y_i^2}{a} - 4b\right)\sqrt{1+\frac{b}{a}}}{2\left(1+\frac{b}{a}-\frac{Y_i^2}{4a^2}\right)}. \tag{3b}$$

Thus, for negative integer values of i, with

$$Z_{-i} = \frac{Y_{-i}}{2a} + \sqrt{1+\frac{b}{a}},$$

we have

$$Z_{-i} = -\cfrac{1}{-\alpha - \cfrac{1}{-\alpha - \cfrac{1}{\ddots - \frac{1}{-\alpha + Z_{0-}}}}},$$

where

$$Z_{0-} = \frac{Y_0}{2a} + \sqrt{1+\frac{b}{a}}.$$

It is possible to give nonrecursive formulas for calculating Y_i and Y_{-i}. For brevity, in the following, we shall consider only Y_i for positive integer indices because, as far as Y_{-i} is concerned, it is enough to change, in all the formulae involved, α into $-\alpha$, Z_i into Z_{-i}, and Z_{0+} into Z_{0-}. Starting from (5), and by considering its particular structure, one can write, for $i = 1,\ 2,\ 3,\ \ldots,$

$$Z_i = -\frac{Q_{i-1}(\alpha)}{Q_i(\alpha)}$$

where $Q_i(\alpha)$ are polynomials with integer coefficients. Here are the first ten of them.

$Q_0(\alpha)$	1
$Q_1(\alpha)$	$\alpha + Z_{0+}$
$Q_2(\alpha)$	$(\alpha^2 - 1) + \alpha Z_{0+}$
$Q_3(\alpha)$	$(\alpha^3 - 2\alpha) + (\alpha^2 - 1)Z_{0+}$
$Q_4(\alpha)$	$(\alpha^4 - 3\alpha^2 + 1) + (\alpha^3 - 2\alpha)Z_{0+}$
$Q_5(\alpha)$	$(\alpha^5 - 4\alpha^3 + 3\alpha) + (\alpha^4 - 3\alpha^2 + 1)Z_{0+}$
$Q_6(\alpha)$	$(\alpha^6 - 5\alpha^4 + 6\alpha^2 - 1) + (\alpha^5 - 4\alpha^3 + 3\alpha)Z_{0+}$
$Q_7(\alpha)$	$(\alpha^7 - 6\alpha^5 + 10\alpha^3 - 4\alpha) + (\alpha^6 - 5\alpha^4 + 6\alpha^2 - 1)Z_{0+}$
$Q_8(\alpha)$	$(\alpha^8 - 7\alpha^6 + 15\alpha^4 - 10\alpha^2 + 1) + (\alpha^7 - 6\alpha^5 + 10\alpha^3 - 4\alpha)Z_{0+}$
$Q_9(\alpha)$	$(\alpha^9 - 8\alpha^7 + 21\alpha^5 - 20\alpha^3 + 5\alpha) + (\alpha^8 - 7\alpha^6 + 15\alpha^4 - 10\alpha^2 + 1)Z_{0+}$

According to a fundamental property of continued fractions [1], these polynomials satisfy the second order linear recurrence

$$Q_i(\alpha) = \alpha Q_{i-1}(\alpha) - Q_{i-2}(\alpha). \tag{7}$$

[1] Equation (3b) can be obtained by solving equation (2) for Y_{i-1}.

We can further write

$$Q_i(\alpha) = P_i(\alpha) + P_{i-1}(\alpha)Z_{0+}, \tag{8}$$

for a sequence of simpler polynomials $P_i(\alpha)$, each either odd or even. In fact, from (7) and (8), we have

$$P_{i+2}(\alpha) = \alpha P_{i+1}(\alpha) - P_i(\alpha).$$

Explicitly,

$$P_i(\alpha) = \begin{cases} 1, & i = 0, \\ \sum_{k=0}^{\frac{i}{2}} (-1)^{\frac{i}{2}+k} \binom{\frac{i}{2}+k}{2k} \alpha^{2k}, & i = 2, 4, 6, \dots, \\ \sum_{k=1}^{\frac{i+1}{2}} (-1)^{\frac{i+1}{2}+k} \binom{\frac{i-1}{2}+k}{2k-1} \alpha^{2k-1}, & i = 1, 3, 5, \dots. \end{cases}$$

From (6), we have

$$Y_i = a\left(\alpha - 2\frac{Q_{i-1}(\alpha)}{Q_i(\alpha)}\right),$$

for $i = 1,\ 2,\ \dots$.

References

[1] H. Davenport, *Higher Arithmetic*, 6-th edition, Cambridge University Press, 1992.

Giovanni Lucca: Via Corvi 20, 29100 Piacenza, Italy
E-mail address: vanni_lucca@inwind.it