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On Three Circles

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Abstract. The classical Three-Circle Problem of Apollonius requires the construction of a fourth circle tangent to three given circles in the Euclidean plane. For circles in general position this may admit as many as eight solutions or even no solutions at all. Clearly, an "experimental" approach is unlikely to solve the problem, but, surprisingly, it leads to a more general theorem. Here we consider the case of a chain of circles which, starting from an arbitrary point on one of the three given circles defines (uniquely, if one is careful) a tangent circle at this point and a tangency point on another of the given circles. Taking this new point as a base we construct a circle tangent to the second circle at this point and to the third circle, and repeat the construction cyclically. For any choice of the three starting circles, the tangency points are concyclic and the chain can contain at most six circles. The figure reveals unexpected connections with many classical theorems of projective geometry, and it admits the Three-Circle Problem of Apollonius as a particular case.

In the third century B.C., Apollonius of Perga proposed (and presumably solved, though the manuscript is now lost) the problem of constructing a fourth circle tangent to three given circles. A partial solution was found by Jean de la Viète around 1600, but here we shall make use of Gergonne's extremely elegant solution, which covers all cases. The closure theorem presented here is a generalization of this classical problem, and it reveals somewhat surprising connections with theorems of Monge, D'Alembert, Pascal, Brianchon, and Desargues.

Unless the three given circles are tangent at a common point, the Problem of Apollonius may have no solutions at all or it may have as many as eight – a Cartesian formulation would have to take into consideration the coordinates of the three centers as well as the three radii, and even after normalization we would still be left with an eighth degree polynomial. Algebraic and geometrical considerations lead us to consider points as circles with radius zero, and lines as circles with infinite radius. Inversion will, of course, permit us to eliminate lines altogether, however

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A biographical note is included at the end of the paper by the Editor, who also provides the annotations.

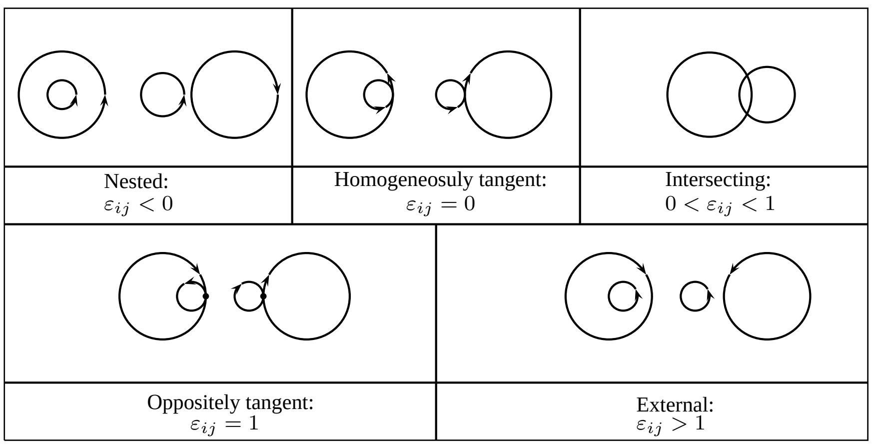
we must take into account the possibility of negative radii[1]. This apparent complication in reality allows us to define general parameters to describe the relationship between pairs of circles:

Notation and Definitions (*Circular Excess*) Let $\mathcal{C}_i = \mathcal{C}_i(x_i, y_i; r_i)$ be the circle with center (x_i, y_i) and radius r_i; define

$$e_i = x_i^2+y_i^2-r_i^2, \quad e_{ij} = (x_i-x_j)^2+(y_i-y_j)^2-(r_i-r_j)^2, \quad \text{and} \quad \epsilon_{ij} = \frac{e_{ij}}{4r_ir_j}.$$

The usefulness of the "excess" quantities e and ϵ will be evident from the following definitions.

Definition. We distinguish five types of relationships between pairs of circles $\mathcal{C}_i$ and $\mathcal{C}_j$ with nonzero radii, as illustrated in the accompanying table.



These descriptions are preserved by inversion – specifically,

Theorem 1. (Inversive Invariants). *The parameter ϵ_{ij} is invariant under inversion in any circle whose center does not lie on either of the two given circles.*

Proof. The circle $\mathcal{C}_0(x_0, y_0; r_0)$ inverts $\mathcal{C}(x, y; r)$ to $\mathcal{C}'(x', y'; r')$, where if d is the Euclidean distance between the centers of $\mathcal{C}$ and $\mathcal{C}_0$, and $I_0 = \frac{r_0^2}{d^2-r^2}$, we find

$$\begin{aligned} x' &= x_0 + I_0(x - x_0), \\ y' &= y_0 + I_0(y - y_0), \\ r' &= rI_0. \end{aligned}$$

[1]There are two common ways to interpret signed radii. They provide an orientation to the circles (as in [6]), so that $r > 0$ would indicate a counterclockwise orientation, $r < 0$ clockwise, and $r = 0$ an unoriented point. In the limit $r = \pm\infty$, and one obtains oriented lines. This seems to be Searby's interpretation. Alternatively, as in [11], one can assume a circle for which $r > 0$ to be a disk (that is, a circle with its interior), while $r < 0$ indicates a circle with its exterior; a line for which $r = \infty$ determines one half plane and $r = -\infty$ the other. This interpretation works especially well in the inversive plane (called the *circle plane* here) which, in the model that fits best with this paper, is the Euclidean plane extended by a single point at infinity that is incident with every line of the plane.

(See [5, p. 79]). Upon applying the formula for ϵ_{ij} to $\mathcal{C}_i'$ and $\mathcal{C}_j'$ then simplifying, we obtain the theorem. □

Theorem 1 permits us to work in a *circle plane* using Cartesian coordinates, the Euclidean definition of circles being extended to admit negative, infinite, and zero radii.

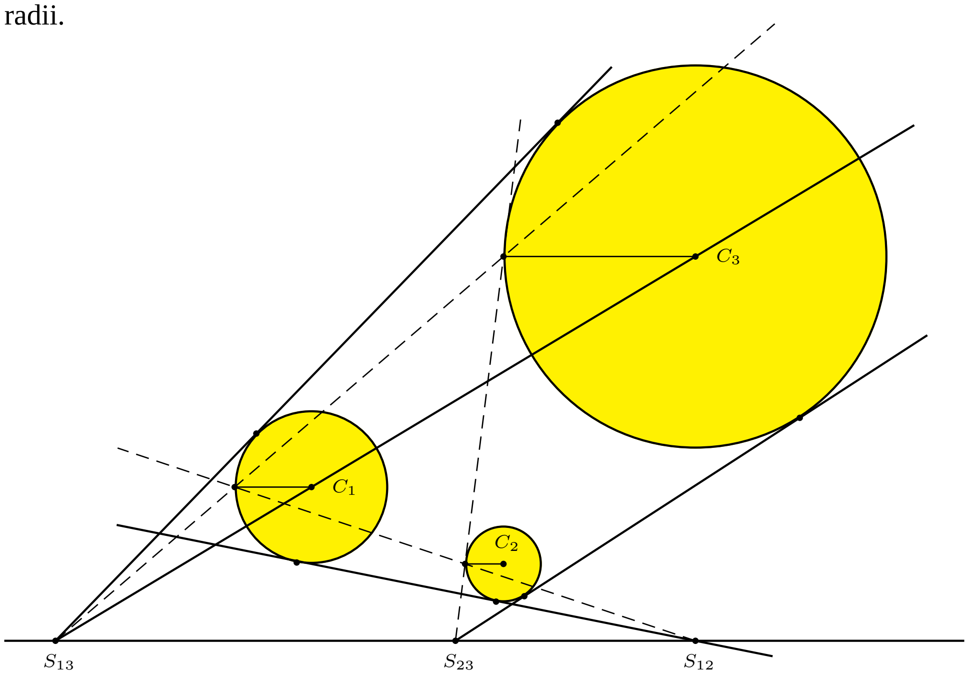


Figure 1. The centers of similitude S_{ij} of three circles lie on the axis of similitude.

Observation (*D'Alembert-Monge*). The centers of similitude S_{ij} of two circles $\mathcal{C}_i$ and $\mathcal{C}_j$ are the points on the line of centers where the common tangents (when they exist) intersect. In Cartesian coordinates we have [7, Art. 114, p.105]

$$S_{ij} = \left(\frac{r_i x_j - r_j x_i}{r_i - r_j}, \frac{r_i y_j - r_j y_i}{r_i - r_j} \right).$$

Note that if the radii are of the same sign these coordinates correspond to the *external* center of similitude; if the signs are opposite the center is *internal*. Moreover, three circles with signed radii generate three collinear points that lie on a line called the *axis of similitude* (or *Monge Line*) σ, whose equation is [7, Art.117, p.107]

$$\sigma = \begin{vmatrix} y_1 & y_2 & y_3 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix} x - \begin{vmatrix} x_1 & x_2 & x_3 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix} y = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ r_1 & r_2 & r_3 \end{vmatrix}.$$

As similar determinants appear frequently, we shall write them as Δ_{abc} if the rows are a_i, b_i, c_i; or simply Δ_{ab} should $c_i = 1$.

Lemma 2 (Second tangency point)**.** *If $P(x_0, y_0)$ is a point on a circle $\mathcal{C}_i$ while $\mathcal{C}_j$ is a second circle, then there exists exactly one circle $\mathcal{C}_a(x_a, y_a; r_a)$ that is homogeneously tangent to $\mathcal{C}_i$ at P and to $\mathcal{C}_j$ at some point $P'(x'_0, y'_0)$. Moreover $\mathcal{C}_a$ has parameters*

$$x_a = x_i + \frac{(x_0 - x_i)e_{ij}}{2f^0_{ij}}, \quad y_a = y_i + \frac{(y_0 - y_i)e_{ij}}{2f^0_{ij}}; \quad r_a = -r_i - \frac{(r_0 - r_i)e_{ij}}{2f^0_{ij}},$$

where

$$r_0 := 0 \quad \textit{and} \quad f^0_{ij} := r_i r_j - (x_0 - x_i)(x_0 - x_j) - (y_0 - y_i)(y_0 - y_j);$$

and the coordinates of P' are

$$x'_0 = x_i + \frac{r_i e_{0j}(x_j - x_i) + r_j e_{ij}(x_0 - x_i)}{r_i e_{0j} + r_j(e_{ij} - e_{0j})},$$
$$y'_0 = y_i + \frac{r_i e_{0j}(y_j - y_i) + r_j e_{ij}(y_0 - y_i)}{r_i e_{0j} + r_j(e_{ij} - e_{0j})},$$

where

$$e_{0j} = (x_0 - x_j)^2 + (y_0 - y_j)^2 - r_j^2$$

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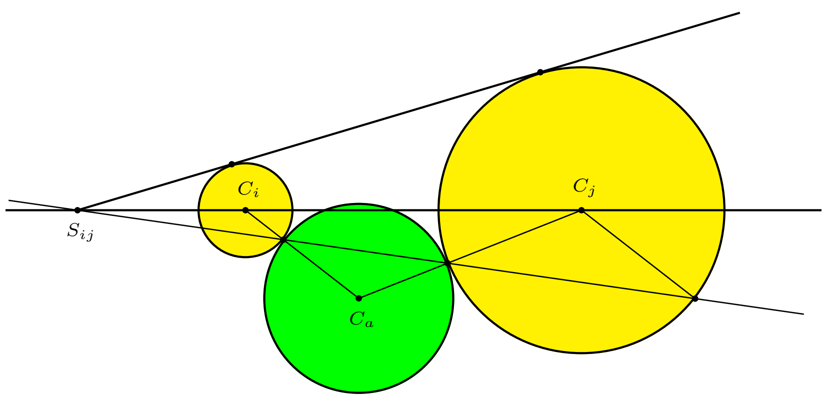

Figure 2. The second tangency point of Lemma 2.

Proof. (Outline)[2] The two tangency points P and P' are collinear with a center of similitude S_{ij}, which will be external or internal according as the radii have the same or different signs [7, Art. 117, p. 108]. It is then sufficient to find the intersections of $S_{ij}P$ with $\mathcal{C}_j$. One of the roots of the resulting quadratic equation

[2]The existence and uniqueness of $\mathcal{C}$ is immediate to anybody familiar with inversive geometry: inversion in a circle with center P sends P to infinity and $\mathcal{C}_i$ to an oriented line; the image of $\mathcal{C}_a$ under that inversion is then the unique parallel oriented line that is homogeneously tangent to the image of $\mathcal{C}_j$. Searby's intent here was to provide explicit parameters, which were especially useful to him for producing accurate figures in the days before the graphics programs that are now common. I, however, drew the figures using *Cinderella*. Searby did all calculations by hand, but they are too lengthy to include here; I confirmed the more involved formulas using *Mathematica*.

represents the point on $\mathcal{C}_j$ whose radius is parallel to that of P on $\mathcal{C}_i$; the other yields the coordinates of P', and the rest follows. □

We are now ready for the main theorem. The first part of the theorem – the closure of the chain of circles – was first proved by Tyrrell and Powell [10], having been conjectured earlier from a drawing.

Theorem 3 (Apollonius Closure)**.** *Let $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$ be three circles in the Circle Plane, and choose a point P_1 on $\mathcal{C}_1$. Define $\mathcal{C}_{12}$ to be the unique circle homogeneously tangent to $\mathcal{C}_1$ at P_1 and to $\mathcal{C}_2$, thus defining $P_2 \in \mathcal{C}_2$. Continue with $\mathcal{C}_{23}$ homogeneously tangent to $\mathcal{C}_2$ at P_2 and to $\mathcal{C}_3$ at P_3, then $\mathcal{C}_{34}$ homogeneously tangent to $\mathcal{C}_3$ at P_3 and to $\mathcal{C}_1$ at P_4, ..., and $\mathcal{C}_{67}$ homogeneously tangent to $\mathcal{C}_3$ at P_6 and to $\mathcal{C}_1$ at P_7. Then this chain closes with $\mathcal{C}_{78} = \mathcal{C}_{12}$ or, more simply, $P_7 = P_1$. Moreover, the points $P_1, \ldots, P_6$ are cyclic* (see Figure 3).

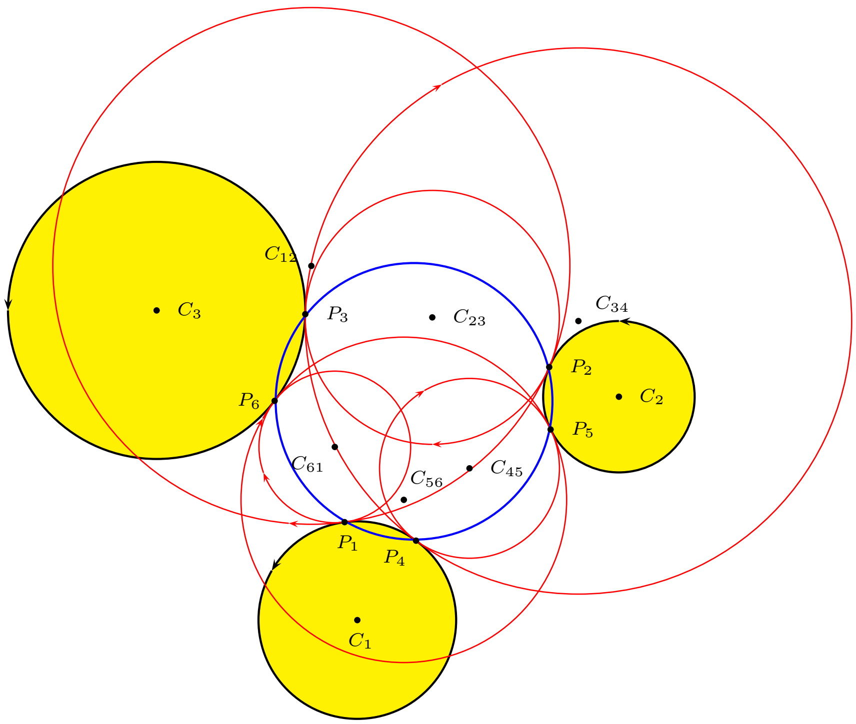

Figure 3. For $i = 1, 2$, and 3 the given circle $\mathcal{C}_i$ (in yellow) is homogeneously tangent at P_i to $\mathcal{C}_{i(i+1)}$ and $\mathcal{C}_{(i+5)i}$, and at P_{i+3} to $\mathcal{C}_{(i+3)(i+4)}$ and $\mathcal{C}_{(i+2)(i+3)}$ (where the subscripts $6 + \ell$ of $\mathcal{C}_{jk}$ are reduced to ℓ).

Proof.[3] We first show that four consecutive P_i's lie on a circle, taking P_1, P_2, P_3, P_4 as a typical example. See Figure 4.

[3]Rigby provides two proofs of this theorem in [6]. Searby independently rediscovered the result around 1987; he showed it to me at that time and I provided yet another proof in [3]. Searby's approach has the virtue of being entirely elementary.

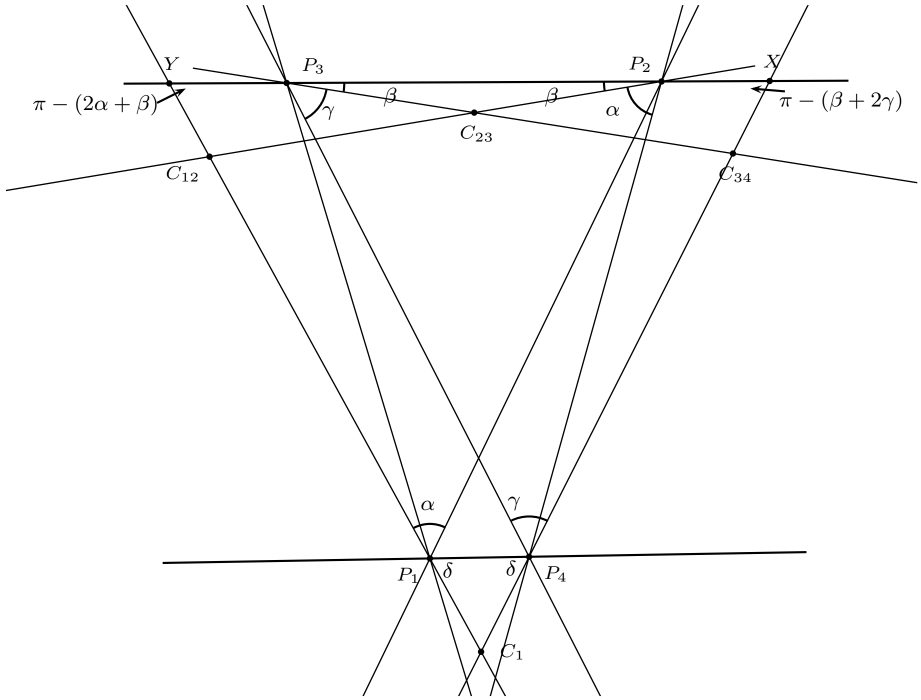

Figure 4. Proof of the Main Theorem 3

Special cases are avoided by using directed angles (so that $\angle ABC$ is the angle between 0 and π through which the line BA must be rotated counterclockwise about B to coincide with BC). Denote by C_i and C_{ij} the centers of the circles $\mathcal{C}_i$ (where $i=1,2,3$) and $\mathcal{C}_{ij}$ (where $1 \leq i < j$). By hypothesis P_i is on the lines joining $C_{(i-1)i}$ to $C_{i(i+1)}$ and C_i to both $C_{i(i+1)}$ and $C_{(i-1)i}$, where we use the convention that $C_{3+k}=C_k$ as shown in Figure 4. In that figure we denote the base angles of the isosceles triangles $\triangle C_{i(i+1)}P_iP_{i+1}$ by α, β, and γ, while δ is the base angle of $\triangle C_1P_4P_1$. Consider $\triangle C_1XY$ formed by the lines $C_1P_4C_{34},\ P_2P_3$, and $C_{12}P_1C_1$. In $\triangle XP_3P_4$, $\angle P_4=\gamma$ and $\angle P_3=\angle P_2P_3C_{23}+\angle C_{23}P_3P_4=\beta+\gamma$, whence $\angle X=\pi-(\beta+2\gamma)$. In $\triangle YP_1P_2$, $\angle P_1=\alpha$ and $\angle P_2=\angle P_1P_2C_{12}+\angle C_{12}P_2P_3 = \alpha+\beta$, whence $\angle Y=\pi-(2\alpha+\beta)$. Consequently, $\angle C_1 = \pi-(\angle X+\angle Y)=2(\alpha+\beta+\gamma)-\pi$. But in $\triangle C_1P_4P_1$, $\angle C_1=\pi-2\delta$; whence, $2(\alpha+\beta+\gamma)-\pi=\pi-2\delta$, or

$$\alpha+\beta+\gamma+\delta=\pi.$$

Because $\angle P_2P_3P_4=\beta+\gamma$ and $\angle P_2P_1P_4=\alpha+\delta$, we conclude that these angles are equal and the points P_1, P_2, P_3, P_4 lie on a circle. By cyclically permuting the indices we deduce that P_5 and P_6 lie on that same circle, which proves the claim in the final statement of the theorem. This new circle already intersects $\mathcal{C}_1$ at P_1 and P_4, so that the sixth circle of the chain, namely the unique circle $\mathcal{C}_{67}$ that is homogeneously tangent to $\mathcal{C}_3$ at P_6 and to $\mathcal{C}_1$, would necessarily be tangent to $\mathcal{C}_1$ at P_1 or P_4. Should the tangency point be P_4, recalling that $\mathcal{C}_{34}$ is the unique circle homogeneously tangent to $\mathcal{C}_1$ at P_4 and to $\mathcal{C}_3$ at P_3, we would necessarily have $P_3 = P_6$. In that case we would necessarily have also $P_1 = P_4$ and $P_2 = P_5$, and the circle $P_1P_2P_3$ would be one of the pair of Apollonius Circles mentioned earlier. In any case, $P_7 = P_1$, and the sixth circle $\mathcal{C}_{67}$ touches $\mathcal{C}_1$ at P_1, closing the chain, as claimed. □

This Euclidean proof is quite general: if any of the circles were straight lines we could simply invert the figure in any appropriate circle to obtain a configuration of ten proper circles. We shall call the circle through the six tangency points the *six-point circle*, and denote it by $\mathcal{S}$. Note the symmetric relationship among the nine circles – any set of three non-tangent circles chosen from the circles of the configuration aside from $\mathcal{S}$ will generate the same figure. Indeed, the names of the circles can be arranged in an array

	P_1	P_5	P_3
P_4	C_1	C_{45}	C_{34}
P_2	C_{12}	C_2	C_{23}
P_6	C_{61}	C_{56}	C_3

so that the circles in any row or column homogeneously touch one another at the point that heads the row or column. Given the configuration of these nine circles without any labels, there are six ways to choose the initial three non-tangent circles. This observation should make clear that the closure of the chain is guaranteed even when the Apollonius Problem has no solution.

Observation (*Apollonius Axis*). The requirement that a circle $\mathcal{C}(x, y; r)$ be tangent to three circles $\mathcal{C}_i(x_i, y_i; r_i)$ yields a system of three quadratic equations which can be simplified to a linear equation in x and y, and which will be satisfied by the coordinates of the centers of two of the solutions of the Apollonius Problem. (The other six solutions are obtained by taking one of the radii to be negative.) We shall call the line through those two centers (whose points satisfy the resulting linear equation) the *Apollonius Axis* and denote it by α; its equation [7, Art. 118, pp.108-110] is

$$\alpha : x\Delta_{xr} + y\Delta_{yr} = \frac{\Delta_{er}}{2}.$$

Note that the two lines σ and α are perpendicular; they are defined even when the corresponding Apollonius Circles fail to exist (or, more precisely, are not real).

Observation (*Radical Center*). The locus of all points having the same power (that is, the square of the distance from the center minus the square of the radius) with respect to two circles is a straight line, the *radical axis* [7, Art. 106, 107, pp.98–99]:

$$\rho_{ij} : 2(x_i - x_j)x + 2(y_i - y_j)y = e_i - e_j.$$

The axes determined by three circles are concurrent at their ***radical center***

$$C_R = \left(\frac{\Delta_{ey}}{2\Delta_{xy}}, \frac{\Delta_{xe}}{2\Delta_{xy}}\right).$$

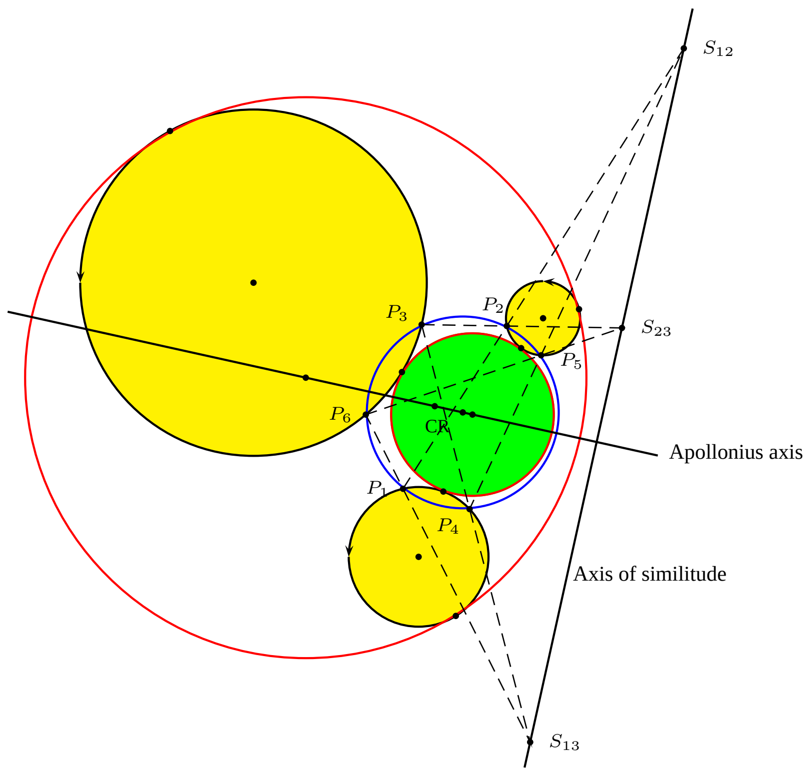

Figure 5. The Apollonius Axis of three oriented circles contains the centers of the two Apollonius circles, the radical center $\mathcal{C}_R$, and the centers S of all six-point circles. It is perpendicular to the axis of similitude.

This point is also known as the *Monge Point*, as it is the center of the circle, called the *Monge Circle*, that is orthogonal to all three given circles whenever such a circle exists. By substitution one sees that C_R lies on α. In summary,

Theorem 4 (Monge Circle)**.** *The Apollonius Axis α of three given circles is the line through their radical center C_R that is perpendicular to the axis of similitude σ; furthermore the Monge circle, if it exists, is a six-point circle that inverts the nine-circle configuration of Theorem 3 into itself.*

Theorem 5 (Centers of Six-Point Circles)**.** *For any three given non-tangent circles, as P_1 moves around $\mathcal{C}_1$ the locus of the center S of the corresponding six-point circle is either the entire Apollonius Axis α, the segment of α between the centers of the two Apollonius Circles (homogeneously tangent to all three of the given circles), or that segment's complement in α.*

Proof. Let $P_1 = (x_0, y_0)$. We saw (while finding the second tangency point) that the line P_1P_2 coincides with $S_{12}P_1$, which (by the formula for S_{12}) has gradient

$$\frac{r_1(y_0 - y_2) - r_2(y_0 - y_1)}{r_1(x_0 - x_2) - r_2(x_0 - x_1)};$$

because the perpendicular bisector β of P_1P_2 passes through C_{12}, its equation must therefore be

$$\beta: \ \frac{y - y_1 - \frac{(y_0-y_1)e_{12}}{2f_{12}^0}}{x - x_1 - \frac{(x_0-x_1)e_{12}}{2f_{12}^0}} = -\frac{r_1(x_0-x_2) - r_2(x_0-x_1)}{r_1(y_0-y_2) - r_2(y_0-y_1)}.$$

We can then use Cramer's rule to find the point where β intersects the Apollonius Axis α, which entails the arduous but rewarding calculation of the denominator,

$$(r_2 - r_1)\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ r_0 & r_1 & r_2 & r_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

It requires only a little more effort to find the coordinates of the desired intersection point, which we claim to be S, namely

$$S = \left(\frac{1}{2}\frac{\begin{vmatrix} e_0 & e_1 & e_2 & e_3 \\ y_0 & y_1 & y_2 & y_3 \\ 0 & r_1 & r_2 & r_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ 0 & r_1 & r_2 & r_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}}, \ -\frac{1}{2}\frac{\begin{vmatrix} e_0 & e_1 & e_2 & e_3 \\ x_0 & x_1 & x_2 & x_3 \\ 0 & r_1 & r_2 & r_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ 0 & r_1 & r_2 & r_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}} \right), \tag{1}$$

where we have used e_0 to stand for $x_0^2 + y_0^2$ (with $r_0 = 0$). Of course, the same calculation could be applied to $\mathcal{C}_{61}$, and we would obtain the same point (1). In other words, the perpendicular bisectors of the chords of $\mathcal{S}$ formed by the tangency points of $\mathcal{C}_{12}$ with $\mathcal{C}_1$ and $\mathcal{C}_2$, and of $\mathcal{C}_{61}$ with $\mathcal{C}_1$ and $\mathcal{C}_3$, must intersect at S, which is necessarily the center of $\mathcal{S}$. As a byproduct of the way its coordinates were calculated, we must have S on α, as claimed. Finally, the Main Theorem guarantees the existence of S, and (1) shows that its coordinates are continuous functions of x_0 and y_0. Since a solution circle to the Problem of Apollonius is obviously a (degenerate) six-point circle, the second part of the theorem is also proved. □

Setting $S = (s_1, s_2)$ and rewriting the first coordinate of (1) as

$$\begin{vmatrix} e_0 & e_1 & e_2 & e_3 \\ y_0 & y_1 & y_2 & y_3 \\ 0 & r_1 & r_2 & r_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 2s_1 \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ 0 & r_1 & r_2 & r_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}, \tag{2}$$

we readily see that this is an equation of the form

$$x_0^2 - 2s_1x_0 + y_0^2 - 2s_2y_0 + \text{ (terms involving neither } x_0 \text{ nor } y_0\text{) } = 0;$$

the only step that cannot be done in one's head is checking that the coefficient of y_0 necessarily equals the second coordinate of (1). In particular, we see that the point (x_0, y_0) satisfies the equation of a circle with center $S = (s_1, s_2)$. But, the unique circle with center S that passes through (x_0, y_0) is our six-point circle $\mathcal{S}$. Because both equation (2) and the corresponding equation using the second coordinate of (1) hold for any point P_1 in the plane, even if P_1 does not lie on $\mathcal{C}_1$, we see that $\mathcal{S}$ is part of a larger family of circles that cover the plane. We therefore deduce that

Theorem 6 (Six-point Pencil)**.** *The equations* (2) *represent the complete set of six-point circles, which is part of a pencil of circles whose radical axis is σ. When the pencil consists of intersecting circles, σ might itself be a six-point circle.*[4]

Proof. S_{12} has the same power[5], namely $\frac{e_{12}r_1r_2}{r_1^2-r_2^2}$, with respect to all circles tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$; but, for any point $P_1 \in \mathcal{C}_1$, the quantity $S_{12}P_1 \times S_{12}P_2$ is also the power of S_{12} with respect to the six-point circle determined by P_1. Since similar claims hold for S_{23} and S_{31}, it follows that σ (the line containing the centers of similitude) is the required radical axis. The rest follows quickly from the definitions. □

Since the tangency points P_1 and P_2 of $\mathcal{C}_{12}$ with $\mathcal{C}_1$ and $\mathcal{C}_2$ are collinear with S_{12}, and similarly for the other pairs, we see immediately that (as in Figure 6)

Theorem 7 (Pascal)**.** *The points where the six-point circle $\mathcal{S}$ meets the given circles form a Pascal hexagon $P_1P_2P_3P_4P_5P_6$ whose axis is the axis of similitude σ.*

Again, the pair of Apollonius circles deriving from Gergonne's construction and (if they are real) delimiting the pencil of Theorem 6 are special positions of the $\mathcal{C}_{ij}$, whence (as in Figure 7)

Theorem 8 (Gergonne-Desargues)**.** *For any given triple of circles, the six tangency points of a pair of Apollonius Circles, the three centers of similitude S_{ij}, and the radical center C_R are ten points of a Desargues Configuration.*

Proof. We should mention for completeness that by Gergonne's construction[6], the poles (x_i', y_i') of σ with respect to $\mathcal{C}_i$ are collinear with the radical center C_R and the tangency points of the two Apollonius Circles with $\mathcal{C}_i$. For those who prefer the use of coordinates,

$$x_i' = x_i + r_i \begin{vmatrix} y_1 & y_2 & y_3 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix}, \qquad y_i' = y_i + r_i \begin{vmatrix} x_1 & x_2 & x_3 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix},$$

[4] One easily sees that each six-point circle cuts the three given circles at equal angles. Salmon [7, Art. 118] derives the same conclusion as our Theorem 6 while determining the locus of the center of a circle cutting three given circles at equal angles.

[5] I wonder if Searby used the definition of *power* that he gave earlier (in the form $d^2 - r^2$), which seems quite awkward for the calculations needed here. The claim about the constant power of S_{12} is clear, however, without such a calculation: the circle, or circles, of inversion that interchange $\mathcal{C}_1$ with $\mathcal{C}_2$, called the *mid-circles* in [2, Sections 5.7 and 5.8] (see, especially, Exercise 5.8.1 on p.126), is the locus of points P such that two circles, tangent to both $\mathcal{C}_1$ and $\mathcal{C}_2$, are tangent to each other at P. The center of this mid-circle is S_{12}, and the square of its radius $S_{12}P$ is the power of S_{12} with respect to any of these common tangent circles.

[6] Details concerning Gergonne's construction can be found in many of the references that deal with the Problem of Apollonius such as [1, Section 10.11.1, p.318], [4, Section 1.10, pp.22–23], or [7, Art. 119 to 121, pp.110–113].

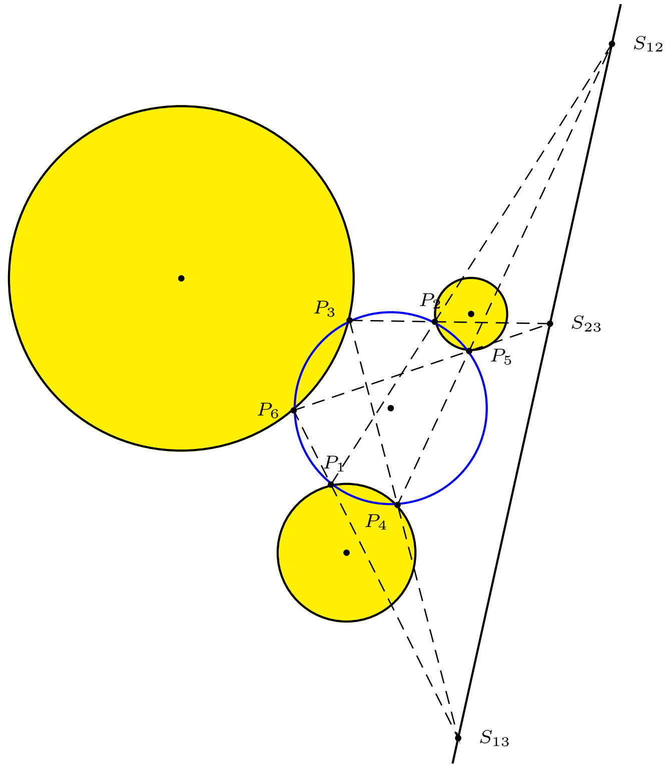

Figure 6. The points where the six-point circle $\mathcal{S}$ (in blue) meets the given circles (in yellow) form a Pascal hexagon whose axis is the axis of similitude

and the equation of the line joining C_R to the points where $\mathcal{C}_i$ is tangent to the Apollonius Circles is

$$(x - x_i)\begin{vmatrix} e_{1i} & e_{2i} & e_{3i} \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} + (y - y_i)\begin{vmatrix} e_{1i} & e_{2i} & e_{3i} \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} + r_i \begin{vmatrix} e_{1i} & e_{2i} & e_{3i} \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Gergonne's construction yields the tangency points in three pairs collinear with C_R, which is, consequently, the center of perspectivity of the triangles inscribed in the Apollonius Circles. The axis is clearly σ because, as with the circles $\mathcal{C}_{ij}$, an Apollonius circle is tangent to the given circles $\mathcal{C}_i$ and $\mathcal{C}_j$ at points whose joining line passes through S_{ij}. □

Finally, on inverting the intersection point of σ and α in $\mathcal{S}$ and tracing the six tangent lines to $\mathcal{S}$ at the points P_i where it meets the given circles, after much routine algebra (which we leave to the reader)[7] we obtain

Theorem 9 (Brianchon)**.** *The inverse image of $\sigma \cap \alpha$ in $\mathcal{S}$ is the Brianchon Point of the hexagon circumscribing $\mathcal{S}$ and tangent to it at the six points where it intersects the given circles $\mathcal{C}_i$, taken in the order indicated by the labels.*

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[7]There is no need for any calculation here: Theorem 9 is the projective dual of Theorem 7 – the polarity defined by $\mathcal{S}$ takes each point P_i to the line tangent there to $\mathcal{S}$, while (because $\sigma \perp \alpha$) it interchanges the axis of similitude σ with the inverse image of $\sigma \cap \alpha$ in $\mathcal{S}$.

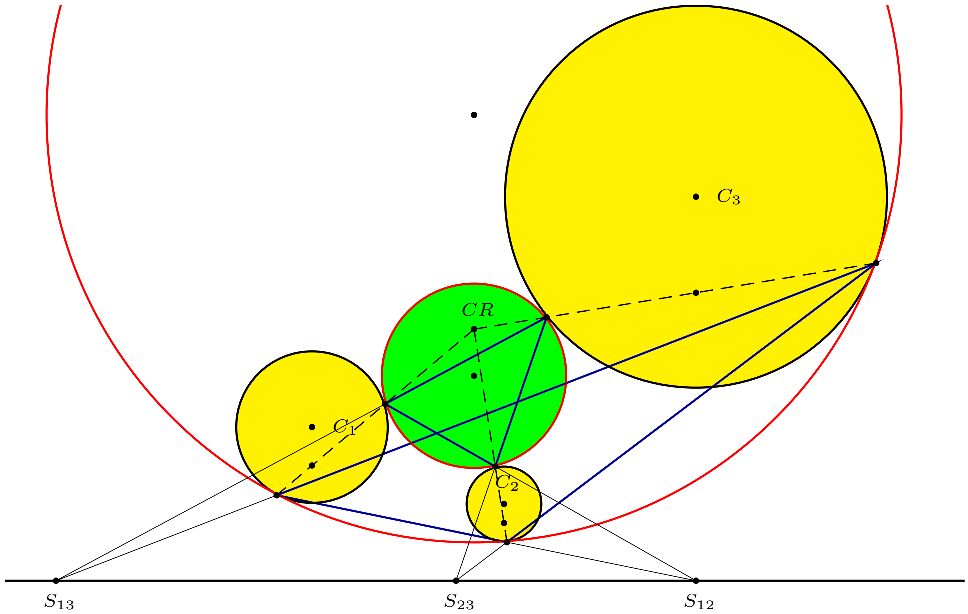

Figure 7. The triangles (shown in blue) whose vertices are the points where the Apollonius circles (red) are tangent to the three given oriented circles (yellow) are perspective from the radical center $\mathcal{C}_R$; the axis of the perspectivity is the axis of similitude of the given circles.

Conclusion. Uniting as it does the classical theorems of Monge, D'Alembert, Desargues, Pascal, and Brianchon together with the problem of Apollonius, we feel that this figure merits to be better known. The ubiquitous and extremely useful e and ϵ symbols take their name from the Einstein-Minkowski metric: in fact, the circle plane (or its three-dimensional analogue) is a vector space which, on substitution of the last coordinate (that is, the radius) by the imaginary distance cti (where $i^2 = -1$) yields interesting analogies with relativity theory. [8]

References

[1] M. Berger, *Geometry I*, Springer-Verlag, 1987.
[2] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Mathematical Association of America, New Mathematical Library No. 19, 1967.
[3] J. Ch. Fisher, Models and theorems of the classical circle planes, *Abh. Math. Sem. Univ. Hamburg*, 63 (1993) 245–264.
[4] D. Pedoe, *Circles*, Dover, 1979.
[5] D. Pedoe, *Geometry: A Comprehensive Course*, Dover,1988.
[6] J. F. Rigby, The geometry of cycles, and generalized Laguerre inversion, in *The Geometric Vein: The Coxeter Festschrift*, ed. by Chandler Davis et al., Springer-Verlag (1982), 355–378.
[7] G. Salmon, *A Treatise on Conic Sections*, 6th ed. Chelsea, 1954.
[8] D. G. Searby and J. W. Wamsley, Minimal presentations for certain metabelian groups. *Proc. Amer. Math. Soc.*, 32 (1972) 342–348.

[8]See, for example, [11, Sections 11 and 15] where there is a discussion of the Lorentz group and further references can be found.

[9] D. G. Searby, A note on the existence of the Pappus configuration in planes of order nine. *Boll. Un. Mat. Ital.*, (4) 10 (1974) 529–531.

[10] J.A. Tyrrel and M. T. Powell, A theorem in circle geometry, *Bull. London Math. Soc.*, 3 (1971) 70–74.

[11] J.B. Wilker, Inversive Geometry, in *The Geometric Vein: The Coxeter Festschrift*, ed. by Chandler Davis et al., Springer-Verlag (1982), 379–442.

David Graham Searby (1948 - 1998): David Searby was born on October 30, 1948 in Melbourne, Australia, and lived in both Melbourne and Adelaide while growing up. He graduated from Flinders University of South Australia with a Bachelor of Science degree (1970) and a Bachelor of Science Honours (1972). He began graduate work there under the supervision of John Wamsley, with whom he published his first paper [8]. He moved to Italy in 1974, and for three years was supported by a foreign-student scholarship at the University of Bologna, funded by the National Research Council (CNR) of Italy. He lived in Bologna for the remainder of his life; although he maintained a tenuous relationship with the university, he survived by giving private lessons in English and in mathematics. He was an effective and inspiring teacher (although he taught relatively little) and scholar – in addition to his mathematics, he knew every corner of Italy, its history and literature; he spoke flawless Italian and even mastered the local Bolognese dialect. He was driven by boundless curiosity and intellectual excitement, and loved to spend long evenings in local bars, sustained by soup, beer and ideas. His lifestyle, however, was unsustainable, and he died on August 19, 1998, just short of his 50th birthday. He left behind a box full of his notes and computations in no apparent order. One of his research interests concerned configurations in projective planes, both classical and finite. An early paper [9] on the existence of Pappus configurations in planes of order nine indicated the direction his research was to take: he found that in the Hughes Plane of order nine there exist triangles which fail to contain a Pappus configuration that has three points on each of its sides. Were coordinates introduced using such a triangle as the triangle of reference, the imposed algebraic structure would be nearly trivial. From this Searby speculated on the existence of finite projective planes whose order is not the power of a prime, and whose coordinates have so little structure that the plane could be discovered only by computer. He collected configuration theorems throughout most of his life with a goal toward finding configurations on which he could base a computer search. Unfortunately, he never had access to a suitable computer. Among his papers was the first draft of a monograph (in Italian) that brought together many of his elementary discoveries on configurations; it is highly readable, but a long way from being publishable. There was also the present paper, almost ready to submit for publication, which brings together several of his discoveries involving configuration theorems. It has been lightly edited by me; I added the footnotes, references, and figures. I would like to thank David's brother Michael and his friends David Glynn, Ann Powell, and David Tiley for their help in preparing the biographical note. (J. Chris Fisher)