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On the Construction of Regular Polygons and Generalized Napoleon Vertices

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Abstract. An algebraic foundation for the derivation of geometric construction schemes transforming arbitrary polygons with n vertices into k-regular n-gons is given. It is based on circulant polygon transformations and the associated eigenpolygon decompositions leading to the definition of generalized Napoleon vertices. Geometric construction schemes are derived exemplarily for different choices of n and k.

1. Introduction

Because of its geometric appeal, there is a long, ongoing tradition in discovering geometric constructions of regular polygons, not only in a direct way, but also by transforming a given polygon with the same number of vertices [2, 6, 9, 10]. In the case of the latter, well known results are, for example, Napoleon's theorem constructing an equilateral triangle by taking the centroids of equilateral triangles erected on each side of an arbitrary initial triangle [5], or the results of Petr, Douglas, and Neumann constructing k-regular n-gons by $n-2$ iteratively applied transformation steps based on taking the apices of similar triangles [8, 3, 7]. Results like these have been obtained, for example, by geometric creativity, target-oriented constructions or by analyzing specific configurations using harmonic analysis.

In this paper the authors give an algebraic foundation which can be used in order to systematically derive geometric construction schemes for k-regular n-gons. Such a scheme is hinted in Figure 1 depicting the construction of a 1-regular pentagon (left) and a 2-regular pentagon (right) starting from the same initial polygon marked yellow. New vertex positions are obtained by adding scaled parallels and perpendiculars of polygon sides and diagonals. This is indicated by intermediate construction vertices whereas auxiliary construction lines have been omitted for the sake of clarity.

The algebraic foundation is derived by analyzing circulant polygon transformations and the associated Fourier basis leading to the definition of eigenpolygons. By choosing the associated eigenvalues with respect to the desired symmetric configuration and determining the related circulant matrix, this leads to an algebraic representation of the transformed vertices with respect to the initial vertices and the eigenvalues. Interpreting this algebraic representation geometrically yields the desired construction scheme.

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The authors would like to thank Bernd Scholz from TWT GmbH, Engineering Department, for pointing out a modified triangle transformation based on the results of [11] which attracted our interest on deriving geometric construction schemes from circulant polygon transformations.

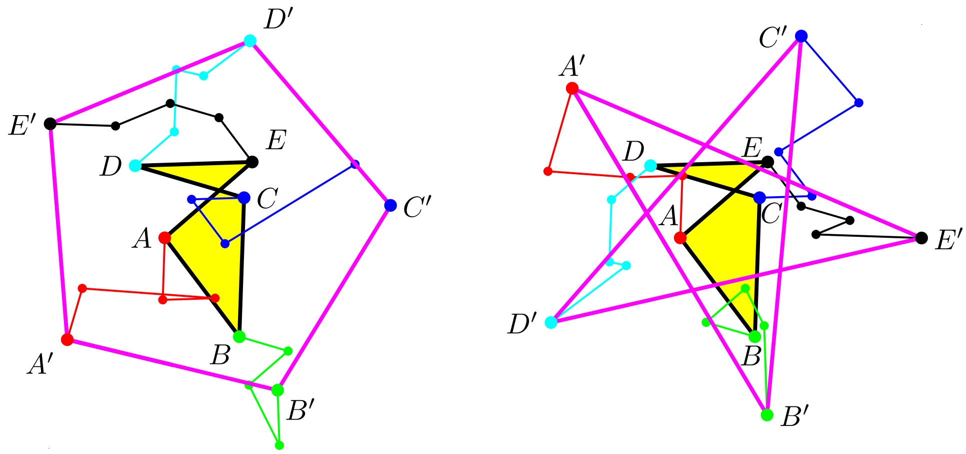

Figure 1. Construction of regular pentagons.

A special choice of parameters leads to a definition of generalized Napoleon vertices, which coincide with the vertices given by Napoleon's theorem in the case of $n = 3$. Geometric construction schemes based on such representations are derived for triangles, quadrilaterals, and pentagons.

2. Eigenpolygon decompositions

Let $z \in \mathbb{C}^n$ denote a polygon with n vertices z_k, $k \in \{0, \dots, n-1\}$, in the complex plane using zero-based indexes. In order to obtain geometric constructions leading to regular polygons, linear transformations represented by complex circulant matrices $M \in \mathbb{C}^{n \times n}$ will be analyzed. That is, each row of M results from a cyclic shift of its preceding row, which reflects that new vertex positions are constructed in a similar fashion for all vertices.

The eigenvectors $f_k \in \mathbb{C}^n$, $k \in \{0, \dots, n-1\}$, of circulant matrices are given by the columns of the Fourier matrix

$$F := \frac{1}{\sqrt{n}} \begin{pmatrix} r^{0 \cdot 0} & \dots & r^{0 \cdot (n-1)} \\ \vdots & \ddots & \vdots \\ r^{(n-1) \cdot 0} & \dots & r^{(n-1) \cdot (n-1)} \end{pmatrix},$$

where $r := \exp(2\pi \mathrm{i}/n)$ denotes the n-th complex root of unity [1]. Hence, the eigenvector $f_k = (1/\sqrt{n})(r^{0 \cdot k}, r^{1 \cdot k}, \dots, r^{(n-1) \cdot k})^{\mathrm{t}}$ represents the k-th Fourier polygon obtained by successively connecting counterclockwise n times each k-th scaled root of unity starting by $r^0/\sqrt{n} = 1/\sqrt{n}$. This implies that f_k is a $\big(n/\gcd(n,k)\big)$-gon with vertex multiplicity $\gcd(n,k)$, where $\gcd(n,k)$ denotes the greatest common divisor of the two natural numbers n and k. In particular, f_0 degenerates to one vertex with multiplicity n, and f_1 as well as f_{n-1} are convex regular n-gons with opposite orientation. Due to its geometric configuration f_k is called k-*regular*, which will also be used in the case of similar polygons.

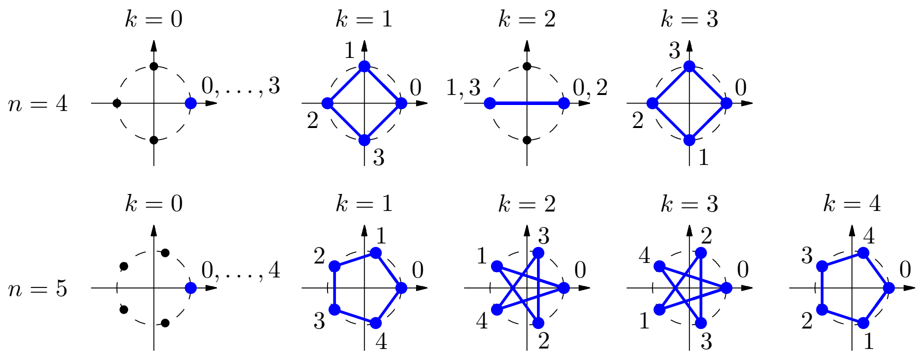

Figure 2. Fourier polygons f_k for $n \in \{4, 5\}$ and $k \in \{0, \ldots, n-1\}$.

Examples of Fourier polygons are depicted in Figure 2. In this, black markers indicate the scaled roots of unity lying on a circle with radius $1/\sqrt{n}$, whereas blue markers denote the vertices of the associated Fourier polygons. Also given is the vertex index or, in the case of multiple vertices, a comma separated list of indexes. If n is a prime number, all Fourier polygons except for $k = 0$ are regular n-gons as is shown in the case of $n = 5$. Otherwise reduced Fourier polygons occur as is depicted for $n = 4$ and $k = 2$.

Since F is a unitary matrix, the diagonalization of M based on the eigenvalues $\eta_k \in \mathbb{C}$, $k \in \{0, \ldots, n-1\}$, and the associated diagonal matrix $D = \operatorname{diag}(\eta_0, \ldots, \eta_{n-1})$ is given by $M = FDF^*$, where F^* denotes the conjugate transpose of F. The coefficients c_k in the representation of $z = \sum_{k=0}^{n-1} c_k f_k$ in terms of the Fourier basis are the entries of the vector $c = F^* z$ and lead to the following definition.

Definition. The k-th *eigenpolygon* of a polygon $z \in \mathbb{C}^n$ is given by

$$e_k := c_k f_k = \frac{c_k}{\sqrt{n}} \left(r^{0 \cdot k}, r^{1 \cdot k}, \ldots, r^{(n-1) \cdot k} \right)^{\mathrm{t}}, \tag{1}$$

where $c_k := (F^* z)_k$ and $k \in \{0, \ldots, n-1\}$.

Since e_k is f_k times a complex coefficient c_k representing a scaling and rotation depending on z, the symmetric properties of the Fourier polygons f_k are preserved. In particular, the coefficient $c_0 = (F^* z)_0 = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} z_k$ implies that $e_0 = \frac{1}{n} \left(\sum_{k=0}^{n-1} z_k \right) (1, \ldots, 1)^{\mathrm{t}}$ is n times the centroid of the initial polygon. This is also depicted in Figure 3 showing the eigenpolygon decomposition of two random polygons. In order to clarify the rotation and orientation of the eigenpolygons, the first three vertices are colored red, green, and blue.

Due to the representation of the transformed polygon

$$z' := Mz = M \left(\sum_{k=0}^{n-1} e_k \right) = \sum_{k=0}^{n-1} M e_k = \sum_{k=0}^{n-1} \eta_k e_k \tag{2}$$

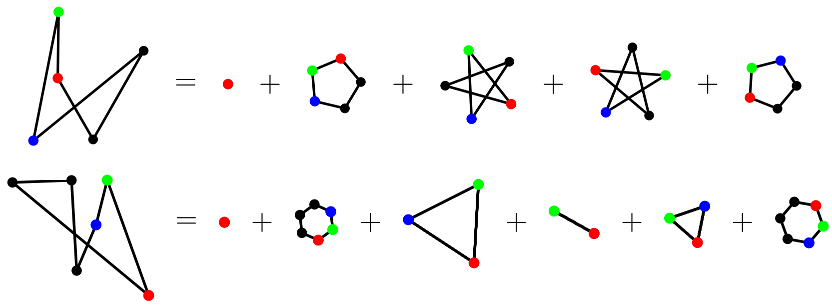

Figure 3. Eigenpolygon decomposition of a 5- and 6-gon.

applying the transformation M scales each eigenpolygon according to the associated eigenvalue $\eta_k \in \mathbb{C}$ of M. This is utilized by geometric construction schemes leading to scaled eigenpolygons. One is the Petr-Douglas-Neumann theorem [8, 3, 7] which is based on $n-2$ polygon transformations each consisting of taking the apices of similar isosceles triangles erected on the sides of the polygon. In each step a different apex angle taken from the set $\{k2\pi/n \,|\, k = 1, \ldots, n-1\}$ is used. The characteristic angles are chosen in such a way that an eigenvalue in the decomposition (2) becomes zero in each case. Since all transformation steps preserve the centroid, $n-2$ steps successively eliminate the associated eigenpolygons until one scaled eigenpolygon with preserved centroid remains. In the case of $n = 3$ this leads to the familiar Napoleon's theorem [5] in which one transformation step suffices to obtain a regular triangle.

3. Construction of regular polygons

The eigenpolygon decomposition presented in the previous section can be used to prove that specific geometric transformations result in regular polygons. Beyond that, it can also be used to find new geometric construction schemes leading to predefined symmetric configurations. This is done by an appropriate choice of the eigenvalues η_k and by interpreting the resulting transformation matrix $M = FDF^*$ geometrically.

3.1. *General case.* In this subsection, a specific choice of eigenvalues will be analyzed in order to derive transformations, which lead to k-regular polygons and additionally preserve the centroid. The latter implies $\eta_0 = 1$ since e_0 already represents the centroid. By choosing $\eta_j = 0$ for all $j \in \{1, \ldots, n-1\} \setminus \{k\}$ and $\eta_k \in \mathbb{C} \setminus \{0\}$, the transformation eliminates all eigenpolygons except the centroid e_0 and the designated eigenpolygon e_k which is scaled by the absolute value of η_k and rotated by the argument of η_k. This implies

$$\begin{aligned} M &= F \operatorname{diag}(1, 0, \ldots, 0, \eta_k, 0, \ldots, 0)\, F^* \\ &= F \operatorname{diag}(1, 0, \ldots, 0)\, F^* + \eta_k F \operatorname{diag}(0, \ldots, 0, 1, 0, \ldots, 0)\, F^* . \end{aligned} \tag{3}$$

Hence, M is a linear combination of matrices of the type $E_k := F I_k F^*$, where I_k denotes a matrix with the only nonzero entry $(I_k)_{k,k} = 1$. Taking into account that $(F)_{\mu,\nu} = r^{\mu\nu}/\sqrt{n}$ and $(F^*)_{\mu,\nu} = r^{-\mu\nu}/\sqrt{n}$, the matrix $I_k F^*$ has nonzero elements only in its k-th row, where $(I_k F^*)_{k,\nu} = r^{-k\nu}/\sqrt{n}$. Therefore, the ν-th column of $E_k = F I_k F^*$ consists of the k-th column of F scaled by $r^{-k\nu}/\sqrt{n}$, thus resulting in $(E_k)_{\mu,\nu} = (F)_{\mu,k} r^{-k\nu}/\sqrt{n} = r^{\mu k} r^{-k\nu}/n = r^{k(\mu-\nu)}/n$. This yields the representation

$$(M)_{\mu,\nu} = (E_0)_{\mu,\nu} + \eta_k (E_k)_{\mu,\nu} = \frac{1}{n}\left(1 + \eta_k r^{k(\mu-\nu)}\right) ,$$

since all entries of E_0 equal $1/n$. Hence, transforming an arbitrary polygon $z = (z_0, \ldots, z_{n-1})^{\mathrm{t}}$ results in the polygon $z' = Mz$ with vertices

$$z'_\mu = (Mz)_\mu = \sum_{\nu=0}^{n-1} \frac{1}{n}\left(1 + \eta_k r^{k(\mu-\nu)}\right) z_\nu ,$$

where $\mu \in \{0, \ldots, n-1\}$. In the case of $\mu = \nu$ the weight of the associated summand is given by $\omega := (1+\eta_k)/n$. Substituting this expression in the representation of z'_μ using $\eta_k = n\omega - 1$, hence $\omega \neq 1/n$, yields the decomposition

$$\begin{aligned} z'_\mu = & \sum_{\nu=0}^{n-1} \frac{1}{n}\left(1 + (n\omega - 1) r^{k(\mu-\nu)}\right) z_\nu \\ = & \underbrace{\frac{1}{n}\sum_{\nu=0}^{n-1}\left(1 - r^{k(\mu-\nu)}\right) z_\nu}_{=:u_\mu} + \omega \underbrace{\sum_{\nu=0}^{n-1} r^{k(\mu-\nu)} z_\nu}_{=:v_\mu} = u_\mu + \omega v_\mu \end{aligned} \tag{4}$$

of z'_μ into a geometric location u_μ not depending on ω, and a complex number v_μ, which can be interpreted as vector scaled by the parameter ω. It should also be noticed that due to the substitution u_μ does not depend on z_μ, since the associated coefficient becomes zero.

A particular choice is $\omega = 0$, which leads to $z'_\mu = u_\mu$. As will be seen in the next section, in the case of $n = 3$ this results in the configuration given by Napoleon's theorem, hence motivating the following definition.

Definition. For $n \geq 3$ let $z = (z_0, \ldots, z_{n-1})^{\mathrm{t}} \in \mathbb{C}^n$ denote an arbitrary polygon, and $k \in \{1, \ldots, n-1\}$. The vertices

$$u_\mu := \frac{1}{n}\sum_{\nu=0}^{n-1}\left(1 - r^{k(\mu-\nu)}\right) z_\nu , \qquad \mu \in \{0, \ldots, n-1\} ,$$

defining a k-regular n-gon are called *generalized Napoleon vertices*.

According to its construction, M acts like a filter on the polygon z removing all except the eigenpolygons e_0 and e_k. The transformation additionally weightens e_k by the eigenvalue $\eta_k \neq 0$. As a consequence, if e_k is not contained in the eigenpolygon decomposition of z, the resulting polygon $z' = Mz$ degenerates to the centroid e_0 of z.

The next step consists of giving a geometric interpretation of the algebraically derived entities u_0 and v_0 for specific choices of n, k, and ω resulting in geometric construction schemes to transform an arbitrary polygon into a k-regular polygon. Examples will be given in the next subsections.

3.2. *Transformation of triangles.* The general results obtained in the previous subsection will now be substantiated for the choice $n = 3$, $k = 1$. That is, a geometric construction is to be found, which transforms an arbitrary triangle into a counterclockwise oriented equilateral triangle with the same centroid. Due to the circulant structure, it suffices to derive a construction scheme for the first vertex of the polygon, which can be applied in a similar fashion to all other vertices.

In the case of $n = 3$ the root of unity is given by $r = \exp(\frac{2}{3}\pi\mathrm{i}) = \frac{1}{2}(-1+\mathrm{i}\sqrt{3})$. By using (4) in the case of $\mu = 0$, as well as the relations $r^{-1} = r^2 = \overline{r}$ and $r^{-2} = r$, this implies

$$\begin{aligned} u_0 &= \frac{1}{3}\sum_{\nu=0}^{2}(1-r^{-\nu})z_\nu = \frac{1}{3}\left[\left(\frac{3}{2}+\mathrm{i}\,\frac{\sqrt{3}}{2}\right)z_1 + \left(\frac{3}{2}-\mathrm{i}\,\frac{\sqrt{3}}{2}\right)z_2\right] \\ &= \frac{1}{2}(z_1+z_2) - \mathrm{i}\,\frac{1}{3}\,\frac{\sqrt{3}}{2}(z_2-z_1) \end{aligned}$$

and

$$\begin{aligned} v_0 &= \sum_{\nu=0}^{2} r^{-\nu}z_\nu = z_0 + \left(-\frac{1}{2}-\mathrm{i}\,\frac{\sqrt{3}}{2}\right)z_1 + \left(-\frac{1}{2}+\mathrm{i}\,\frac{\sqrt{3}}{2}\right)z_2 \\ &= z_0 - \frac{1}{2}(z_1+z_2) + \mathrm{i}\,\frac{\sqrt{3}}{2}(z_2-z_1)\,. \end{aligned}$$

Thereby, the representations of u_0 and v_0 have been rearranged in order to give geometric interpretations as depicted in Figure 4.

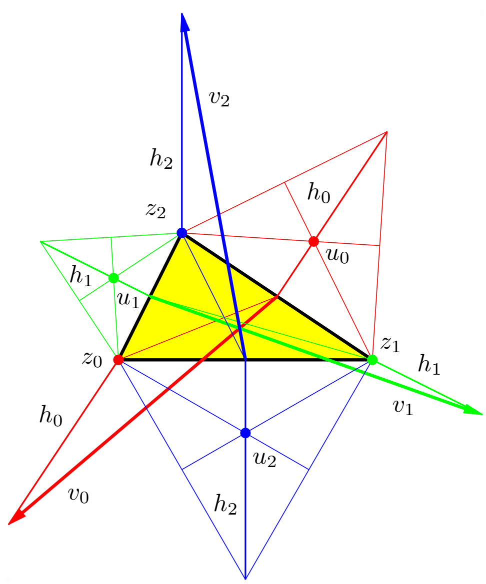

Figure 4. Napoleon vertices u_μ and directions v_μ in the case $n = 3$, $k = 1$.

Since multiplication by $-\mathrm{i}$ denotes a clockwise rotation by $\pi/2$, u_0 represents the centroid of a properly oriented equilateral triangle erected on the side z_1z_2. In a vectorial sense, v_0 represents the vector from the midpoint of the side z_1z_2 to z_0 added by the opposite directed height h_0 of the equilateral triangle erected on z_1z_2. Due to the circulant structure of M, the locations u_1, u_2 and the vectors v_1, v_2 can be constructed analogously. Using this geometric interpretation of the algebraically derived elements, the task is now to derive a construction scheme which combines the elements of the construction.

Algebraically, an obvious choice in the representation $z'_\mu = u_\mu + \omega v_\mu$ is $\omega = 0$, which leads to the familiar Napoleon configuration since in this case $z'_\mu = u_\mu$. Geometrically, an alternative construction is obtained by parallel translation of v_μ to u_μ. This is equivalent to the choice $\omega = 1$, hence $z'_0 = z_0 + \frac{\mathrm{i}}{\sqrt{3}}(z_2 - z_1)$. An according geometric construction scheme is depicted in Figure 5.

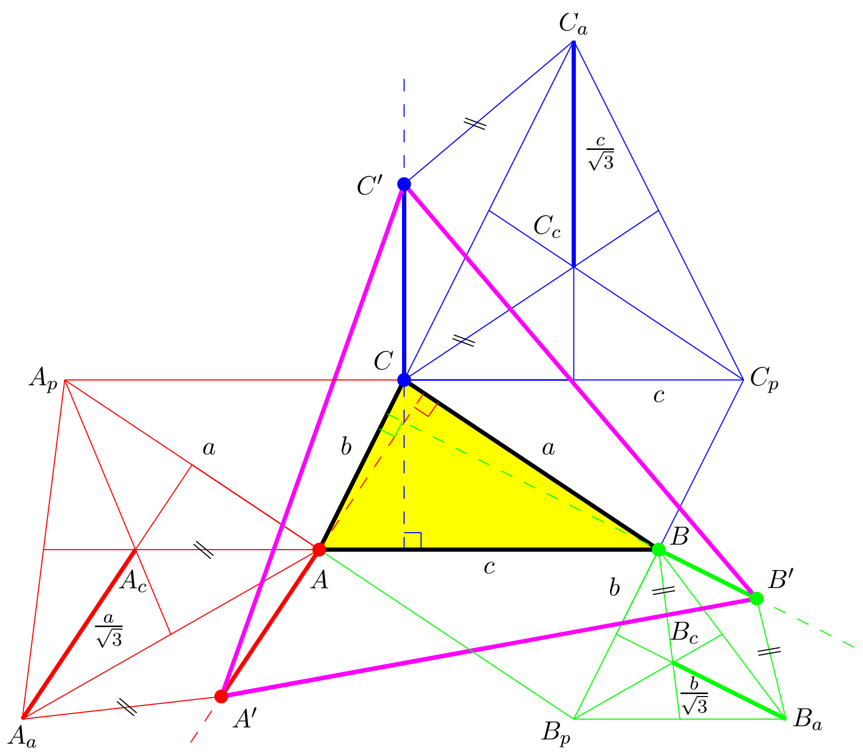

Figure 5. Construction of an equilateral triangle.

Thereby, the new position A' of A is derived as follows. First, the parallelogram $ABCA_p$ is constructed and an equilateral triangle is erected on AA_p. Since the distance from the centroid A_c to the apex A_a of this triangle is of the required length $a/\sqrt{3}$, where $a = |BC|$, one can transfer it by parallel translation to the vertical line on BC through A. The other vertices are constructed analogously as is also depicted in Figure 5.

According to the choice of parameters in the definition of M, the resulting triangle $A'B'C'$ is equilateral and oriented counterclockwise. A geometric proof is given by the fact that the triangle $A_pB_pC_p$ of the associated parallelogram vertices is similar to ABC with twice the side length. Due to their construction the new vertices A', B', and C' are the Napoleon vertices of $A_pB_pC_p$, hence $A'B'C'$ is equilateral. In particular, the midpoints of the sides of $A'B'C'$ yield the Napoleon triangle of ABC. Thus, A' can also be constructed by intersecting the line through A_p and A_c with the vertical line on BC through A.

3.3. *Transformation of quadrilaterals.* As a second example, the generalized Napoleon configuration in the case of $n = 4$, $k = 1$ is presented, that is $\omega = 0$ resulting in $z'_\mu = u_\mu$, $\mu \in \{0, \ldots, 3\}$. Using $r = \mathrm{i}$ and the representation (4) implies

$$\begin{aligned} u_0 &= \frac{1}{4}\sum_{\nu=0}^{3}(1 - r^{-\nu})z_\nu = \frac{1}{4}\Big((1+\mathrm{i})z_1 + (1+1)z_2 + (1-\mathrm{i})z_3\Big) \\ &= z_1 + \frac{1}{2}(z_2 - z_1) + \frac{1}{4}(z_3 - z_1) - \mathrm{i}\,\frac{1}{4}(z_3 - z_1)\,, \end{aligned}$$

which leads to the construction scheme depicted in Figure 6.

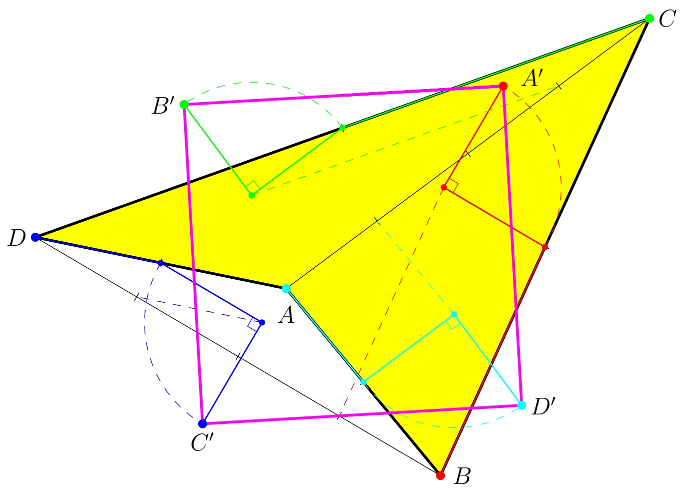

Figure 6. Construction of a regular quadrilateral.

As in the case of $n = 3$, the generalized Napoleon vertices can be constructed with the aid of scaled parallels and perpendiculars. Figure 6 depicts the intermediate vertices obtained by successively adding the summands given in the representation of u_μ from left to right. Parallels, as well as rotations by $\pi/2$ are marked by dashed lines. Diagonals, as well as subdivision markers are depicted by thin black lines.

3.4. *Transformation of pentagons.* In the case of $n = 5$, the root of unity is given by $r = (-1 + \sqrt{5})/4 + \mathrm{i}\sqrt{(5 + \sqrt{5})/8}$. The pentagon depicted on the left of Figure 1 has been transformed by using $k = 1$ and $\omega = 1$ resulting in

$$\begin{aligned} u_0 + v_0 = \; & z_0 + \frac{1}{\sqrt{5}}(z_1 - z_2) + \frac{1}{\sqrt{5}}(z_4 - z_3) \\ & - \mathrm{i}\,\frac{\sqrt{10 + 2\sqrt{5}}}{5}(z_1 - z_4) - \mathrm{i}\,\frac{\sqrt{10 - 2\sqrt{5}}}{5}(z_2 - z_3)\,. \end{aligned}$$

Due to the choice $k = 1$, z' is a regular convex pentagon. The same initial polygon transformed by using $k = 2$ and $\omega = 1$ resulting in

$$\begin{aligned} u_0 + v_0 = \; & z_0 + \frac{1}{\sqrt{5}}(z_2 - z_1) + \frac{1}{\sqrt{5}}(z_3 - z_4) \\ & - \mathrm{i}\,\frac{\sqrt{10 - 2\sqrt{5}}}{5}(z_1 - z_4) - \mathrm{i}\,\frac{\sqrt{10 + 2\sqrt{5}}}{5}(z_3 - z_2) \end{aligned}$$

is depicted on the right. Since $k = 2$ is not a divisor of $n = 5$, a star shaped nonconvex 2-regular polygon is constructed. Again, the representation also gives the intermediate constructed vertices based on scaled parallels and perpendiculars, which are marked by small markers. Thereby, auxiliary construction lines have been omitted in order to simplify the figure.

3.5. *Constructibility.* According to (4) the coefficients of the initial vertices z_μ in the representation of the new vertices z'_μ are given by $1 - r^{k(\mu-\nu)}$ and $\omega r^{k(\mu-\nu)}$ respectively. Using the polar form of the complex roots of unity, these involve the expressions $\cos(2\pi\xi/n)$ and $\sin(2\pi\xi/n)$, where $\xi \in \{0, \ldots, n-1\}$. Hence a compass and straightedge based construction scheme can only be derived if there exists a representation of these expressions and ω only using the constructible operations addition, subtraction, multiplication, division, complex conjugate, and square root.

Such representations are given exemplarily in the previous subsections for the cases $n \in \{3, 4, 5\}$. As is well known, Gauß proved in [4] that the regular polygon is constructible if n is a product of a power of two and any number of distinct Fermat prime numbers, that is numbers $F_m = 2^{(2^m)} + 1$ being prime. A proof of the necessity of this condition was given by Wantzel [13]. Thus, the first non constructible case using this scheme is given by $n = 7$. Nevertheless, there exists a neusis construction using a marked ruler to construct the associated regular heptagon.

4. Conclusion

A method of deriving construction schemes transforming arbitrary polygons into k-regular polygons has been presented. It is based on the theory of circulant matrices and the associated eigenpolygon decomposition. Following a converse approach, the polygon transformation matrix is defined by the choice of its eigenvalues representing the scaling and rotation parameters of the eigenpolygons. As has been shown for the special case of centroid preserving transformations leading to k-regular polygons, a general representation of the vertices of the new polygon can be derived in terms of the vertices of the initial polygon and an arbitrary transformation parameter ω. Furthermore, this leads to the definition of generalized Napoleon vertices, which are in the case of $n = 3$ identical to the vertices given by Napoleon's theorem.

In order to derive a new construction scheme, the number of vertices n and the regularity index k have to be chosen first. Since the remaining parameter ω has influence on the complexity of the geometric construction it should usually be chosen in order to minimize the number of construction steps. Finally giving a geometric interpretation of the algebraically derived representation of the new vertices is still a creative task. Examples for $n \in \{3, 4, 5\}$ demonstrate this procedure. Naturally, the problems in the construction of regular convex n-gons also apply in the presented scheme, since scaling factors of linear combinations of vertices have also to be constructible.

It is evident that construction schemes for arbitrary linear combinations of eigenpolygons leading to other symmetric configurations can be derived in a similar fashion. Furthermore, instead of setting specific eigenvalues to zero, causing the associated eigenpolygons to vanish, they could also be chosen in order to successively damp the associated eigenpolygons if the transformation is applied iteratively. This has been used by the authors to develop a new mesh smoothing scheme presented in [11, 12]. It is based on successively applying transformations to low quality mesh elements in order to regularize the polygonal element boundary iteratively. In this context transformations based on positive real valued eigenvalues are of particular interest, since they avoid the rotational effect known from other regularizing polygon transformations.

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